Classical Electrodynamics

Week 10

0. *Lévi-Civitá symbol*
The Lévi-Civitá symbol is defined by

$$\varepsilon^{\mu\nu\rho\sigma} = \begin{cases} 1 & \text{if } (\mu\nu\rho\sigma) \text{ is an even permutation of } (0123) \\ -1 & \text{if } (\mu\nu\rho\sigma) \text{ is an odd permutation of } (0123) \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

a) Defining the Lorentz transformation of $\varepsilon^{\mu\nu\rho\sigma}$ by

$$\varepsilon'^{\mu\nu\rho\sigma} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\Lambda^{\rho}{}_{\gamma}\Lambda^{\sigma}{}_{\delta}\varepsilon^{\alpha\beta\gamma\delta}\,, \tag{2}$$

show that $\varepsilon'^{\mu\nu\rho\sigma} = \det(\Lambda)\varepsilon^{\mu\nu\rho\sigma}$ and that $\det(\Lambda) = \pm 1$.

b) Verify the following identities and find the value of the constants N_1, N_2 and N_3:

$$\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\mu\nu\rho\sigma} = N_1 \tag{3}$$

$$\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\alpha\nu\rho\sigma} = N_2\,\delta^{\mu}_{\alpha} \tag{4}$$

$$\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\alpha\beta\rho\sigma} = N_3\left(\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} - \delta^{\mu}_{\beta}\delta^{\nu}_{\alpha}\right) \tag{5}$$

Solution

a) The Lorentz transformation of $\varepsilon^{\mu\nu\rho\sigma}$ is a completely antisymmetric object:

$$\begin{aligned} \varepsilon'^{\mu\nu\rho\sigma} &= \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\Lambda^{\rho}{}_{\gamma}\Lambda^{\sigma}{}_{\delta}\varepsilon^{\alpha\beta\gamma\delta} \\ &= \Lambda^{\nu}{}_{\beta}\Lambda^{\mu}{}_{\alpha}\Lambda^{\rho}{}_{\gamma}\Lambda^{\sigma}{}_{\delta}(-\varepsilon^{\beta\alpha\gamma\delta}) \\ &= -\varepsilon'^{\nu\mu\rho\sigma} \end{aligned} \tag{6}$$

and the same can be verified for every pair of indices. This implies that $\varepsilon'^{\mu\nu\rho\sigma} = C\varepsilon^{\mu\nu\rho\sigma}$, with C some constant. In order to determine the value of this constant, we can consider the entry $\{0123\}$:

$$\varepsilon'^{0123} = \Lambda^{0}{}_{\alpha}\Lambda^{1}{}_{\beta}\Lambda^{2}{}_{\gamma}\Lambda^{3}{}_{\delta}\varepsilon^{\alpha\beta\gamma\delta} = \det(\Lambda) = C \tag{7}$$

by definition of the determinant of a matrix. Taking the determinant of

$$\Lambda\eta\Lambda^T = \eta$$

we see that $(\det(\Lambda))^2 = 1$, $\det(\Lambda) = \pm 1$.

b)

$$\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\mu\nu\rho\sigma} = \eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\rho\gamma}\eta_{\sigma\delta}\varepsilon^{\alpha\beta\gamma\delta}\varepsilon^{\mu\nu\rho\sigma}$$

The only non vanishing terms are the ones for which $\mu = \alpha$, $\nu = \beta$, $\rho = \gamma$, $\sigma = \delta$, with $\alpha \neq \beta \neq \gamma \neq \delta$: this corresponds to summing over all permutations of $\{0123\}$, which are in the number of 4!. For each of these terms $\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\rho\gamma}\eta_{\sigma\delta} = \eta_{00}\eta_{11}\eta_{22}\eta_{33} = -1$. One obtains:

$$\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\mu\nu\rho\sigma} = -\sum_{\pi}(\varepsilon^{\pi(0)\pi(1)\pi(2)\pi(3)})^2 = -4! = N_1 \tag{8}$$

For the second identity we have:

$$\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\alpha\nu\rho\sigma} = \eta_{\alpha\zeta}\eta_{\nu\beta}\eta_{\rho\gamma}\eta_{\sigma\delta}\varepsilon^{\zeta\beta\gamma\delta}\varepsilon^{\mu\nu\rho\sigma}$$

The only non vanishing terms are the ones for which $\nu = \beta$, $\rho = \gamma$, $\sigma = \delta$, with $\alpha \neq \beta \neq \gamma \neq \delta$ and $\alpha = \mu$. Given a fixed μ this corresponds to summing over all permutations of the three remaining indices, there are 3! such permutations:

$$\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\alpha\nu\rho\sigma} = -\sum_{\nu,\rho,\sigma}\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\nu\rho\sigma} = -3!\delta^{\mu}_{\alpha} \tag{9}$$

$N_2 = -6$.

