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Simplified derivation of the Fokker-Planck equation

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An alternative derivation of the Fokker-Planck equation for the probability density of a random noise process is presented, starting from the Langevin equation. The derivation makes use of the first two derivatives of the Dirac delta function. The derivation of the Fokker-Planck equation then becomes simpler and more transparent, at least for those willing to accept these singularity functions.

I. INTRODUCTION

The Fokker-Planck equation describes the diffusive spread and drift of the probability density for a random variable under the influence of a potential plus random noise forces. Fokker-Planck equations are widely used in probability and statistics,[1,2] communications and noise theory,[3] and most recently in quantum electronics and quantum noise theory.[4]

The mathematical form of the Fokker-Planck equation is usually derived either by considering some specific model for the discrete random walk of a particle and then taking the limit of very many, very small steps;[1,4] or by a more complicated approach that involves defining a somewhat arbitrary and mysterious integral, interchanging orders of integration, manipulating the integral, and arguing that since the integral vanishes part of its integrand must also vanish.[2,3] The first approach is lacking in generality, while the second approach is lacking in motivation and transparency.

We present here a rather compact and physically transparent derivation from which the Fokker-Planck equation falls out directly. The derivation is or is not mathematically acceptable depending on whether or not you are willing to accept derivatives of the Dirac delta function as mathematically (and physically) useful representations of reality.

II. DERIVATION

Consider a continuous random process $x(t)$ which may be viewed for descriptive purposes as the position of a particle acted upon by a macroscopic potential plus microscopic random noise forces (collisions, Brownian, motion, etc.). The position of an individual particle versus time then obeys a Langevin equation of the general form

$$\dot{x}(t) + \beta(x) = f(t), \tag{1}$$

where $\beta(x)$ describes the macroscopic potential in which the particle moves and $f(t)$ describes the microscopic random noise forces acting on the particle. This particle description is, of course, merely a graphic shorthand for describing any observable physical quantity $x(t)$, such as particle position, particle velocity, signal voltage, or whatever, which obeys an equation of the Langevin form.

Since the function $x(t)$ is a random process we cannot simply solve Eq. (1) but must instead consider the statistical properties of the process $x(t)$, such as its probability distribution, ensemble or time averages, autocorrelation function, and so forth. In particular, consider the probability density $p(x,t)$ for the process $x(t)$. In particle terms $p(x,t)$ dx for an ensemble of particles gives the probability that any one particle will be found within a small range dx centered about x at time t. More generally, of course, $p(x,t)$ dx expresses the probability that the value of the process $x(t)$ at time t will be within the small range dx about x.

Specifically, we want to consider the conditional probability or Green's function $p(x,t|x_0,t_0)$, which is the probability density over x at time t for a collection of particles all of which were at point x_0 at time t_0. This function thus has the initial condition at $t = t_0$ that

$$p(x,t = t_0|x_0,t_0) = \delta(x - x_0), \tag{2}$$

where $\delta(x)$ is the Dirac delta function. For an uncorrelated process with some specified initial probability density $p_0(x_0,t_0)$ at time t_0, the probability density at any later time t can then be obtained from

$$p(x,t) = \int_{-\infty}^{\infty} p(x,t|x_0,t_0)p_0(x_0,t_0)dx_0. \tag{3}$$

If the underlying process $x(t)$ is governed by a Langevin equation like (1), then the Green's function $p(x,t|x_0,t_0)$ is found to be governed by the Fokker-Planck equation

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial x}[\beta(x)p] + D\frac{\partial^2 p}{\partial x^2}, \tag{4}$$

where D is a diffusion coefficient associated with the microscopic random noise forces $f(t)$. Our objective here is to derive this last equation.

We first note that different particles reaching point x at time t from point x_0 at time t_0 may have been at various different points x_1 at any intermediate time t_1, where $t_0 \le t_1 \le t$. From the usual arguments of conditional probability theory we can write the probability for a particle being at (x,t) having been at (x_0,t_0) in terms of the probability of getting from (x_0,t_0) to (x_1,t_1), times the probability of getting from (x_1,t_1) to (x,t), summed over all possible intermediate positions (x_1,t_1). This gives the so-called Smoluchowski equation

$$p(x,t|x_0,t_0) = \int_{-\infty}^{\infty} p(x,t|x_1,t_1)p(x_1,t_1|x_0,t_0)dx_1. \tag{5}$$

This form of the equation does assume that the evolution of the probability distribution from the intermediate time t_1 to the final time t is uncorrelated with the previous history from t_0 to t_1. This is equivalent to assuming a Markoff condition, which says in simple terms that the value of the random force $f(t)$ between t_1 and t is entirely uncorrelated

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with its value during the earlier time interval from t_0 to t_1.

