Review problems

Problem 1. In this problem, we'll analyze the Poisson effect for the extension of a homogeneous and isotropic elastic body which originally has the shape of a cube with sides of length L. We choose coordinates such that the origin is at the center of the cube, and the axes are aligned with its sides. We apply a tensile stress along the x axis, causing the cube to deform into a rectangular block, with the dimension along the x axis now $L+2\Delta L$, and along the y and z axes now $L-2\Delta L'$.

a) Consider a point inside the cube at position $\boldsymbol{r}=(x,y,z)$. After the deformation, the point has moved to $\boldsymbol{r}'=\boldsymbol{r}+\mathrm{d}\boldsymbol{r}$. As the origin stays fixed and the sides of the cube move, the deformation is not constant throughout the block. Argue why the local strain is given by

$$\mathrm{d}\gamma_x=\frac{\mathrm{d}x}{x},\quad \mathrm{d}\gamma_y=-\frac{\mathrm{d}y}{y},\quad \mathrm{d}\gamma_z=-\frac{\mathrm{d}z}{z}.$$

where we've taken the deformations $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$ to be all positive.

b) By definition of the Poisson ratio ν (and symmetry in y and z), we have $-\nu\mathrm{d}\gamma_x=\mathrm{d}\gamma_y=\mathrm{d}\gamma_z$. As our material is homogeneous, we can integrate the resulting differential equation, to get

$$-v\int_L^{L+\Delta L}\frac{\mathrm{d}x}{x}=-\int_{L-\Delta L'}^{L}\frac{\mathrm{d}y}{y}=\int_L^{L-\Delta L'}\frac{\mathrm{d}y}{y}$$

(We're leaving out the integral over z as it is identical to the one over y). Carry out the integration, to find an equation containing v and the ratios $\Delta L/L$ and $\Delta L'/L$.

c) Using the expansion $(1+x)^n=1+nx+O\left(x^2\right)$ (valid for small x, and even for n not an integer and/or negative), simplify the relation you found in (b) to get the approximate expression for the Poisson ratio we use mostly in practice.

Problem 2. The two-dimensional steady flow of a fluid with density ρ is given by

$$\boldsymbol{v}(x,y)=K\frac{-y\hat{\boldsymbol{x}}+x\hat{\boldsymbol{y}}}{x^2+y^2},$$

where K is a constant.

a) Can this flow correspond to the flow of an incompressible fluid?

b) Determine and sketch the streamlines of this flow and the acceleration field.

c) Determine the pressure difference between two points, a distance r_1 and $r_2>r_1$ away from the origin. (Hint: Start by taking two points on the positive x-axis and then generalize using a symmetry argument.) What happens at the origin?

Problem 3. In this exercise we analyze the laminar regime of pipe flow, called Poiseuille flow. We consider the steady laminar flow in a cylindrical pipe with length L and radius R ($R\ll L$) in the absence of gravity. A pressure difference $\Delta p=p_0-p_L$ is applied across the pipe and no-slip boundary conditions apply at the wall. Because of the rotational symmetry of the system, the flow profile can be described simply by a radial function $v(r)$ for $0\leq r\leq R$.

a) Take the Navier-Stokes equation for this case, write it in cylindrical coordinates, and show that it leads to a linear differential equation for the flow profile $v(r)$.

b) Calculate $v(r)$ from the differential equation, using the proper boundary conditions.

Problem 4. An explosion occurs in open air, creating a nearly spherical blast wave. The radius R of the blast wave is measured at different times t. The density of air is ρ. Assume the explosion releases energy E instantaneously. Using dimensional analysis (Buckingham π theorem), determine a dimensionless relationship between the blast radius R, the energy E, the air density ρ, and time t. Show that the blast radius follows a scaling law of the form $R(t) \propto \left(\frac{Et^2}{\rho}\right)^{1/5}$.

