

# Classical Electrodynamics

## Week 9

1. Consider an infinite cylinder of radius  $a$  carrying the current  $I$ . The cylinder is surrounded by an insulator with magnetic permeability  $\mu$ . A metallic cylindrical surface of radius  $b > a$  conducts the current in the opposite direction.

Determine the magnetic field  $\mathbf{H}$ , the magnetic induction  $\mathbf{B}$  and the magnetization  $\mathbf{M}$  in every point in space. Find the free current density  $\mathbf{J}$  and the average microscopic current density  $\langle \mathbf{j} \rangle$ .

*Solution*

The Maxwell equations of magnetostatics are

$$\begin{aligned}\nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{H} &= \mathbf{J}\end{aligned}\tag{1}$$

For  $a < r < b$ , when we calculate the circulation of  $H$  along a circle of radius  $r$  we obtain

$$\int \mathbf{H} \cdot d\mathbf{l} = \int \nabla \times \mathbf{H} \cdot d\mathbf{S} = \int \mathbf{J} \cdot d\mathbf{S} = I ,\tag{2}$$

where we have used the Stokes theorem, the second equation above and the definition of current density. Since we know, because of the cylindrical symmetry, that  $\mathbf{H} = H(r)\mathbf{e}_\phi$ , we can write

$$\int \mathbf{H} \cdot d\mathbf{l} = H(r)2\pi r ,\tag{3}$$

and thus

$$\mathbf{H} = \frac{I}{2\pi r}\mathbf{e}_\phi .\tag{4}$$

For  $r > b$  the total current is zero, so we have  $\mathbf{H} = 0$ . For  $r < a$ , the current relevant for  $H$  becomes  $J\pi r^2 = \frac{Ir^2}{a^2}$ . Now that we have  $\mathbf{H}$ , we can calculate  $\mathbf{B}$  using

$$\mathbf{B} = \mu\mathbf{H} .\tag{5}$$

The magnetic induction is

$$\begin{aligned}\mathbf{B} &= \frac{\mu_0 I r}{2\pi a^2}\mathbf{e}_\phi & \text{for } r < a \\ \mathbf{B} &= \frac{\mu I}{2\pi r}\mathbf{e}_\phi & \text{for } a < r < b \\ \mathbf{B} &= 0 & \text{for } r > b ,\end{aligned}\tag{6}$$

and the magnetization is given by  $\mathbf{M} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{H}$  and we obtain as expected a non-vanishing component only in the insulator

$$\begin{aligned}\mathbf{M} &= 0 & \text{for } r < a \\ \mathbf{M} &= \frac{I}{2\pi r} \frac{\mu - \mu_0}{\mu_0}\mathbf{e}_\phi & \text{for } a < r < b \\ \mathbf{M} &= 0 & \text{for } r > b .\end{aligned}\tag{7}$$

In order to find the average microscopic current density we start by writing  $\mathbf{M}$  in a more synthetic way:

$$\frac{I}{2\pi r} \frac{\mu - \mu_0}{\mu_0} \mathbf{e}_\phi (\Theta(r - a) - \Theta(r - b)) . \quad (8)$$

We can now apply the formula

$$\langle \mathbf{j} \rangle = \mathbf{J} + \nabla \times \mathbf{M} , \quad (9)$$

and when we do so, we must remember that the derivative of the Heaviside function  $\Theta$  is the Dirac  $\delta$ , as we have shown in a previous exercise. We obtain

$$\nabla \times \mathbf{M} = \frac{1}{r} \frac{\partial}{\partial r} (rM(r)) \mathbf{e}_z = \frac{I}{2\pi r} \frac{\mu - \mu_0}{\mu_0} \mathbf{e}_z (\delta(r - a) - \delta(r - b)) . \quad (10)$$

Note that, since  $\mu > \mu_0$  for a paramagnetic insulator and  $\mu < \mu_0$  for a diamagnetic insulator, we can see that a paramagnetic insulator gives a current in the same direction of  $\mathbf{J}$ , so that  $\mathbf{B}$  is increased, whereas in a diamagnetic insulator the direction is the opposite and  $\mathbf{B}$  is decreased.

