Exercise sheet #6

Problem 1. Deriving Stokes' law $\boldsymbol{F}_{\mathrm{d}} = -6\pi\eta R\boldsymbol{v}$, for the drag force on a sphere at low Reynolds number is a nontrivial exercise if you try to solve the whole problem in one go. However, because the Stokes equation is linear, there is an alternative way, as we can use the superposition principle to write the drag force as a sum over point forces on the surface of the sphere. As we know the solution of Stokes' equation for a point force (given by the Oseen tensor $v_i = \frac{1}{8\pi\eta r}\left(\delta_{ij} + \frac{r_i r_j}{r^2}\right) F_j = J_{ij}(\boldsymbol{r})F_j$.), we can then sum up all flows to get the total velocity for a given applied force, or inversely, the force needed to sustain a given velocity. Because our sphere is a continuous object, the point forces will be distributed continuously over the surface, turning our sum into an integral.

To set the stage, suppose we have a sphere of radius R with its center at the origin, embedded in an incompressible fluid with viscosity η. We exert a force $\boldsymbol{F} = F\hat{\boldsymbol{z}}$ on the sphere, causing it to move with (currently unknown) velocity $\boldsymbol{v}$ (of course we'll find that $\boldsymbol{v}$ will also be in the z direction). We make sure that our force and resulting velocity are low enough that we are in the low Reynolds number regime, allowing us to use Stokes law. When the sphere is moving at constant velocity, it exerts a net force $\boldsymbol{F}$ on the surrounding fluid; by Newton's third law, the surrounding fluid exerts an equal but opposite net drag force on the sphere. These sphere-fluid forces act at the surface of the sphere, resulting in an effective force per unit area of the sphere of $F\hat{\boldsymbol{z}}/4\pi R^2$ [1]. For the velocity we then find

$$\boldsymbol{v} = \oint_{\partial V} \mathrm{d}S \boldsymbol{J}(\boldsymbol{r}) \cdot \frac{F\hat{\boldsymbol{z}}}{4\pi R^2} \tag{1}$$

The integral in eq (1) is taken over the surface of the sphere.

a) Substitute the expression for the Oseen tensor:

$$J_{ij}(\boldsymbol{x}) = \frac{1}{\eta}\frac{1}{8\pi x}\left(\delta_{ij} + \frac{x_i x_j}{x^2}\right) \tag{2}$$

in eq (1), simplifying as much as possible.

b) Write the metric $\mathrm{d}S$ and the position vector $\boldsymbol{r}$ in spherical coordinates.

c) Show that the integrals over the x and y components are zero (you can do this without actually calculating an integral, but evaluating the integral can also be done).

d) Evaluate the integral over the z component to get

$$v = \frac{F}{6\pi\eta R}\hat{z} \tag{3}$$

e) Argue why Stokes law follows from eq. 3.

A slightly more general form of eq. 1 allows for arbitrary surface shape and center position:

$$\boldsymbol{v} = \oint_{\partial V} \mathrm{d}S \boldsymbol{J}\left(\boldsymbol{r} - \boldsymbol{r}'\right) \cdot \boldsymbol{f}(\boldsymbol{r}),$$

where $\boldsymbol{f}$ is still a force per unit (surface) area, and the object is centered at $\boldsymbol{r}'$. This equation can be used to calculate the drag force on an ellipsoid or on a cylinder (as a limit of a long sequence of spheres).

[1] To see that this is true, simply integrate $F\hat{\boldsymbol{z}}/4\pi R^2$ over the area of the sphere, and we retrieve the net force $\boldsymbol{F}$.

Problem 2. Derive an estimate for the acceleration/deceleration timescale T of a swimmer operating at low Reynolds number. The swimmer has characteristic length L, mass m, swims with typical speed U, and is immersed in a fluid of dynamic viscosity μ and density ρ. Express T in terms of m, μ, L and show how T compares with the swimming timescale L/U and with an oscillation period $1/\omega$. Finally, obtain a simple estimate of the extra (coasting) distance Δx travelled after the propulsive stroke stops and evaluate the order of magnitude of T and Δx for a typical bacterium ($L \sim 1$ μm, $U \sim 10^{-5}$–10^{-4} m/s).

Problem 3. How do the scaling results for acceleration and deceleration timescales in the last problem change at high Reynolds number?