

Classical Electrodynamics

Week 11

1. Lorentz transformation of electromagnetic fields

Show that a Lorentz boost leads to the following transformation:

$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel} \quad (1)$$

$$\mathbf{E}'_{\perp} = \gamma (\mathbf{E}_{\perp} + c\boldsymbol{\beta} \times \mathbf{B}) \quad (2)$$

$$\mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel} \quad (3)$$

$$\mathbf{B}'_{\perp} = \gamma \left(\mathbf{B}_{\perp} - \frac{1}{c}\boldsymbol{\beta} \times \mathbf{E} \right) \quad (4)$$

Consider now a constant electromagnetic field $\{\mathbf{E}, \mathbf{B}\}$ in the reference frame \mathcal{R} .

- Find a reference frame \mathcal{R}' where the electromagnetic fields $\{\mathbf{E}', \mathbf{B}'\}$ are parallel to each other: $\mathbf{E}' \parallel \mathbf{B}'$.
- Is this always possible? Is the solution unique?
- Compute the magnitudes of \mathbf{E}' and \mathbf{B}' in the reference frame \mathcal{R}' .

Solution

Let us consider for example a boost along the direction x . The fields transform as follows:

$$\begin{aligned} E'_{\parallel} &= E'_x = F'^{01} \\ &= \Lambda^0_{\alpha} \Lambda^1_{\beta} F^{\alpha\beta} \\ &= \Lambda^0_0 \Lambda^1_1 F^{01} + \Lambda^0_1 \Lambda^1_0 F^{10} \\ &= \gamma^2 (1 - \beta^2) F^{01} \\ &= F^{01} \\ &= E_x = E_{\parallel} \end{aligned} \quad (5)$$

For the orthogonal component ($j \neq 1$):

$$\begin{aligned} (\mathbf{E}'_{\perp})_j &= E'_j = F'^{0j} \\ &= \Lambda^0_{\alpha} \Lambda^j_{\beta} F^{\alpha\beta} \\ &= \Lambda^0_0 F^{0j} + \Lambda^0_i F^{ij} \\ &= \gamma (\mathbf{E}_{\perp})_j - c\beta\gamma \varepsilon^{1jk} B_k \\ &= \gamma (\mathbf{E}_{\perp})_j - c\gamma \varepsilon^{ijk} B_k \beta_i \\ &= \gamma (\mathbf{E}_{\perp} + c\boldsymbol{\beta} \wedge \mathbf{B})_j \end{aligned} \quad (6)$$

Where we used that $\Lambda^j_{\beta} = \delta^j_{\beta}$, and that $\Lambda^0_i = -\delta^1_i \beta \gamma$, $\boldsymbol{\beta} = (\vec{\beta})_x$.

An analogous computation gives the result for the magnetic field \mathbf{B} :

$$\begin{aligned}
cB'_{\parallel} &= -cB_x = F'^{23} \\
&= \Lambda_{\alpha}^2 \Lambda_{\beta}^3 F^{\alpha\beta} \\
&= F'^{23} \\
&= cB_x = cB_{\parallel}
\end{aligned} \tag{7}$$

$$\begin{aligned}
2c(\mathbf{B}'_{\perp})_j &= 2cB'_j = \varepsilon_{jlm} F'^{lm} \\
&= 2\varepsilon_{j1m} F'^{1m} \\
&= 2\varepsilon_{j1m} \Lambda_{\alpha}^1 \Lambda_{\beta}^m F^{\alpha\beta} \\
&= 2\varepsilon_{j1m} (\Lambda_1^1 F^{1m} + \Lambda_0^1 F^{0m}) \\
&= 2\varepsilon_{j1m} (-\gamma\beta E_m + \gamma c \varepsilon^{1ml} B_l) \\
&= -2\varepsilon_{j1m} \gamma\beta E_m + 2\gamma c B_j \\
&= 2c\gamma (\mathbf{B}'_{\perp} - \frac{1}{c} \boldsymbol{\beta} \wedge \mathbf{E})_j
\end{aligned}$$

Let us now consider a constant electromagnetic field $\{\mathbf{E}, \mathbf{B}\}$ in the reference frame \mathcal{R} .