Problem 5. Derive an estimate for the acceleration/deceleration timescale T of a swimmer operating at low Reynolds number. The swimmer has characteristic length L, mass m, swims with typical speed U, and is immersed in a fluid of dynamic viscosity μ and density ρ. Express T in terms of m, μ, L and show how T compares with the swimming timescale L/U and with an oscillation period $1/\omega$. Finally, obtain a simple estimate of the extra (coasting) distance Δx travelled after the propulsive stroke stops and evaluate the order of magnitude of T and Δx for a typical bacterium ($L \sim 1$ μm, $U \sim 10^{-5}$–10^{-4} m/s).

Problem 6. Use dimensional analysis to argue why it's much easier to bend than to stretch a sheet of paper. Hint: Consider a sheet or rod of thickness d. Compare the scaling of stretching and bending energies.

Problem 7. In lecture we discussed the two simplest models for viscoelastic materials, Maxwell fluids and Kelvin Voigt solids, which we can construct pictorially by putting a dashpot and spring in series or in parallel, respectively. While both models capture an essential component of viscoelastic materials, they also each lack one, as a Maxwell fluid has no creep, and a Kelvin-Voigt solid no stress relaxation. We can however construct a three-component model that has both these features. This model is known as the standard linear solid model. It consists of one dashpot, with viscosity η, and two springs, with moduli G_1 and G_2. Given these four elements, we could construct (in principle) four different models, by putting them various series or parallel arrangements.

a) Explain why we don't get qualitatively 'new' models by either putting all three elements in series, or all three elements in parallel to each other.

b) One of the two remaining options is to put the new spring (with modulus G_2) in parallel to a Maxwell model (which consists of the spring with modulus G_1 in series with the dashpot). Derive the constitutive relation (the relation between the stress σ and the strain γ) for this model.

Problem 8. You release a tiny drop of ink in still water. After some time, the ink spreads out smoothly. This process is described by the diffusion equation:

$$\frac{\partial p}{\partial t} = D\frac{\partial^2 p}{\partial x^2},$$

where $p(x,t)$ is the concentration (or probability density) at position x and time t. What does this equation describe physically, and what does D represent? Why does the ink's spreading slow down over time even as it continues to expand?

For a single Brownian particle starting at the origin, the solution for this equation is Gaussian:

$$p(x,t) = \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{x^2}{4Dt}\right).$$

Why does this shape make sense physically, and how does it evolve in time? Give an argument based on this solution for hy the mean squared displacement (MSD) satisfies $\langle x^2(t)\rangle = 2Dt$. What does a graph of $\langle x^2(t)\rangle$ versus t look like, and what does its slope mean?

Problem 9. Now suppose a gentle breeze pushes the Brownian particle to the right. The Fokker–Planck equation describing its behavior becomes

$$\frac{\partial p}{\partial t} = -v\,\frac{\partial p}{\partial x} + D\,\frac{\partial^2 p}{\partial x^2}.$$

What does the new drift term $-v\,\partial_x p$ represent physically? How would the probability distribution evolve compared to pure diffusion? Imagine many identical particles, all starting at $x=0$. What does $p(x,t)$ represent for this ensemble, and how would a histogram of their positions compare to $p(x,t)$? Why can we think of $p(x,t)$ as a "density of states" in phase space?

Problem 10. A suspension of spherical polystyrene particles (density $\rho_p = 1050$ kg m^{-3}) is dispersed in water ($\rho_w = 1000$ kg m^{-3}, viscosity $\eta = 1.0 \times 10^{-3}$ Pa s) at room temperature $T = 298$ K. Each particle has radius $a = 500$ nm. Estimate the time to sediment a distance $h = 1$ cm by advection at velocity v. Comment on whether the particle suspension is well mixed.

Problem 11. In this problem, we'll make an attempt at estimating the force-extension relation of the worm-like chain model using scaling arguments. At relatively large forces, we expect the individual segments of our chain to 'more or less' line up with the direction of the force; the 'more or less' here is of course colloquial, but also accurately represents the fact that we still have a distribution of alignments. We can now introduce a unit of length ξ in between the large total chain length L and the small size b of a single segment as we'd have in a freely-jointed chain. At very short length scales, the polymer is simply straight; at our new intermediate length scales, the polymer is still relatively straight (as it resists bending) but not necessarily aligned with the external force. Therefore, this segment consists of a number of correlated links, which we'll call 'clinks'. Clinks are deformed through thermal fluctuations (of magnitude $k_{\mathrm{B}}T$), and resist bending because of the bending energy. Their resulting shape is an arc-like structure, deforming a distance h away from a straight line, reducing the end-to-end distance of that straight line from ξ to $\xi - \Delta$, see figure below.