**2.** In electrostatics, the  $n$ -th pole is given by

$$Q_{i_1 \dots i_n} = \int d^3x \rho(\mathbf{x}) T_{i_1 \dots i_n}(\mathbf{x}) , \quad (11)$$

where the totally symmetric tensor  $T_{i_1 \dots i_n}$  can be defined by

$$T_{i_1 \dots i_n} = (2n - 1)!! x_{i_1} \dots x_{i_n} - A_{i_1 \dots i_n} , \quad (12)$$

with the double factorial  $(2n - 1)!! = (2n - 1)(2n - 3)(2n - 5) \dots (5)(3)(1)$  and  $(-1)!! = 1$ . The tensor  $A_{i_1 \dots i_n}$  is an homogeneous polynomial of degree  $n$  in the components of  $\mathbf{x}$  and it contains at least one Kronecker- $\delta$  so that the trace vanishes:

$$T_{i_1 \dots i_n} \delta_{i_k i_l} = 0 , \quad \forall k, l \in \{1, 2, \dots, n\} , \quad k \neq l . \quad (13)$$

It is convenient to introduce the notation  $B_{(i_1 \dots i_n)}$  for the total symmetrization of a tensor  $B_{i_1 \dots i_n}$ . More precisely,

$$B_{(i_1 \dots i_n)} \equiv \frac{1}{n!} \sum_{perm \ \sigma} B_{\sigma(i_1 \dots i_n)} , \quad (14)$$

where the sum runs over all permutations  $\sigma$  of the  $n$  indices  $i_1 \dots i_n$ . For example,

$$B_{(ij)} \equiv \frac{1}{2} (B_{ij} + B_{ji}) , \quad v_{(i} w_j) \equiv \frac{1}{2} (v_i w_j + v_j w_i) \quad (15)$$

$$B_{(ijk)} \equiv \frac{1}{6} (B_{ijk} + B_{ikj} + B_{jik} + B_{jki} + B_{kij} + B_{kji}) . \quad (16)$$

a) Argue that the tensors  $A_{i_1 \dots i_n}$ , for  $n = 2, 3, 4$ , must be of the form

$$A_{ij} = c_2 x^2 \delta_{ij} \quad (17)$$

$$A_{ijk} = c_3 x^2 \delta_{(ij} x_k) \quad (18)$$

$$A_{ijkl} = c_4 x^2 \delta_{(ij} x_k x_l) + c'_4 x^4 \delta_{(ij} \delta_{kl)}, \quad (19)$$

where  $c_2, c_3, c_4$  and  $c'_4$  are numerical constants.

b) Determine the constants  $c_2, c_3, c_4$  and  $c'_4$  imposing the trace condition (13).

c) How many independent components does the tensor  $Q_{i_1 \dots i_n}$  have? Start by working out the cases  $n = 0, 1, 2$ . Can you guess the formula for general  $n$ ?

**Hint:** Start by counting the number of independent components in a totally symmetric tensor with  $n$  indices and then impose the trace constraint.

d) Generalize the previous question to  $d$  space dimensions. Show that the number of independent components of a traceless symmetric tensor  $Q_{i_1 \dots i_n}$  in  $d$  dimensions (*i.e.*, each index can take the values  $1, 2, \dots, d$ ) is given by

$$(2n + d - 2) \frac{(n + d - 3)!}{n!(d - 2)!}. \quad (20)$$

### Solution

a) The building blocks for the tensors  $A_{i_1 \dots i_n}$  are two elementary tensors:  $\delta_{ij}$  and  $x_j$ . We need to put at least one  $\delta$ , and we need to build a tensor with  $n$  indices. The most general structure is obtained by multiplying  $k$  Kronecker- $\delta$ s and  $n - 2k$  components of  $\mathbf{x}$ , with  $k = 1, 2, \dots, [n/2]$ ,  $[n/2]$  being the integer part of  $n/2$ . This yields tensors with  $n$  indices, and degree  $n - 2k$  in the components of  $\mathbf{x}$ . To compensate, we multiply the result by  $x^{2k}$ . Finally, we symmetrize the indices as described in the text. This fixes the rank  $n$  tensor up to  $[n/2]$  coefficients  $c_{k,n}$ :

$$A_{i_1 \dots i_n} = \sum_{k=1}^{[n/2]} c_{k,n} x^{2k} \underbrace{\delta_{(i_1 i_2} \dots \delta_{i_{2k-1} i_{2k}})}_k \underbrace{x_{i_{2k+1}} \dots x_{i_n})}_{n-2k}. \quad (21)$$