- (a) When the two vectors are parallel to each other their vector product is equal to zero:

$$\mathbf{E}' \times \mathbf{B}' = 0 . \tag{8}$$

It is sufficient to consider a boost $\boldsymbol{\beta}$ perpendicular to \mathbf{E} and \mathbf{B} . Indeed, a boost in the plane of \mathbf{B} and \mathbf{E} can be written as a composition of a boost in the direction parallel to \mathbf{E} and a boost in the direction parallel to \mathbf{B} , and we can demonstrate that the angle θ' between \mathbf{B}' and \mathbf{E}' cannot be zero: we have

$$\cos \theta' = \frac{\mathbf{E}' \cdot \mathbf{B}'}{E'B'} = \frac{\mathbf{E} \cdot \mathbf{B}}{E'B'} = \frac{EB}{E'B'} \cos \theta . \tag{9}$$

Moreover, when $\mathbf{E}_{\perp} = 0$, we have

$$\frac{EB}{E'B'} = \frac{E_{\parallel} \sqrt{B_{\parallel}^2 + B_{\perp}^2}}{\sqrt{(E_{\parallel}^2 + \gamma^2 \beta^2 c^2 B_{\perp}^2) (B_{\parallel}^2 + \gamma^2 B_{\perp}^2)}} < 1 , \tag{10}$$

which implies

$$|\cos \theta'| < |\cos \theta| . \tag{11}$$

We will thus choose a boost such that $\mathbf{E}_{\parallel} = \mathbf{B}_{\parallel} = 0$. We can now develop the calculation of the cross product:

$$\begin{aligned}
\mathbf{E}' \times \mathbf{B}' &= \gamma (\mathbf{E}_{\perp} + c\boldsymbol{\beta} \times \mathbf{B}) \times \gamma \left(\mathbf{B}_{\perp} - \frac{1}{c} \boldsymbol{\beta} \times \mathbf{E} \right) = \\
&= \gamma^2 \left(EB \sin \theta - c\beta B^2 - \frac{1}{c} \beta E^2 + \beta^2 EB \sin \theta \right) \frac{\boldsymbol{\beta}}{\beta} .
\end{aligned} \tag{12}$$

The solutions of $\mathbf{E}' \times \mathbf{B}' = 0$ are

$$\begin{aligned}\beta_{\pm} &= \frac{1}{2\alpha \sin \theta} \left(1 + \alpha^2 \pm \sqrt{(1 - \alpha^2)^2 + 4\alpha^2 \cos^2 \theta} \right) \\ &= \frac{1}{2\alpha \sin \theta} \left(1 + \alpha^2 \pm \sqrt{(1 + \alpha^2)^2 - 4\alpha^2 \sin^2 \theta} \right)\end{aligned}\quad (13)$$

with $\alpha = \frac{cB}{E}$.

(b) We must check if $|\beta| \in [0, 1]$. This is true only for

$$\beta = \beta_- = \frac{1}{2\alpha \sin \theta} \left(1 + \alpha^2 - \sqrt{(1 + \alpha^2)^2 - 4\alpha^2 \sin^2 \theta} \right). \quad (14)$$

The solution we have found is degenerate because we can do an arbitrary boost in the direction of \mathbf{E}' and \mathbf{B}' and the new electric and magnetic field will still be parallel.

(c) We can use equation (9) with $\cos \theta' = 1$, which gives us B' as a function of E' , and another equation that has been proven previously, that is:

$$E'^2 - c^2 B'^2 = E^2 - c^2 B^2. \quad (15)$$

The solution for E' of the system of these two equations is

$$E'_{\pm} = \frac{1}{2} \left(E^2 - c^2 B^2 \pm \sqrt{(E^2 - c^2 B^2)^2 + 4(EcB \cos \theta)^2} \right), \quad (16)$$

and we have to keep only E'_+ , since the other solution is negative. The corresponding magnetic field is

$$B'^2 = \frac{1}{2c^2} \left(c^2 B^2 - E^2 + \sqrt{(E^2 - c^2 B^2)^2 + 4(EcB \cos \theta)^2} \right). \quad (17)$$

2. Synchrotron radiation

Consider a non-relativistic electron ($v \ll c$) in circular movement due to a magnetic field \mathbf{B} orthogonal to the plane of the movement.

a) Calculate the Poynting vector $\mathbf{S} = \epsilon_0 c^2 \mathbf{E}_e \times \mathbf{B}_e$, using the radiative part of the electromagnetic fields \mathbf{E}_e , \mathbf{B}_e produced by the accelerated electron. You can use the formulas of Liénard-Wiechert in the non-relativistic limit ($v \ll c$).

b) Calculate the time average of the Poynting vector

$$\langle \mathbf{S} \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt \mathbf{S}(t), \quad (18)$$

and determine the total radiated power.

c) Study the angular distribution of the radiation.

