Exercise sheet #3

Problem 1. Which of the following equations are valid expressions using index notation? If you decide an expression is invalid, state which rule is violated.

a) $S_{ij} = \sigma_{ij} - \sigma_{kk}\delta_{ij}$

b) $\in_{ijk}\in_{kkj}= 0$

c) $\rho\frac{\partial u_i}{\partial t} = \frac{\partial}{\partial x_j}C_{ijkl}\frac{\partial u_k}{\partial x_l}$

Problem 2. Use index notation rules to show that $\nabla \times \nabla \times \mathbf{u} = \nabla\nabla \cdot \mathbf{u} - \nabla^2\mathbf{u}$

Problem 3. In this problem we will use the summation convention (repeated indices are presumed to be summed over). Here $\boldsymbol{A}$ and $\boldsymbol{B}$ are two-tensors (which can be represented by matrices), and $\boldsymbol{x}, \boldsymbol{v}$ and $\boldsymbol{w}$ are vectors. The Levi-Civita symbol (not a tensor) is defined as:

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } ijk \text{ is an even permutation of 123,} \\ -1 & \text{if } ijk \text{ is an odd permutation of 123,} \\ 0 & \text{otherwise.} \end{cases}$$

a) Write $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{B}$ in index notation.

b) Write the eigenvector equation $\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v}$ in index notation.

c) Calculate:

$$\frac{\partial x_i}{\partial x_j} \quad \text{and} \quad \left(\delta_{ij} - \frac{x_i x_j}{x^2}\right)\delta_{ij}$$

where x is the length of the vector $\boldsymbol{x}$ with n independent components x_i.

d) Express $\boldsymbol{v} \times \boldsymbol{w}$ and $\det(\boldsymbol{A})$ in index notation using the Levi-Civita symbol; for the determinant you only need to do the two-dimensional case (so $\boldsymbol{A}$ can be represented by a 2×2 matrix).

e) Write $\partial_k A_{ik}$ and $v_k \partial_k v_i$ in index-free notation.

Problem 4. Consider the most general symmetric 2-tensor $\boldsymbol{A}$. The tensor can be written as the sum of two components, $\boldsymbol{A} = \alpha\boldsymbol{I} + \beta\boldsymbol{\Phi}$, with $\mathrm{Tr}(\boldsymbol{\Phi}) = 0$ (where Tr is the trace) and (without loss of generality) $\det(\boldsymbol{\Phi}) = -1$. Writing the decomposition in components, we have:

$$A_{ij} = \alpha\delta_{ij} + \beta\Phi_{ij}$$

In this problem, we'll work in two dimensions, in which case the 2-tensor can be represented as a 2×2 matrix (note that it would be an $n \times n$ matrix in n dimensions):

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} = \alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \beta\Phi$$

Find α and β in terms of a, b and c.