Choosing  $n = 2, 3, 4$  we get the specific cases shown in the text. You may still object: why did we write  $x^{2k}$  rather than picking some specific components, say  $(x_1)^{2k}$ ? The resulting tensor would have been of the right degree in  $\mathbf{x}$ . The easy answer is that the purpose in life of  $A_{i_1 \dots i_n}$  is to cancel the traces from the tensor  $x_{i_1} \dots x_{i_n}$ . It is easy to see that all traces of the latter generate factors of  $x^2$  and not specific components. There is also a fancier explanation. The tensor  $T_{i_1 \dots i_n}$  in eq. (2) appears in the Taylor expansion of a function of  $\mathbf{x}$  and  $\mathbf{x}'$ :

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \sum_{n=0}^{\infty} \frac{1}{n!} T^{i_1 \dots i_n}(\mathbf{x}') \frac{x_{i_1} \dots x_{i_n}}{x^{2n+1}}. \quad (22)$$

The function on the left hand side is invariant under rotations of the reference frame, so each term on the right hand side must be as well. Specific components of the vector  $\mathbf{x}'$  are not invariant under rotations, of course, and so cannot appear. Only scalar products can appear on the right hand side. In particular,  $x'^2 = \mathbf{x}' \cdot \mathbf{x}'$  is allowed, as are  $\mathbf{x} \cdot \mathbf{x}'$  and  $x^2$ .

b) Let's start with  $n = 2$ :

$$(2n - 1)!! \delta^{ij} x_i x_j - c_2 x^2 \delta^{ij} \delta_{ij} = 3x^2 - 3c_2 x^2 = 0, \quad (23)$$

that is,

$$c_2 = 1. \quad (24)$$

Starting from  $n = 3$ , we need to symmetrize. Since  $\delta_{ij}$  is already symmetric, we do not need to add up all nine permutations of the three indices:

$$\delta_{(ij)x_k} = \frac{1}{3} (\delta_{ij} x_k + \delta_{ik} x_j + \delta_{jk} x_i). \quad (25)$$

Therefore,

$$5 \cdot 3 x^2 x_k = \frac{c_3}{3} x^2 (3x_k + x_k + x_k), \quad (26)$$

that is,

$$c_3 = 9. \quad (27)$$

In the  $n = 4$  case, the permutations of four elements are 24, but we can again use symmetry to reduce the needed number.  $\delta_{ij} x_k x_l$  is symmetric under the exchanges  $i \leftrightarrow j$  and  $k \leftrightarrow l$ , so we only need 6 permutations:

$$\delta_{(ij)x_k x_l} = \frac{1}{6} (\delta_{ij} x_k x_l + \delta_{ik} x_j x_l + \delta_{il} x_k x_j + \delta_{jk} x_i x_l + \delta_{jl} x_k x_i + \delta_{kl} x_i x_j). \quad (28)$$

Finally,  $\delta_{ij} \delta_{kl}$  has the same symmetries of the previous case, plus the exchange  $(ij) \leftrightarrow (kl)$ . We only need 3 permutations:

$$\delta_{(ij)\delta_{kl}} = \frac{1}{3} (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}). \quad (29)$$

Taking the trace of  $T_{ijkl}$  leads to the following:

$$7 \cdot 5 \cdot 3 x_k x_l = \frac{c_4}{6} (7x_k x_l + \delta_{kl} x^2) + \frac{5}{3} c'_4 x^2 \delta_{kl}. \quad (30)$$

This implies

$$c_4 = 90, \quad c'_4 = -\frac{c_4}{10} = -9. \quad (31)$$

c) and d) We directly give the solution for general dimension. Since the tensor  $T_{i_1 \dots i_n}$  is symmetric, we are free to order the indices, say in increasing order, and count all the components of the kind

$$T_{\underbrace{1 \dots 1}_{k_1 \text{ times}} \underbrace{2 \dots 2}_{k_2 \text{ times}} \dots \underbrace{d \dots d}_{k_d \text{ times}}}, \quad \sum_{i=1}^d k_i = n. \quad (32)$$