Solution

- a) Let us use cartesian coordinates where \mathbf{B} points in the z direction: $\mathbf{B} = B_0 \hat{z}$. The electron has a uniform circular trajectory of coordinates:

$$\mathbf{r}_0(t) = r_0 \begin{pmatrix} \cos(\omega_s t) \\ \sin(\omega_s t) \\ 0 \end{pmatrix}, \quad (19)$$

where ω_s is the Larmor frequency:

$$\omega_s = \frac{eB_0}{m_e}. \quad (20)$$

Remember the Liénard-Wiechert formula giving the electric and magnetic field produced by a moving point charge:

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0} \frac{1}{R^2} \frac{1}{(1 - \boldsymbol{\beta} \cdot \mathbf{n})^3} \left((\mathbf{n} - \boldsymbol{\beta})(1 - \beta^2) + \frac{R}{c} \mathbf{n} \times \left((\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right) \right)$$

where all the quantities are evaluated at the retarded time $t' = t - \frac{R}{c}$, R is the distance between the point of observation and the position of the charge $R = |\mathbf{r} - \mathbf{r}_0(t')|$ and \mathbf{n} is the unit vector defined as $\mathbf{n} = \frac{\mathbf{r} - \mathbf{r}_0(t')}{R}$. The magnetic field is given by:

$$\mathbf{B} = \frac{1}{c} \mathbf{n} \times \mathbf{E}. \quad (21)$$

Now we take the non-relativistic limit $\beta \ll 1$ of this formula so we can neglect $\boldsymbol{\beta}$ in front of \mathbf{n} and we consider the radiative regime $R \gg |\mathbf{r}_0|$ so the first term in the parenthesis (the Coulomb term) drops off. We are left with:

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0} \frac{1}{Rc} \mathbf{n} \times (\mathbf{n} \times \dot{\boldsymbol{\beta}})$$

Far from the point charge, we have $R = |\mathbf{r} - \mathbf{r}_0(t')| \approx |\mathbf{r}| = r$ and $\mathbf{n} = \frac{\mathbf{r} - \mathbf{r}_0(t')}{R} \approx \frac{\mathbf{r}}{R} \approx \hat{r}$. Doing the cross-products explicitly, we find the formulas:

$$\mathbf{B}_e = \frac{q}{4\pi\epsilon_0 c^3} \frac{\mathbf{a} \times \hat{r}}{r} \quad \mathbf{E}_e = \frac{q}{4\pi\epsilon_0 c^2} \frac{(\mathbf{a} \times \hat{r}) \times \hat{r}}{r}, \quad (22)$$

where \mathbf{a} is the acceleration of the electron, in this case of a circular motion it is simply $\mathbf{a}(t) = -\omega_s^2 \mathbf{r}_0(t)$. In principle, the above formula should be evaluated at the retarded time $t' = t - \frac{r}{c}$ but for fixed r this is just a shift in t so it will not change the following discussion.

The magnetic field is given by:

$$\begin{aligned} \mathbf{B}_e &= \frac{e\omega_s^2 r_0}{4\pi\epsilon_0 c^3 r} \begin{pmatrix} \cos(\omega_s t') \\ \sin(\omega_s t') \\ 0 \end{pmatrix} \times \begin{pmatrix} \sin \theta \cos \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{pmatrix} \\ &= \frac{e^3 B_0^2 r_0}{4\pi\epsilon_0 m_e^2 c^3 r} \begin{pmatrix} \sin(\omega_s t') \cos \theta \\ -\cos(\omega_s t') \cos \theta \\ -\sin(\omega_s t' - \varphi) \sin \theta \end{pmatrix} \end{aligned} \quad (23)$$

and the electric field is $\mathbf{E}_e = c\mathbf{B}_e \times \hat{r}$.

