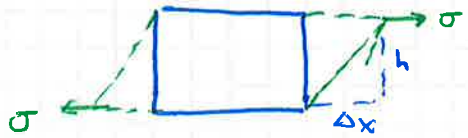
Lecture 3. Strain and stress tensors
Fluid dynamics

We have considered ~~the~~ two types of stress applied to continuum media:

Tensile stress

strain $\gamma = \frac{\Delta L}{L}$

Shear stress



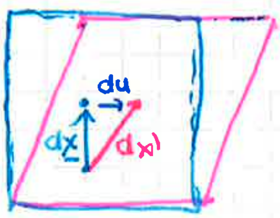
strain $\gamma = \frac{\Delta x}{h}$

How do we describe a force that is neither applied perpendicular or parallel to any surface of the object?
How can we describe more generally the deformation of the material in response to this force?

Treat stress and strain as tensors.

Tensor: Linear map from vectors to vectors represented by matrices. (If you are interested in their formal mathematical definition see last week's post on moodle)

Strain tensor



- Before the deformation we have two points in the material close together, with the vector $d\underline{x}$ connecting them
- After the deformation, the vector connecting them is
$d\underline{x}' = d\underline{x} + d\underline{u}$

The length of this vector is $(d\underline{x}' \cdot d\underline{x}')^{1/2}$

If we calculate $d\underline{x}' \cdot d\underline{x}'$ we will find the strain tensor:

$$d\underline{x}' \cdot d\underline{x}' = dx_i' dx_i' = (dx_i + du_i)(dx_i + du_i)$$

$$= dx_i dx_i + 2 du_i dx_i + du_i du_i$$

using the chain rule
$du_i = \frac{\partial u_i}{\partial x_k} dx_k$

$$= dx_i dx_i + 2\frac{\partial u_i}{\partial x_k} dx_i dx_k + \frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_l} dx_k dx_l$$

$$= dx_i dx_i + \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) dx_i dx_k + \frac{\partial u_j}{\partial x_i}\frac{\partial u_j}{\partial x_k} dx_i dx_k$$

$$= dx_i dx_i + \underbrace{\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} + \frac{\partial u_j}{\partial x_i}\frac{\partial u_j}{\partial x_k}\right)}_{\gamma_{ik}} dx_i dx_k$$

So we have that the infinitesimal strain tensor is given by:

$$\boxed{\gamma_{ik} = \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} + \frac{\partial u_j}{\partial x_i}\frac{\partial u_j}{\partial x_k}\right)}$$

Or in coordinate free notation:

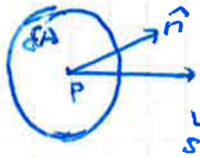
$$\underline{\underline{\gamma}} = \nabla \underline{u} + (\nabla \underline{u})^T + (\nabla \underline{u})^T \cdot (\nabla \underline{u})$$

For almost all cases we can assume that deformations are small so they have small gradients, such that we can limit ourselves to the linearized version of $\underline{\underline{\gamma}}$:

$$\gamma_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \quad ; \quad \underline{\underline{\gamma}} = \nabla \underline{u} + (\nabla \underline{u})^T$$

Stress tensor

The stress tensor maps the normal to a surface, $\hat{n}$, to the force per unit area, $\underline{P}$, acting on it:

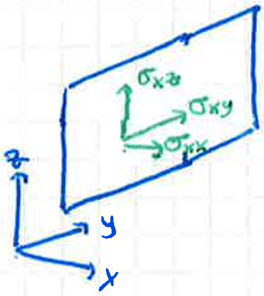


$$\underbrace{\underline{P}(\hat{n}, \underline{x}, t)}_{\text{stress vector}} = \underline{\underline{\sigma}} \cdot \hat{n}$$

$\uparrow\ \underline{P} = \underline{F}/A$

The force per unit area at any point P associated with a plane with normal $\hat{n}$ can be expressed in terms of the stresses acting on 3 mutually ⊥ planes alligned with the coordinate axes.

Consider the yz plane.