In other words, we need to count the partitions of  $n$  in a maximum of  $d$  addends - the  $k'_i$ s are allowed to be zero. The problem can be rephrased in a way which makes the solution easy. Let us draw a wall each time we increase the value of the index:

$$\underbrace{\boxed{1} \dots \boxed{1} \boxed{2} \dots \boxed{2} \dots \boxed{d} \dots \boxed{d}}_{n+d-1 \text{ boxes}} \quad (33)$$

We include the  $d-1$  walls in the count of the boxes. Now, different partitions are completely specified by the position of the walls. So here is the new problem: count in how many ways we can choose  $d-1$  boxes out of  $n+d-1$ . But this is just the definition of a binomial, so we arrive at the following partial answer:

$$\begin{array}{l} \# \text{ of independent components of} \\ \text{a symmetric tensor of rank } n \text{ in } d \\ \text{dimensions} \end{array} = \binom{n+d-1}{d-1} = \frac{(n+d-1)!}{n!(d-1)!}.$$

In setting the traces to zero, we can again take advantage of the symmetry. Once the following equation is satisfied:

$$\delta^{i_1 i_2} T_{i_1 i_2 i_3 \dots i_n} = 0, \quad (34)$$

all other traces are zero as well. Formula (34) amounts to a set of homogeneous linear equations for the components of the tensor. The number of equations is equal to the number of independent ways to choose the indices  $i_3$  to  $i_n$  which are not contracted. This is the number of independent components of a symmetric tensor of rank  $n-2$ . Each equation provides a constraint, and so we get the final answer by subtracting the number of constraints:

$$\begin{array}{l} \# \text{ of independent components of} \\ \text{a symmetric traceless tensor of} \\ \text{rank } n \text{ in } d \text{ dimensions} \end{array} = \binom{n+d-1}{d-1} - \binom{n+d-3}{d-1} \\ = (2n+d-2) \frac{(n+d-3)!}{n!(d-2)!}.$$

### 3. Velocity's transformations in Special Relativity

In this exercise, you will study how velocities  $v_1$  and  $v_2$  transform from one reference frame  $\mathcal{R}_1$  to another  $\mathcal{R}_2$  in the context of Special Relativity, in different scenarios.

- a) The reference frame  $\mathcal{R}_1$  is moving along the  $x$ -axis with speed  $v_0$  in the reference frame of the laboratory  $\mathcal{R}_0$ , and a particle is moving with speed  $v_1$  along the  $x$ -axis in the reference frame  $\mathcal{R}_1$ . What is the velocity of the particle in the reference frame  $\mathcal{R}_0$ ?
- b) Two particles (with the same mass) are moving in the same direction with velocities  $v_1$  and  $v_2 > v_1$  in the reference frame  $\mathcal{R}_1$ . At what speed should the reference frame  $\mathcal{R}_2$  move with respect to  $\mathcal{R}_1$  so that the center of mass condition  $v'_1 + v'_2 = 0$  is obeyed? What is the value of  $v'_1$ ? Is your result compatible with your non-relativistic intuition?
- c) The trajectory of a particle moving at constant velocity makes an angle  $\theta$  with the  $x$ -axis of a reference frame  $\mathcal{R}_1$ . Compute the corresponding angle  $\theta'$  in a reference frame  $\mathcal{R}_2$  moving with speed  $v$  along the  $x$ -axis of  $\mathcal{R}_1$ .

- d) Consider two particles (with the same mass) moving at the same speed  $v$ . The angle between their trajectories is  $\theta$ . Find a reference frame in which  $\mathbf{v}'_1 + \mathbf{v}'_2 = 0$ .