The Poynting vector is now:

$$\begin{aligned} \mathbf{S}(t) &= \epsilon_0 c^2 \mathbf{E}_e \times \mathbf{B}_e = \epsilon_0 c^3 (\mathbf{B}_e \times \hat{r}) \times \mathbf{B}_e = \epsilon_0 c^3 (|\mathbf{B}_e|^2 \hat{r} - (\mathbf{B}_e \cdot \hat{r}) \mathbf{B}_e) \\ &= \epsilon_0 c^3 |\mathbf{B}_e|^2 \hat{r} = \frac{e^6 B_0^4 r_0^2}{16\pi^2 \epsilon_0 m_e^4 c^3 r^2} (\cos^2 \theta + \sin^2(\omega_s t' - \varphi) \sin^2 \theta) \hat{r}. \end{aligned} \quad (24)$$

We can understand better the term in brackets by noticing that:

$$(\cos^2 \theta + \sin^2(\omega_s t' - \varphi) \sin^2 \theta) = |\hat{a}(t') \times \hat{r}|^2 = \sin^2 \alpha(t) \quad (25)$$

where $\alpha(t)$ is the angle between $\mathbf{a}(t')$ and \mathbf{r} .

b) The time average of the Poynting vector is:

$$\langle \mathbf{S} \rangle = \frac{e^6 B_0^4 r_0^2}{16\pi^2 \epsilon_0 m_e^4 c^3 r^2} \hat{r} \left(\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \sin^2 \alpha(t) dt \right), \quad (26)$$

and

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \sin^2 \alpha(t) dt &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (\cos^2 \theta + \sin^2(\omega_s t' - \varphi) \sin^2 \theta) dt \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \left[t \cos^2 \theta + \sin^2 \theta \frac{2(\omega_s t - \varphi) - \sin(2(\omega_s t - \varphi))}{4\omega_s} \right]_{-\frac{r}{c}}^{T - \frac{r}{c}} \\ &= \cos^2 \theta + \frac{1}{2} \sin^2 \theta \end{aligned}$$

so we get

$$\langle \mathbf{S} \rangle = \frac{e^6 B_0^4 r_0^2}{32\pi^2 \epsilon_0 m_e^4 c^3 r^2} (1 + \cos^2 \theta) \hat{r}. \quad (27)$$

Notice that the Poynting vector (24) depend on θ and φ whereas the time averaged Poynting vector (27) depend only on θ . This is because the radiated field depend on the precise position of the electron but once we average over the periodic trajectory we gain a cylindrical symmetry of axis z .

The Poynting vector is the flux density of radiated energy. The total power is given by the flux of the Poynting vector through a surface enclosing the system that we choose to be a sphere of radius R .

$$\begin{aligned} P(t) &= \int_S \mathbf{S}(t) \cdot d\boldsymbol{\sigma} \\ &= \frac{e^6 B_0^4 r_0^2}{16\pi^2 \epsilon_0 m_e^4 c^3 R^2} \int_{-1}^1 d\cos \theta \int_0^{2\pi} d\varphi R^2 (\cos^2 \theta + \sin^2(\omega_s t' - \varphi) \sin^2 \theta) \\ &= \frac{e^6 B_0^4 r_0^2}{16\pi^2 \epsilon_0 m_e^4 c^3} \int_{-1}^1 d\cos \theta 2\pi \cos^2 \theta + \sin^2 \theta \left[-\frac{2(\omega_s t' - \varphi) - \sin(2(\omega_s t' - \varphi))}{4} \right]_0^{2\pi} \\ &= \frac{e^6 B_0^4 r_0^2}{16\pi^2 \epsilon_0 m_e^4 c^3} \int_{-1}^1 d\cos \theta (2\pi \cos^2 \theta + \pi \sin^2 \theta) \\ &= \frac{e^6 B_0^4 r_0^2}{16\pi \epsilon_0 m_e^4 c^3} \int_{-1}^1 d\cos \theta (1 + \cos^2 \theta) \\ &= \frac{e^6 B_0^4 r_0^2}{6\pi \epsilon_0 m_e^4 c^3} = \frac{e^2 \omega_s^4 r_0^2}{6\pi \epsilon_0 c^3}. \end{aligned} \quad (28)$$

We see that the total radiated power does not depend on time: this is because we can see the integral over the solid angle as an average over all directions and in this circular uniform movement, averaging over φ is equivalent to time-averaging.