The normal to this plane is $\hat{e}_x$

For this plane the stress has 3 components
- normal one $\sigma_{xx} \rightarrow$ pressure
- two in plane ones σ_{xy} and $\sigma_{xz} \rightarrow$ shear stress

For a continuum rather than looking at the total force $\underline{F}$ we'll be interested in the force/unit volume $\underline{f}$. To find the relation between $\underline{f}$ and $\underline{\underline{\sigma}}$:

$$\underline{F} = \int_V \underline{f}\, dV$$

$\uparrow$ entire volume of the body

Now, the total force on the body is the sum of the volume/internal forces plus the surface forces. The internal forces cancel due to Newton's third law. While the surface forces of a volume V can be writen using the stress tensor as:

$$\underline{F}_s = \oint_{\partial V} \underline{P}\, dA = \oint_{\partial V} (\underline{\underline{\sigma}} \cdot \hat{n})\, dA \underset{\substack{\uparrow \\ \text{using the} \\ \text{divergence theorem}}}{=} \int_V (\nabla \cdot \underline{\underline{\sigma}})\, dV$$

So we have $\underline{F} = \underline{F}_s$

$$\Rightarrow \int_V \underline{f}\, dV = \int_V (\nabla \cdot \underline{\underline{\sigma}})\, dV$$

$$\Rightarrow \underline{f} = \nabla \cdot \underline{\underline{\sigma}}$$

the force per unit volume equals the divergence of the stress tensor

$\left(\text{or } f_i = \partial_j \sigma_{ij} \right)$

Note that unlike the ~~[illegible]~~ strain tensor which has no dimensions, the stress tensor has dimensions of force/area, or pressure.

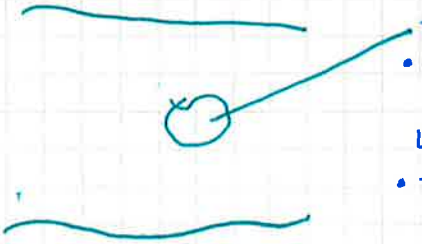
In the case where we have a force applied uniformly to the body (i.e radially from all sides & with the same magnitude), the stress tensor takes the form:

$$\sigma_{ij} = -p\delta_{ij}$$

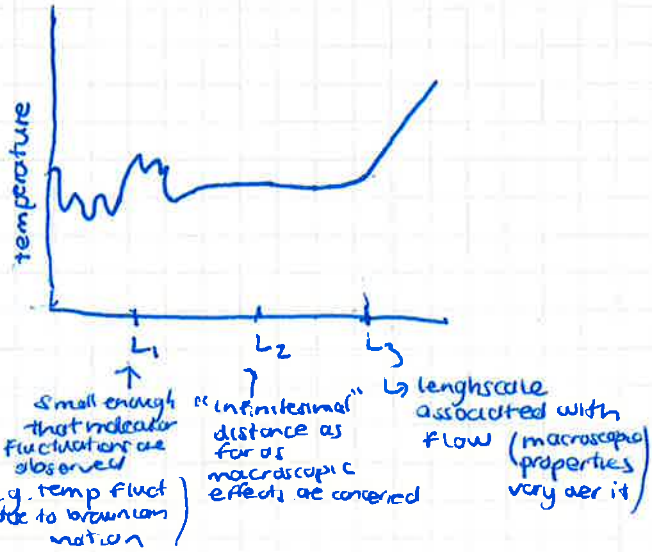
where p is the hydrostatic pressure.
(minus sign since pressure is exerted along the inwards pointing normal)

Fluid dynamics

we are now ready to consider the motion of a "fluid particle".



fluid particle
- ~~[illegible]~~ dimension α L_2 such that $L_1 << L_2 << L_3$
- fixed mass

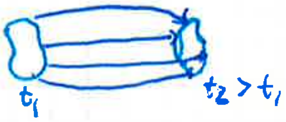


To set up the equations of motion for this fluid particle we need to decide on a coordinate system, there are 2 posibilities:

* Follow individual particles as they move
  Each particle is labelled by its initial position ($t=0$) → Lagrangian

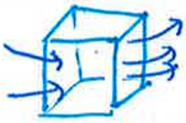
* stay fixed in space and watch particles as they flow through. Quantities are described as a function of their spatial location & time → Eulerian

Lagrangian description



Moving fluid ~~[illegible]~~ element

Eulerian description



Fixed volume element

Material pathline

$$\frac{dx}{dt} = v(x,t)$$

Stream line

$$v(\underline{x}, t)\Big|_{t=t_0}$$

A Lagrangian approach seems ideal to derive the equations of motion of a fluid particle, but in practice this formulation is not convenient to do numerical calculations (require fixed ~~[illegible]~~ BC) or experimental measurements (e.g. measuring pressure), so its easier to adopt the Eulerian description instead.

