

### Exercise sheet #4

**Problem 1.** In this exercise we analyze the laminar regime of pipe flow, called Poiseuille flow. We consider the steady laminar flow in a cylindrical pipe with length  $L$  and radius  $R$  ( $R \ll L$ ) in the absence of gravity. A pressure difference  $\Delta p = p_0 - p_L$  is applied across the pipe and no-slip boundary conditions apply at the wall. Because of the rotational symmetry of the system, the flow profile can be described simply by a radial function  $v(r)$  for  $0 \leq r \leq R$ .

- Take the Navier-Stokes equation for this case, write it in cylindrical coordinates, and show that it leads to a linear differential equation for the flow profile  $v(r)$ .
- Calculate  $v(r)$  from the differential equation, using the proper boundary conditions.

**Problem 2.** In the last problem we solved the Navier-Stokes equations for fluid in a cylindrical pipe. To do so, we used the version of the equations in which we already knew what the viscous stress tensor is. We can also find an equation for the stress tensor itself, which will be valid also for non-Newtonian fluids. This derivation will also illustrate how the Navier-Stokes equations are an expression of conservation of momentum.

We again consider the flow through a cylindrical pipe of radius  $R$  and length  $L$ , ignoring any effects at the end of the pipe. We will generalize the problem to also allow for gravitationally driven flow, by tilting the cylinder such that its axis makes an angle  $\theta$  with the horizontal.

- Identify the nonzero components of the velocity, pressure, and stress tensor, and on which of the coordinates they can depend.
- Verify that there can be no convective flow in the cylinder.

We now consider a thin ring-shaped cylinder inside the pipe (see figure below). The ring has inner radius  $r$ , thickness  $\Delta r$ , and is as long as the pipe itself. Because the velocity can depend on the radial coordinate, there is a difference in the momentum of the fluid at the inner and outer surface of our thin cylinder. Because there is no convective term, and the fluid is moving faster inside than outside (which we know because the fluid is stationary at  $r = R$  due to the no-slip boundary condition), momentum must effectively be generated inside our thin cylinder. The 'generation' of momentum is due to the net force acting on the fluid (due to the pressure difference and gravitational force). By Newton's second law, the force equals the rate of momentum generation.

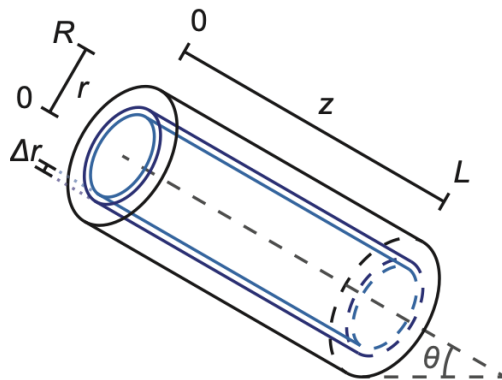


Figure 1: Cylindrical pipe of radius  $R$  with an inner cylindrical tube of inner radius  $r$  and thickness  $\Delta r$ .

- c) Express the rate of momentum flow across the inner surface of the cylinder in terms of the cylinder's properties and the viscous stress tensor of the fluid (remember that the stress tensor represents a force per unit area; what we need is the effective force on the total inner surface, as that equals the momentum rate across that surface).
- d) Also write down the rate of momentum flow across the outer surface of the cylinder (doing so should be a trivial extension of your answer at (c)).
- e) Write down the total amount of momentum generated in our thin cylinder by the external forces (pressure difference and gravity).
- f) In steady state, there should be no net momentum change in the cylinder; we therefore get (momentum flow out) - (momentum flow in) = (momentum generated). Substitute your expressions from (c-e) to get an equation for the stress tensor.
- g) Now take the limit that  $\Delta r \rightarrow 0$ , and show that we get

$$\frac{1}{r} \frac{d}{dr} (r\sigma_{rz}) = \frac{\Delta p}{L} + \rho g \sin(\theta) \quad (1)$$

- h) Show that if we consider the viscous stress tensor  $\sigma'_{ij} = \eta\gamma'_{ij}$  for an incompressible flow in cylindrical coordinates, equation 1 predicts the same flow as you found in the previous problem.