The time evolution of $p(x,t)$ from time t to time $t + \Delta t$, a very small time later, may be written

$$\frac{\partial p(x,t|x_0,t_0)}{\partial t} = \lim_{\Delta t \to 0} \frac{p(x,t + \Delta t|x_0,t_0) - p(x,t|x_0,t_0)}{\Delta t} \quad (6)$$

Let us drop the limit notation for simplicity and use the Smoluchowski equation (5) for $p(x,t + \Delta t)$. We then have

$$\Delta t \; \frac{\partial p(x,t|x_0,t_0)}{\partial t} = \int p(x,t + \Delta t|x_1,t) \times p(x_1,t|x_0,t_0)dx_1 - p(x,t|x_0,t_0). \quad (7)$$

In the integral for $p(x,t + \Delta t|x_0,t_0)$ we have chosen t_1 (which is arbitrary so long as $t_0 \leq t_1 \leq t + \Delta t$) to have the value $t_1 = t$.

Now, if Δt is a very short time interval then $p(x,t + \Delta t|x_1,t)$ cannot have changed very much from its initial delta function value. That is, from Eq. (2) it must be that in the limit for $\Delta t \to 0$,

$$\lim_{\Delta t \to 0} p(x,t + \Delta t|x_1,t) = \delta(x - x_1). \quad (8)$$

Therefore, as the key step in this derivation we assume that it is meaningful to expand p at $t + \Delta t$ to first order in Δt in the form

$$p(x,t + \Delta t|x_1,t) \approx \delta^{(0)}(x - x_1) + \Delta t[a_1(x_1)\delta^{(1)}(x - x_1) + a_2(x_1)\delta^{(2)}(x - x_1) + \cdots], \quad (9)$$

where $\delta^{(1)}(x)$ and $\delta^{(2)}(x)$ are the first and second derivatives of the dirac delta function $\delta^{(0)}(x)$. That is, $\delta^{(1)}(x)$ is an asymmetric function which can be viewed as two delta functions of opposite sign displaced apart by a very small amount, while $\delta^{(2)}(x)$ is the derivative of this function in turn.[5,6] Figure 1 illustrates one possible way of representing these functions using finite rectangular impulse approximations.

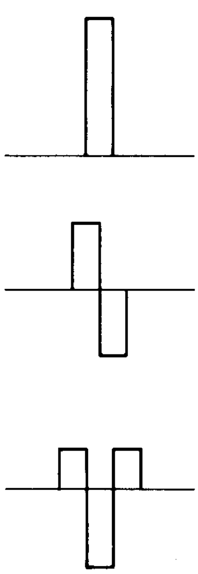

Fig. 1. Rectangular impulse representation of the delta function $\delta^{(0)}(x)$ and its first two derivatives $\delta^{(1)}(x)$ and $\delta^{(2)}(x)$.

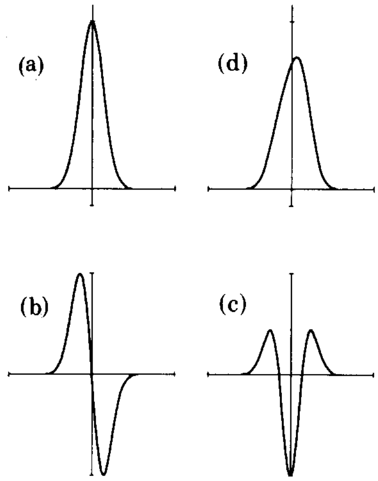

Fig. 2. (a) Gaussian approximation to a unit impulse function; (b) and (c) analytic first and second derivative; (d) combination of original impulse plus small amounts of the first two derivatives.

We will not attempt to debate the mathematical validity of this expansion here. Glauber has commented on some of the mathematical weaknesses of such an expansion.[7] It seems physically clear that adding a small component of the delta-function first derivative $\delta^{(1)}(x)$ will shift the center of gravity of the initial delta function slightly to one side or the other, simulating a drift of the density distribution; while adding a small component of $\delta^{(2)}(x)$ will, in effect, depress the center and raise the wings of $p(x,t)$, simulating a diffusive broadening. Both of these effects may be linear to first order in Δt. The coefficients $a_1(x_1)$ and $a_2(x_1)$ may depend on the initial position coordinate x_1 since the drift velocity and the noise-induced diffusive spreading may in general be different at different initial points x_1.

Figure 2 shows as another example a Gaussian representation of the delta function (top left), the first two analytic derivatives of this gaussian (bottom row), and the shifted and broadened function (top right) that results from combining small amounts of the first two derivative functions with the initial Gaussian function.