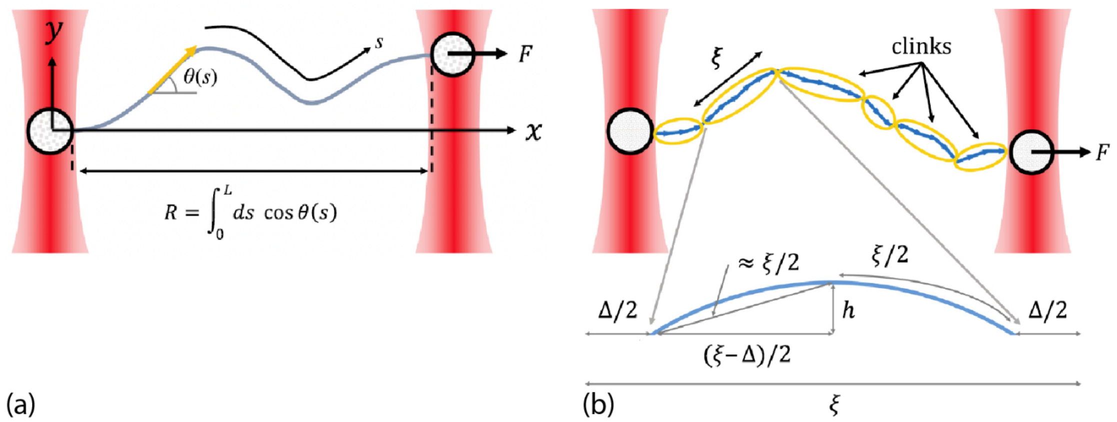

Figure 4.3: Force-extension experiment of a polymer. (a) The positions of the ends of the polymer are constrained by two beads; the distance between the beads is given by equation (4.29). (b) 'clinks' as used in the scaling approach of problem 8. Figure from Marantan and Mahadevan, Am. J. Phys. 2018.

a) If we approximate the shape of the clink as a circular arc, find its curvature in terms of h and ξ, in the approximation where both Δ and h are significantly smaller than ξ.

b) Calculate the bending energy of a clink, making use of appropriate approximations. You should get that the bending energy scales with $K_{\text{eff}}\, h^2/\xi^3$.

c) By comparing the bending energy in equilibrium with the thermal energy $k_{\mathrm{B}}T$, estimate $\langle h^2 \rangle$, and from that estimate, show that the end-to-end shrinkage of the clink is approximately given by $\Delta \sim \xi^2/\xi_{\mathrm{p}}$ (where, as before, $\xi_{\mathrm{p}} = K_{\text{eff}}/k_{\mathrm{B}}T$).

Problem 12. In a nematic liquid crystal, the rod-like molecules tend to align along a common direction called the director. However, in many situations-such as when the material is confined, surfaces impose different orientations, or temperature changes occur-this alignment cannot be maintained everywhere. As a result, topological defects (also called disclinations) appear, where the director field becomes undefined.

Consider a thin film of nematic liquid crystal confined between two parallel plates. The plates impose different anchoring directions: the bottom plate forces the director to lie horizontally, while the top plate forces the director to lie vertically. Somewhere in the film, the director field must change from one direction to the other.

a) Explain why defects must form in this situation.

b) A common defect in nematics is a $\pm\frac{1}{2}$ disclination, where the director rotates by $\pm 180^\circ$ around a loop enclosing the defect core. Explain qualitatively why such half-integer defects are allowed in nematic liquid crystals, and why they occur more frequently than integer defects.

c) Describe how the elastic energy of the liquid crystal tends to influence the size and structure of the region near a defect, and why the core of the defect often behaves more like an isotropic fluid.