*Solution*

- a) **First method** We will write  $\tilde{\mathbf{x}} = (ct, x)$  the coordinates in  $\mathcal{R}_0$  and  $\tilde{\mathbf{x}}' = (ct', x')$  the coordinates in  $\mathcal{R}_1$ . The particle is moving with speed  $v_1$  along  $x$  in  $\mathcal{R}_1$  so by definition:

$$v_1 = \frac{dx'}{dt'} \quad (35)$$

and we want to express the speed of the particle in  $\mathcal{R}_0$  which is:

$$v = \frac{dx}{dt}. \quad (36)$$

We want to express  $(ct, x)$  as function of  $(ct', x')$  so let's write the change of coordinates from  $\mathcal{R}_1$  to  $\mathcal{R}_0$ . The laboratory frame is moving along the  $x$ -axis with velocity  $-v_0$  in the frame  $\mathcal{R}_1$  so:

$$\begin{cases} ct = \gamma_0(ct' + \beta_0 x') \\ x = \gamma_0(x' + \beta_0 ct') \end{cases} \quad (37)$$

where

$$\beta_0 = \frac{v_0}{c} \quad \text{and} \quad \gamma_0 = \frac{1}{\sqrt{1 - \frac{v_0^2}{c^2}}}. \quad (38)$$

Then we have

$$\begin{aligned} \frac{dx}{dt} &= c \frac{dx' + \beta_0 c dt'}{cdt' + \beta_0 dx'} = c \frac{v_1 + c\beta_0}{c + \beta_0 v_1} \\ &= \frac{v_1 + v_0}{1 + \frac{v_1 v_0}{c^2}}. \end{aligned} \quad (39)$$

Notice that in the non-relativistic limit, this reduces to the usual formula  $v = v_0 + v_1$ .

**Second method** Let us call  $\mathcal{R}_2$  the rest frame of the particle.  $\mathcal{R}_2$  is moving along the  $x$ -axis with speed  $v_1$  in the frame  $\mathcal{R}_1$  and we want to express its speed  $v$  in the frame  $\mathcal{R}_0$ .

We can express  $\tilde{\mathbf{x}}'' = (ct'', x'')$ , the coordinates in  $\mathcal{R}_2$ , as function of coordinates of  $\mathcal{R}_0$  by making two successive boosts, the first of velocity  $v_0$  to go from  $\mathcal{R}_0$  to  $\mathcal{R}_1$ , the second of velocity  $v_1$  to go from  $\mathcal{R}_1$  to  $\mathcal{R}_2$ . In matrix notation (keeping only the  $(ct, x)$  coordinates), the transformations are defined as

$$\tilde{\mathbf{x}}' = \Lambda(v_0)\tilde{\mathbf{x}}, \quad \tilde{\mathbf{x}}'' = \Lambda(v_1)\tilde{\mathbf{x}}' \quad \Rightarrow \quad \tilde{\mathbf{x}}'' = \Lambda(v_1)\Lambda(v_0)\tilde{\mathbf{x}}. \quad (40)$$

The transformation can be written as:

$$\begin{aligned} \Lambda(v_1)\Lambda(v_0) &= \begin{pmatrix} \gamma_1 & -\beta_1\gamma_1 \\ -\beta_1\gamma_1 & \gamma_1 \end{pmatrix} \begin{pmatrix} \gamma_0 & -\beta_0\gamma_0 \\ -\beta_0\gamma_0 & \gamma_0 \end{pmatrix} \\ &= \begin{pmatrix} \gamma_0\gamma_1(1 + \beta_0\beta_1) & -\gamma_0\gamma_1(\beta_0 + \beta_1) \\ -\gamma_0\gamma_1(\beta_0 + \beta_1) & \gamma_0\gamma_1(1 + \beta_0\beta_1) \end{pmatrix}. \end{aligned} \quad (41)$$

But Lorentz transformation form a group, which means that:

$$\Lambda(v_1)\Lambda(v_0) = \Lambda(v) = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}, \quad (42)$$

which corresponds to doing one boost from the frame  $\mathcal{R}_0$  to  $\mathcal{R}_2$

$$\tilde{\mathbf{x}}'' = \Lambda(v)\tilde{\mathbf{x}}. \quad (43)$$

We need to solve only one component of the matrix equation. For example, writing  $\Lambda(v)_{11}$  using equations (42) and (41) one has:

$$\gamma = \gamma_0\gamma_1(1 + \beta_0\beta_1). \quad (44)$$

After inverting and squaring this reduces to:

$$1 - \frac{v^2}{c^2} = \frac{\left(1 - \frac{v_0^2}{c^2}\right)\left(1 - \frac{v_1^2}{c^2}\right)}{\left(1 + \frac{v_0v_1}{c^2}\right)^2}, \quad (45)$$