To ~~make~~ write down the equations of motion using the Eulerian description we need to find a way to express the rate of change of the properties of a fluid particle at a fixed point in the flow field.

Let's first consider the rate of change of a scalar quantity (for simplicity) $T(x,y,z,t)$, which can be the temperature in cartesian coordinates. Quite generally:

$$\left.\delta T = \frac{\partial T}{\partial t}\delta t + \frac{\partial T}{\partial x}\delta x + \frac{\partial T}{\partial y}\delta y + \frac{\partial T}{\partial z}\delta z\right\}$$

≡ small change δT produced by a change in time δt and small changes in position $\delta x, \delta y, \delta z$

To get the rate of change of this quantity we divide it by δt:

$$\frac{\delta T}{\delta t} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\overset{v_x}{\cancel{\frac{\delta x}{\delta t}}} + \frac{\partial T}{\partial y}\overset{v_y}{\cancel{\frac{\delta y}{\delta t}}} + \frac{\partial T}{\partial z}\overset{v_z}{\cancel{\frac{\delta z}{\delta t}}}$$

$$= \frac{\partial T}{\partial t} + (\underline{v}\cdot\nabla)T$$

$$= \underbrace{\left(\frac{\partial}{\partial t} + \underline{v}\cdot\nabla\right)}T$$

This gives us a way to go between descriptions

$$\underset{\text{Lagrangian}}{\frac{d}{dt}} = \underset{\text{Eulerian}}{\frac{\partial}{\partial t} + \underline{v}\cdot\nabla}$$

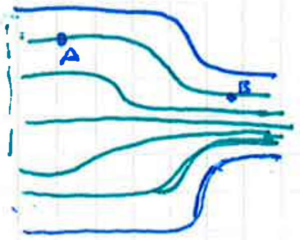
$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \underline{v}\cdot\nabla$$

convective or material or substantive derivative

$\frac{\partial}{\partial t}$:
- variation in time at a given point
- "local" part of total derivative
- only source of variation if field is uniform in space

$\underline{v}\cdot\nabla$:
- variation that occurs in space as a fluid particle passes through a given point
- Any quantity changes in response to the spatial variations in velocity (including the velocity itself)

To better understand the material derivative consider the following flow:



consider a steady flow going through a constriction. The rate of change of velocity at a point is:

$$\frac{D\underline{v}}{Dt} = \frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\nabla)\underline{v}$$

we know that the velocity of point B is greater because the crossectional area of B < ~~[illegible]~~ crossectional area of A ⇒ Fluid particle experiences an acceleration when it moves from A to B since the flow is steady $\frac{\partial \underline{v}}{\partial t} = 0$ so this acceleration is due to $(\underline{v}\cdot\nabla)$ term

to understand these concepts better watch video posted on moodle highly recomended!

Now we are ready to write the equation of motion for a fluid particle (of volume V):

$(\underline{F} = m\underline{a})$ .. First: $m\underline{a} = m\frac{D\underline{v}}{Dt} = \int \rho\, dV \frac{D\underline{v}}{Dt}$ (*) (where $\rho \equiv$ mass density)

Now all we need to determine is what are the forces $\underline{F}$ acting on the fluid element.
From our previous discussion we saw that the force per unit volume equals the divergence of the stress tensor ($f_i = \partial_j \sigma_{ij}$). If we consider the case of an ideal fluid (no dissipation), it cannot support shear, just compression, so the only contribution to the stress is the hydrostatic pressure $\sigma_{ij} = -p\delta_{ij}$ so

Now. $\underline{F} = \int_V \nabla\cdot\underline{\underline{\sigma}}\, dV = -\int_V \nabla p\, dV$ (**)

Equating (*) and (**) for an infinitesimal volume element we get:

$$-\nabla p = \rho\frac{D\underline{v}}{Dt} = \rho\left[\frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\nabla)\underline{v}\right]$$

$$\boxed{-\nabla p = \rho\left[\frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\nabla)\underline{v}\right] \quad \text{Euler's equation}}$$

