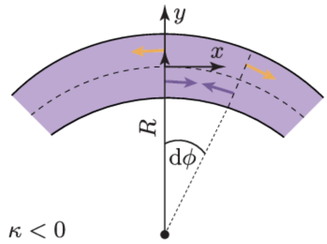
# Exercise sheet #7

**Problem 1.** Derivation of the bending energy of a bent rod/sheet




a) Argue for the situation sketched in the figure that for a sheet bent in the $x$ direction only, the fact that there is no external pressure applied to the surface of the sheet implies that at the upper and lower side $\sigma_{yi} = 0$ for $i = x, y, z$.

b) Argue that since in particular $\sigma_{yy} = 0$ at the upper and lower sides, to lowest order in the curvature we must have $\sigma_{vy} = 0$ everywhere within the sheet.

c) Use the linear stress-strain equation $\sigma_{ij} = K\gamma_{kk}\delta_{ij} + 2G(\gamma_{ij} - \frac{1}{3}\gamma_{kk}\delta ij)$ for $\sigma_{yy} = 0$ to derive $\gamma_{yy} \approx \frac{-\nu}{1-\nu}\frac{y}{R}$.
Hint: It is easiest to express $K$ and $G$ immediately in terms of $E_\gamma$ and $\nu$ using equations $K = \frac{E_\gamma}{3(1-2\nu)}$ and $G = \frac{E_\gamma}{2(1+\nu)}$.

d) Show with these results that $\sigma_{xx} = \frac{E_\gamma y}{(1-\nu^2)R}$ and that $\sigma_{ij}u_{ij} = E_\gamma y^2 / \left(1-\nu^2\right) R^2$.

**Problem 2.** Use dimensional analysis to argue why it's much easier to bend than to stretch a sheet of paper. Hint: Consider a sheet or rod of thickness $d$. Compare the scaling of stretching and bending energies.

**Problem 3.** In biological systems, elastic materials are often constructed as networks of elastic filaments. A good example is the spectrin network underlying the plasma membrane of a red blood cell, which has a roughly triangular structure. In this problem, we'll study an (ideal) two-dimensional triangular network of springs. Each spring has spring constant $k$, rest length $s_0$, and potential energy $V(s) = \frac{1}{2}k\left(s - s_0\right)^2$. Note that because we are dealing with a two-dimensional system here, we will use the free energy not per unit volume but per unit area; the two-dimensional version of Hooke's law reads

$$\sigma_{ij} = \frac{1}{2}K_2\gamma_{kk}\delta_{ij} + G_2\left(\gamma_{ij} - \frac{1}{2}\gamma_{kk}\delta_{ij}\right),$$

where $K_2$ and $G_2$ are the two-dimensional bulk and shear modulus.

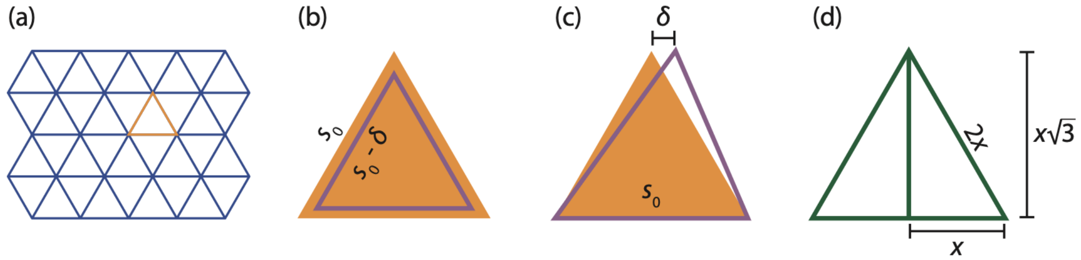



Figure 2.1: Triangular network of springs, as a model for the e.g. the spectrin network in a red blood cell. (a) Configuration of the equilateral triangles. (b) Pure compression by decreasing the length of each spring from $s_0$ to $s_0 - \delta$. (c) Pure shear by moving the top vertex of a triangle by a distance $\delta$ in the direction parallel to the bottom line. (d) Some geometric properties of an equilateral triangle.

a) Suppose we apply a pure compression to our triangular spring network, starting from its equilibrium configuration. The compression results in shortening the length of each spring by a small amount $\delta = s_0 - s$ (see figure 2.1b). Express the change in potential energy per triangle, $\Delta V$, in terms of $\delta$. Note that each spring is shared by two triangles.

b) By dividing your expression in (a) by the area per triangle (in the equilibrium configuration), find the change in the free energy density $\Delta f$ for the small compression.

c) Express the strain tensor for the pure compression in terms of $s_0$ and $\delta$. You may ignore any higherorder terms in $\delta$.

d) Substitute the expression for the strain tensor in the expression of the free energy for a continuous material, $\Delta f = \frac{1}{4}\sigma_{ij}\gamma_{ij}$, to obtain an expression for $\Delta f$ in $s_0$ and (to lowest order in) $\delta$. You can use the 2D version of Hooke's law from equation $V(u) = -\int F(u)\mathrm{d}u = \frac{1}{2}ku^2$ here. You can use the tensor version of Hooke's law to relate the stress and the strain, but note that here we have a 2D material whereas the expressions in equation $\sigma_{ij} = \frac{1}{2}K\gamma_{kk}\delta_{ij} + G\left(\gamma_{ij} - \frac{1}{3}\gamma_{kk}\delta_{ij}\right)$ is for 3D.

e) Compare the expression at (d) with your answer at (b) to show that the bulk modulus of the spring network is given by $K_2 = \frac{1}{2}\sqrt{3}k$.

f) From the pure shear deformation in figure 2.1c, show that the shear modulus of the network is given by $G_2 = \frac{1}{4}\sqrt{3}k$.