- c) The angular distribution of the radiation is proportional to $\sin^2 \alpha(t) = (\cos^2 \theta + \sin^2(\omega_s t' - \varphi) \sin^2 \theta)$ where $\alpha(t)$ is the angle between $\mathbf{a}(t')$ and \mathbf{r} . There is no radiation when \mathbf{r} is parallel to $\mathbf{a}(t)$ and the radiation is maximum for \mathbf{r} perpendicular to $\mathbf{a}(t)$.

The time average of the radiation is maximum in a direction always perpendicular to $\mathbf{a}(t)$, this is the case in the directions $\theta = 0$ and $\theta = \pi$, i.e. in the direction z of the external magnetic field. More generally, we can see the angular dependence $1 + \cos^2 \theta$ of the emitted power in equation (27).

Note on coordinate systems In principle the problems can be solved in any system of coordinates. However, in radiation problems, spherical coordinates (r, θ, φ) are usually more adapted because we take the large r limit and we often calculate the flux of the Poynting vector through a sphere.

However in order to take the scalar product and cross product of vectors, a fixed orthonormal triplet $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ is more convenient than the triplet $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi)$ which depend on θ and φ .

This is why in this exercise we use the parametrization $\mathbf{r} = r \sin \theta \cos \varphi \mathbf{e}_x + r \sin \theta \sin \varphi \mathbf{e}_y + r \cos \theta \mathbf{e}_z$ in the calculation (23) before switching back to $\mathbf{r} = r \mathbf{e}_r$.

3. Classical atom

Consider the classical model of the hydrogen atom:

- The proton, of charge $e = 1.60 \times 10^{-19}$ C and mass $m_p = 1.67 \times 10^{-27}$ kg, is at rest at the center of the atom.
- The electron, of charge $-e$ and mass $m_e = 9.11 \times 10^{-31}$ kg $\ll m_p$, moves around the proton in a circular orbit of radius $r_0 = 5.29 \times 10^{-11}$ m.

- a) Calculate the frequency ν of this rotation.
- b) Calculate the total power radiated by the system. Recall the formula

$$\mathbf{S}(t) = \frac{e^2}{16\pi^2 \varepsilon_0 c^3} \frac{|\mathbf{a}|^2 \sin^2 \alpha(t)}{r^2} \mathbf{e}_r \quad (29)$$

for the Poynting vector of a non-relativistic electron. Here, $\alpha(t)$ is the angle between the acceleration vector \mathbf{a} of the electron and the observation direction \mathbf{e}_r .

- c) Estimate the life time of the classical atom. Why are you still alive?

Solution

- a) We are considering the electron as moving in a circular orbit around a fixed proton. Newton's law gives us immediately:

$$m_e \omega^2 r_0 = \frac{e^2}{4\pi\epsilon_0 r_0^2}, \quad (30)$$

from which we can deduce the frequency:

$$\nu = \frac{\omega}{2\pi} = \sqrt{\frac{e^2}{16\pi^3\epsilon_0 m_e r_0^3}} = 6.57 \times 10^{15} \text{ Hz}. \quad (31)$$

- b) We have already computed the Poynting vector for an electron in a circular motion. It is:

$$\mathbf{S}(t) = \frac{e^2}{16\pi^2\epsilon_0 c^3} \frac{|\mathbf{a}|^2 \sin^2 \alpha(t)}{r^2} \mathbf{e}_r. \quad (32)$$

Notice that the velocity of the electron is $v = \omega r_0 = 2.2 \times 10^6 \text{ m} \cdot \text{s}^{-1} \ll c$ so our non-relativistic approximation is valid.