Here we have only taken into account internal forces acting on the fluid. If there are other external forces acting on the fluid we also need to add them to $\underline{F}$, just remember they need to be expressed as forces per unit volume. For example if we add gravity we express it as $\rho\underline{g}$, so Euler's equation becomes:

$$\boxed{-\nabla p + \rho\underline{g} = \rho\left[\frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\nabla)\underline{v}\right]}$$; with $\underline{g}$ pointing downwards (as usual)

Note that from this eq we can derive the eq. for hydrostatic pressure, setting $\underline{v} = 0$ we get the equation $\nabla p = \rho\underline{g}$, which has the familiar solution: $p = p_0 + \rho g z$, with $z \equiv$ depth, $p_0 \equiv$ surface pressure

Equations of motion for a viscous flow

If we now take friction into account, for each component of the force on a fluid element we need to consider its action not only perpendicular to the surface, but also parallel to it. Therefore we need to consider the general form of the stress tensor

$$\underline{f} = \nabla\cdot\underline{\underline{\sigma}} \quad \text{or} \quad f_i = \partial_j \sigma_{ij}$$

We can write down the stress tensor for viscous flow as the sum of the hydrostatic plus viscous terms:

$$\sigma_{ij} = \underbrace{-p\delta_{ij}}_{\text{hydrostatic part}} + \underbrace{\sigma'_{ij}}_{\text{viscous part}}$$

(shear flow)

We saw in the first lecture that friction between fluid layers occurs when they move at different speeds. Therefore the viscous stress tensor σ'_{ij} should scale with gradients of the velocity.

Let's consider the velocity gradient tensor:

$A_{ij} = \frac{\partial v_i}{\partial x_j}$ This tensor contains the information about how velocity varies in space (field)

We can decompose this tensor into a symmetric and antisymmetric part:

$$A_{ij} = \underbrace{\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)}_{\text{strain rate tensor}} + \underbrace{\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)}_{\text{vorticity tensor}}$$

Describes the changes in shape or size of a fluid element

Describes rigid body rotations of fluid elements. It's the angular velocity.
Because it's antisymmetric in 3D it's defined by 3 independent components, given by the vorticity vector

$$w_k = \epsilon_{ijk}\Omega_{ij} \quad (\text{or } \underline{w} = \nabla \times \underline{v})$$

Antisymmetric combinations of velocity gradients cannot occur in the viscous stress tensor. In a purely rotational flow all radial layers have the same angular velocity, so there is no friction between them (even if layers further out move with higher linear velocity).

So the viscous stress tensor will be proportional to the strain rate tensor $\dot{\gamma}_{ij} = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}$

We can split the strain rate tensor into a pure shear and a pure compression (or expansion) part:

$$\dot{\gamma}_{ij} = \dot{\gamma}_{ij}^{shear} + \dot{\gamma}_{ij}^{vol} \quad ; \quad \dot{\gamma}_{ij}^{shear} = \dot{\gamma}_{ij} - \frac{1}{3}\mathrm{Tr}(\underline{\underline{\dot{\gamma}}})\delta_{ij}$$

- Traceless part of the strain rate tensor
- Element changes shape w/o changing volume

$$\dot{\gamma}_{ij}^{vol} = \frac{1}{3}\mathrm{Tr}(\underline{\underline{\dot{\gamma}}})\delta_{ij}$$

- Isotropic part of strain rate tensor proportional to the trace
- Element changes its volume w/o changing its shape

With this we can write the viscous stress that in general depends on either part with different coefficients as:

$$\boxed{\sigma_{ij}' = \eta\left[\dot{\gamma}_{ij} - \frac{1}{3}\mathrm{Tr}(\underline{\underline{\dot{\gamma}}})\delta_{ij}\right] + \frac{1}{3}\frac{\xi}{2}\mathrm{Tr}(\underline{\underline{\dot{\gamma}}})\delta_{ij}}$$

where η and ξ are the first and second viscosity.

Note that for incompressible fluids (there is no change in volume) the trace of σ_{ij}' vanishes so the second viscosity is ill defined. For liquids we only talk about η.