The nth order delta function derivative $\delta^{(n)}(x)$ has the mathematical property that

$$\int f(x)\delta^{(n)}(x - x_0)\, dx = \frac{(-1)^n}{n!} f^{(n)}(x_0) = \frac{(-1)^n}{n!} \left(\frac{\partial^n f(x)}{\partial x^n} \right)_{x=x_0}. \quad (10)$$

Therefore, putting the expansion (9) into the integral in Eq. (7) leads to

$$\frac{\partial p(x,t|x_0,t_0)}{\partial t} = -\frac{\partial}{\partial x}[a_1(x)p(x,t)|x_0,t_0)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[a_2(x)p(x,t|x_0,t_0)]. \quad (11)$$

We can also use the expansion (9) directly to find the moments about x_1 that the density function develops in the short time Δt, namely

$$\overline{\Delta x}(\Delta t) = \int_{-\infty}^{\infty} (x - x_1)p(x,t + \Delta t|x_1,t)\, dx \approx a_1(x_1)\Delta t \quad (12)$$

and

$$\overline{\Delta x^2}(\Delta t) = \int_{-\infty}^{\infty} (x - x_1)^2 p(x,t + \Delta t|x_1,t)dx$$
$$\approx a_2(x_1)\Delta t. \tag{13}$$

On the other hand we can also solve for these same moments by directly solving the Langevin equation for the motion $x(t + \Delta t)$ of a particle starting at x_1 at time t. Integrating the Langevin equation over a short time interval Δt gives to first order

$$\Delta x(\Delta t) = x(t + \Delta t) - x_1$$
$$\approx -\beta(x_1)\Delta t + \int_t^{t+\Delta t} f(t')\, dt'. \tag{14}$$

We assume that the noise forces $f(t)$ have zero mean

$$\overline{f(t)} = 0 \tag{15}$$

and hence an ensemble average of (14) over different examples of Langevin noise term leads to

$$\overline{\Delta x}(\Delta t) = -\beta(x_1)\Delta t. \tag{16}$$

Squaring Eq. (14), dropping terms of order Δt^2, and again using the zero mean value of $f(t)$ then leads to the squared ensemble average

$$\overline{\Delta x^2}(\Delta t) = \int_t^{t+\Delta t} dt' \int_t^{t+\Delta t} dt'' \overline{f(t')f(t'')}. \tag{17}$$

Now under the Markoff assumption $f(t)$ is assumed to be delta-function correlated, i.e., it has a squared ensemble average of the form

$$\overline{f(t')f(t'')} = D\delta(t' - t''). \tag{18}$$

That is, $f(t')$ and $f(t'')$ are completely uncorrelated unless $t' = t''$, and there the mean-square value is infinite. Note that this implies that the random forces $f(t)$ have a white noise spectrum with a finite power per unit bandwidth. In reality, of course, the noise process $f(t)$ will presumably have an autocorrelation function which is a finite approximation to the delta function, but with a width in $t' - t''$ much narrower than any finite time interval Δt over which it would be reasonable to make measurements. To put this in another way, the spectrum of $f(t)$ is flat to frequencies much above any frequency at which the particle's motion $x(t)$ can respond.

The ensemble-averaged second moment then becomes

$$\overline{\Delta x^2}(\Delta t) = D\, \Delta t. \tag{19}$$

Combining Eqs. (11) through (19) then gives us immediately the desired Fokker-Planck result

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial x}[\beta(x)p] + D\frac{\partial^2 p}{\partial x^2}. \tag{20}$$

The noise forces $f(t)$ and the related diffusion coefficient D are in most cases of interest not dependent on the coordinate x, and hence we have ignored this dependence, as is usually done.

III. EXAMPLE

The simplest nontrivial example of a specific Fokker-Planck equation is the case where β is constant, $\beta = \beta_0$, leading to

$$\frac{\partial p}{\partial t} = \beta_0 \frac{\partial p}{\partial x} + D\frac{\partial^2 p}{\partial x^2} \tag{21}$$

The solution to this is

$$p(x,t) = \frac{\exp\{-[x - \overline{x}(t)]^2/2\sigma^2(t)\}}{[2\pi^2(t)]^{1/2}}, \tag{22}$$

with

$$\overline{x}(t) = x_0 - \beta_0 t, \tag{23}$$

$$\sigma^2(t) = 2Dt. \tag{24}$$

The distribution is a Gaussian whose center drifts linearly with time, while the variance also increases linearly with time.

A second simple example is a linear restoring force which produces a restoring velocity proportional to displacement from the origin, i.e.,

$$\beta(x) = \beta_1 x. \tag{25}$$

The solution in this case is the same Gaussian as above, but with

$$\overline{x}(t) = x_1 e^{-\beta_1 t}, \tag{26}$$

$$\sigma^2(t) = (D/\beta_1)(1 - e^{-2\beta_1 t}). \tag{27}$$

Starting from a delta function at an initial displacement x_1, the center of the distribution moves exponentially in to the origin, while the variance grows from an initial value of zero to a constant final value $\sigma^2 = D/\beta_1$ for which the outward diffusion term D just balances the inward force term β_1.

IV. DISCUSSION

Most of the preceding derivation closely parallels the standard derivations, as it must since the underlying physics is constant. The primary gimmick is using the delta-function derivative expansion (9) for the evolution of $p(x,t + \Delta t|x,t)$ at small times Δt. This expansion seems (at least to this author) to be physically meaningful, and it very readily yields both the mathematical form of the Fokker-Planck equation and the first and second moments linear in Δt.

Given the usefulness of the delta-function-derivative expansion here, perhaps we should be on the lookout for other situations where this same expansion can be used again to tell us something we already know.

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