We can now compute the radiated power by integrating the flux of the Poynting vector over a sphere of radius R ($R \gg r_0$):

$$\begin{aligned} P &= \int_{S_R} R^2 \mathbf{S}(t) \cdot \mathbf{e}_r d\Omega = \frac{e^2}{16\pi^2\epsilon_0 c^3} \frac{r_0^2 \omega^4}{R^2} \int_{S_R} R^2 \sin^2 \alpha(t) d\Omega \\ &= \frac{e^2 r_0^2 \omega^4}{16\pi^2\epsilon_0 c^3} \int_{-1}^1 d\cos\theta \int_{-\pi}^{\pi} d\varphi (\cos^2\theta + \sin^2\theta \sin^2(\omega t' - \varphi)) \\ &= \frac{e^2 r_0^2 \omega^4}{16\pi^2\epsilon_0 c^3} \int_{-1}^1 d\cos\theta [2\pi \cos^2\theta + \pi(1 - \cos^2\theta)] \\ &= \frac{e^2 r_0^2 \omega^4}{6\pi\epsilon_0 c^3} = \frac{e^6}{96\pi^3\epsilon_0^3 c^3 m_e^2 r_0^4} \end{aligned} \quad (33)$$

We find the same result as in exercise 1 equation (28): the power radiated in a certain direction depends on time but the total energy does not. This is due to the circular uniform movement of the electron.

- c) An electron in a circular motion radiates energy so we expect the electron to spiral down towards the proton to compensate for the energy lost in radiation. While the problem is very complicated, we can treat it with the following approximations:

- At each time t we can consider the electron to be in a circular motion around the proton of radius $r(t)$, i.e. the electron falls slowly towards the proton.
- At each time t we can approximate the radiated power to be given by the result of question **b**).

At each time t the energy of the system is:

$$E(t) = \frac{m_e}{2} r(t)^2 \omega(t)^2 - \frac{e^2}{4\pi\epsilon_0 r(t)} = -\frac{e^2}{8\pi\epsilon_0 r(t)}. \quad (34)$$

Then we can write:

$$\begin{aligned}\frac{dr}{dt} &= \frac{dr}{dE} \frac{dE}{dt} = \left(\frac{dE}{dr} \right)^{-1} (-P) = -\frac{8\pi\epsilon_0 r^2}{e^2} \frac{e^6}{96\pi^3 \epsilon_0^3 c^3 m_e^2 r^4} \\ &= -\frac{e^4}{12\pi^2 \epsilon_0^2 c^3 m_e^2 r^2} \equiv -\frac{\alpha}{r^2}.\end{aligned}\quad (35)$$

This is a separable differential equation.

$$\begin{aligned}-r^2 dr &= \alpha dt \\ \left[-\frac{r^3}{3} \right]_{r_0}^0 &= \alpha \tau \\ \tau &= \frac{r_0^3}{3\alpha} = \frac{4\pi^2 \epsilon_0^2 c^3 m_e^2 r_0^3}{e^4}.\end{aligned}\quad (36)$$

Numerically, we have $\tau = 1.6 \times 10^{-11}$ s so it seems that the hydrogen atom is highly unstable.

The stability of atoms and thus of the world we know cannot be explained by classical electrodynamics. One needs to treat the problem in quantum mechanics. In this framework, one can show that the hydrogen atom has discrete energy levels with a lowest energy level at $E = -13.6$ eV which explains why the atom is stable.

Physically, our result (36) does not make sense and should only be taken as an order of magnitude because our approximations break down well before the radius of the trajectory goes to zero.

- First the non-relativistic assumption becomes wrong as $v \approx c$. In our circular trajectory approximation, this is when

$$v = \omega r = \sqrt{\frac{e^2}{4\pi\epsilon_0 m_e r}} \approx c \implies r \approx \frac{e^2}{4\pi\epsilon_0 m_e c^2} \approx 3 \times 10^{-15} \text{ m}.\quad (37)$$

- Second, our assumption that at any moment the trajectory can be considered circular. This is true if the relative rate of change of the radius $\frac{1}{r} \frac{dr}{dt}$ is small compared to the frequency ν (or ω). This breaks down when

$$\begin{aligned}\left| \frac{1}{r} \frac{dr}{dt} \right| &= \frac{e^4}{12\pi^2 \epsilon_0^2 c^3 m_e^2 r^3} \approx \nu = \sqrt{\frac{e^2}{16\pi^3 \epsilon_0 m_e r_0^3}} \\ &\implies r \approx \frac{e^2}{\sqrt[3]{9\pi\epsilon_0 m_e c^2}} \approx 1 \times 10^{-14} \text{ m}.\end{aligned}\quad (38)$$

In any case there is still a large range of values $r = 10^{-11} - 10^{-14}$ m where our approximations are valid so it does not change our conclusion: the hydrogen atom seen classically is not stable because of energy loss by radiation.