We are finally ready to write the equation of motion for a viscous flow:

$$\underline{F} = m\underline{a} \qquad \text{where } \underline{F} = \int_V \nabla \cdot \underline{\underline{\sigma}}\, dV \quad \text{and } \underline{\underline{\sigma}} = -p\underline{\underline{I}} + \underline{\underline{\sigma}}'$$

$$m\underline{a} = \int_V \rho\, dV \frac{D\underline{v}}{Dt} = \int_V \rho\, dV \left[\frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\nabla)\underline{v}\right]$$

$$\Rightarrow \nabla \cdot \underline{\underline{\sigma}} = \rho\left[\frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\nabla)\underline{v}\right],$$

More generally we can add other body forces, $\underline{F}_B$:

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{F}_B = \rho\left[\frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\nabla)\underline{v}\right] \quad \text{or in components:}$$

$$\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = \rho\left(\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}\right).$$

We need to subs the expression for σ_{ij} to get the eq of motion.

$$-\frac{\partial p}{\partial x_i} + \frac{\partial \sigma_{ij}'}{\partial x_i} + b_i = \rho\left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}\right)$$

Let's calculate $\frac{\partial \sigma_{ij}'}{\partial x_i}$:

$$\frac{\partial \sigma_{ij}'}{\partial x_j} = \frac{\partial}{\partial x_j}\left\{\eta\left[\dot{\gamma}_{ij} - \frac{1}{3}Tr(\dot{\underline{\underline{\gamma}}})\delta_{ij}\right] + \frac{1}{3}\frac{\xi}{2}Tr(\dot{\underline{\underline{\gamma}}})\delta_{ij}\right\}$$

$$= \eta\frac{\partial}{\partial x_j}\dot{\gamma}_{ij} + \frac{1}{3}\frac{\partial}{\partial x_j}Tr(\dot{\underline{\underline{\gamma}}})\delta_{ij}\left[-\eta + \frac{\xi}{2}\right]$$

$$= \eta\frac{\partial}{\partial x_j}\dot{\gamma}_{ij} + \frac{1}{3}\left(-\eta + \frac{\xi}{2}\right)\frac{\partial}{\partial x_i}\left(Tr(\dot{\underline{\underline{\gamma}}})\right] \quad ; \text{ now } \dot{\gamma} = \left(\frac{\partial}{\partial x_j}v_i + \frac{\partial v_j}{\partial x_i}\right)$$

$$= \eta\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}v_i + \eta\frac{\partial}{\partial x_j}\frac{\partial v_j}{\partial x_i} + \frac{1}{3}\left(-\eta + \frac{\xi}{2}\right)\frac{\partial}{\partial x_i}Tr(\dot{\underline{\underline{\gamma}}})$$

$$= \eta\Delta v_i + \eta\frac{\partial}{\partial x_i}\frac{\partial v_j}{\partial x_j} + \frac{1}{3}\left(-\eta + \frac{\xi}{2}\right)\frac{\partial}{\partial x_i}Tr(\dot{\underline{\underline{\gamma}}})$$

$$= \eta\Delta v_i + \eta\frac{\partial}{\partial x_i}Tr(\dot{\underline{\underline{\gamma}}}) + \frac{1}{3}\left(-\eta + \frac{\xi}{2}\right)\frac{\partial}{\partial x_i}Tr(\dot{\underline{\underline{\gamma}}})$$

$$= \eta\Delta v_i + \frac{1}{3}\frac{\partial}{\partial x_i}Tr(\dot{\underline{\underline{\gamma}}})\left[3\eta - \eta + \frac{\xi}{2}\right]$$

$$= \eta\Delta v_i + \frac{1}{3}\frac{\partial}{\partial x_i}Tr(\dot{\underline{\underline{\gamma}}})\left(2\eta + \frac{\xi}{2}\right)$$

$$= \eta\nabla^2\underline{v} + \frac{1}{3}\left(2\eta + \frac{\xi}{2}\right)\nabla(\nabla\cdot\underline{v})$$

So our equation of motion becomes:

$$-\nabla p + \eta\nabla^2\underline{v} + \frac{1}{3}\left(2\eta + \frac{\xi}{2}\right)\nabla(\nabla\cdot\underline{v}) + \underline{F}_{ext} = \rho\left[\frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\nabla)\underline{v}\right]$$

Now if we consider an incompressible fluid subject to NO external forces:

$$\boxed{\begin{aligned} &\nabla\cdot\underline{v} = 0 \\ &-\frac{\nabla p}{\rho} + \nu\nabla^2\underline{v} = \frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\nabla)\underline{v} \end{aligned}}$$

Navier-Stokes equations

where we have defined $\nu \equiv \frac{\eta}{\rho}$

the kinematic viscosity