and finally

$$v = \frac{v_1 + v_0}{1 + \frac{v_1v_0}{c^2}}. \quad (46)$$

- b)** The reference frame  $\mathcal{R}_2$  is boosted with respect to  $\mathcal{R}_\infty$  by a velocity  $v$  in the same direction as  $v_1$  and  $v_2$ . The velocity of the particles in the reference frame  $\mathcal{R}_2$  was derived in the previous point:

$$v'_{1,2} = \frac{v_{1,2} - v}{1 - \frac{vv_{1,2}}{c^2}}. \quad (47)$$

We find  $v$  through the equation  $v'_1 + v'_2 = 0$ , which gives:

$$\frac{\beta_1 - \beta}{1 - \beta_1\beta} + \frac{\beta_2 - \beta}{1 - \beta_2\beta} = 0, \quad (48)$$

where, as usual, we denote  $\beta_i = v_i/c$  the velocities measured in units of  $c$ . This gives the following second order equation:

$$\beta^2 - 2\beta \left( \frac{1 + \beta_1\beta_2}{\beta_1 + \beta_2} \right) + 1 = 0, \quad (49)$$

whose solutions are

$$\beta_\pm = \frac{1}{\beta_1 + \beta_2} \left( 1 + \beta_1\beta_2 \pm \sqrt{(1 - \beta_1^2)(1 - \beta_2^2)} \right). \quad (50)$$

It is not difficult to show that the plus sign leads to an unphysical velocity  $\beta > 1$ , therefore we choose the minus sign. Equivalently, one can notice that, when  $\beta_1 \rightarrow -\beta_2$ ,  $\beta_+ \rightarrow \infty$  while  $\beta_- \rightarrow 0$ , which is the obvious correct result. Plugging the solution back in eq. (47), we get

$$\beta'_1 = \frac{1}{\beta_2 - \beta_1} \left( -1 + \beta_1\beta_2 + \sqrt{(1 - \beta_1^2)(1 - \beta_2^2)} \right). \quad (51)$$

In the non-relativistic limit, these formulae are reduced to the usual ones:

$$v = \frac{v_1 + v_2}{2}, \quad v'_1 = \frac{v_1 - v_2}{2}. \quad (52)$$

- c) As we know, in the reference frame  $\mathcal{R}_2$  the positions and the time are given by

$$\begin{aligned}x' &= \gamma(x - vt) \\y' &= y\end{aligned}\tag{53}$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)\tag{54}$$

Therefore we have

$$u'_x \equiv \frac{dx'}{dt'} = \frac{u_x - v}{1 - \frac{vu_x}{c^2}},\tag{55}$$

and

$$u'_y \equiv \frac{dy'}{dt'} = \frac{dy}{\gamma\left(dt - \frac{vdx}{c^2}\right)} = \frac{u_y}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}.\tag{56}$$

Therefore, the angles  $\theta$  and  $\theta'$  are related in the following way:

$$\tan \theta' \equiv \frac{u'_y}{u'_x} = \frac{u_y}{\gamma(u_x - v)} = \frac{u \sin \theta}{\gamma(u \cos \theta - v)}.\tag{57}$$

- d) Suppose that  $\theta < \pi$  and choose a boost of speed  $v_0$  along a direction which lies in the plane of the two trajectories and makes an angle  $\theta/2$  with each one of them. We denote by  $x$  this direction and by  $y$  the orthogonal one in the plane, and we use the result of the previous exercises to write the condition  $\mathbf{v}'_1 + \mathbf{v}'_2 = 0$  in components:

$$\frac{\beta_x - \beta_0}{1 - \beta_x \beta_0} + \frac{\beta_x - \beta_0}{1 - \beta_x \beta_0} = 0\tag{58}$$

$$\frac{\beta_y}{\gamma(1 - \beta_x \beta_0)} - \frac{\beta_y}{\gamma(1 - \beta_x \beta_0)} = 0.\tag{59}$$

The second equation is automatically satisfied, as a consequence of our choice for the direction of the boost. The first equation is satisfied by the choice

$$v_0 = v_x = v \cos \frac{\theta}{2}.\tag{60}$$