Finally when two indices are left uncontracted,

$$\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\alpha\beta\rho\sigma} = -\sum_{\rho,\sigma}\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\rho\sigma}$$

The only non vanishing terms are the ones with $\mu \neq \nu$ and the other two indices are either $\{\mu = \alpha, \nu = \beta\}$ or $\{\mu = \beta, \nu = \alpha\}$. There are two possible permutations for two fixed indices, we then obtain:

$$\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\alpha\beta\rho\sigma} = -2(\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} - \delta^{\mu}_{\beta}\delta^{\nu}_{\alpha}) \tag{10}$$

$N_3 = -2$.

1. *Maxwell equations*

Using the definition of the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, where $A^\mu = (\Phi, c\mathbf{A})^T$, verify that the equations

$$\partial_\mu F^{\mu\nu} = -\frac{1}{c\varepsilon_0}j^\nu \tag{11}$$

$$\partial_\mu \varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} = 0 \tag{12}$$

describe Maxwell equations. Show that equations (11) and (12) are Lorentz invariant.

Solution

We start with the first equation. It may be a good idea to consider separately the spatial and the temporal components. For the temporal component, namely $\nu = 0$, the left hand side can be written in this way:

$$\partial_0 F^{00} + \partial_i F^{i0} = \partial_i(-E^i) = -\nabla \cdot \vec{E} \tag{13}$$

The right hand side is

$$-\frac{1}{c\epsilon_0} j^0 \equiv -\frac{\rho}{\epsilon_0} \,. \tag{14}$$

As a consequence, if we use (13) and (14), we obtain

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \,. \tag{15}$$

If instead, for example, $\nu = 1$, we obtain a component of another equation:

$$\partial_0 F^{01} + \partial_1 F^{11} + \partial_2 F^{21} + \partial_3 F^{31} = \frac{1}{c}\partial_t E_x - c\partial_y B_z + c\partial_z B_y \,, \tag{16}$$

and this is the x component of the term

$$\frac{1}{c}\partial_t \vec{E} - c\nabla \times \vec{B} \,. \tag{17}$$

On the right hand side we simply have

$$-\frac{1}{c\epsilon_0} j^1 \equiv -\frac{1}{c\epsilon_0} j_x = -c\mu_0 j_x \,. \tag{18}$$

A similar derivation can be done for the other components, so that we end up with

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2}\partial_t \vec{E} \,. \tag{19}$$

It is now time to derive the other two. When $\nu = 0$, the second equation becomes

$$\begin{aligned}
\partial_\mu \varepsilon^{\mu 0 \rho\sigma} F_{\rho\sigma} &= \partial_i \varepsilon^{i0jk} F_{jk} \\
&= -\partial_i \varepsilon^{0ijk} F_{jk} \\
&= -\partial_i \varepsilon^{ijk} F_{jk} = \\
&= -2\partial_i \varepsilon^{ijk} \partial_j A_k \\
&= -2c\partial_i \varepsilon^{ijk} \partial_j (\vec{A})_k \\
&= -2c\partial_i \left(\nabla \times \vec{A}\right)^i \\
&= -2c\nabla \cdot \left(\nabla \times \vec{A}\right) \\
&= -2c\nabla \cdot \vec{B} \,.
\end{aligned} \tag{20}$$

Therefore, we have derived the relation

$$\nabla \cdot \vec{B} = 0 \,. \tag{21}$$

If $\nu = 1$, we have

$$\begin{aligned}
\partial_\mu \varepsilon^{\mu 1 \rho\sigma} F_{\rho\sigma} &= \partial_0 \varepsilon^{01ij} F_{ij} + \partial_i \varepsilon^{i10j} F_{0j} + \partial_i \varepsilon^{i1j0} F_{j0} = \\
&= \partial_0 \varepsilon^{01ij} F_{ij} + 2\partial_i \varepsilon^{i10j} F_{0j} = \\
&= \partial_0 \varepsilon^{0123} F_{23} + \partial_0 \varepsilon^{0132} F_{32} + 2\partial_2 \varepsilon^{2103} F_{03} + 2\partial_3 \varepsilon^{3102} F_{02} = \\
&= 2\partial_0 \varepsilon^{0123} F_{23} + 2\partial_2 \varepsilon^{2103} F_{03} + 2\partial_3 \varepsilon^{3102} F_{02} = \\
&= 2\left(\partial_0 F_{23} - \partial_2 F_{03} + \partial_3 F_{02}\right) = 2\left(\partial_t \vec{B}_1 + \left[\nabla \times \vec{E}\right]_1\right) . \qquad (22)
\end{aligned}$$

If we also do an analogous calculation for the other components, we obtain

$$\nabla \times \vec{E} = -\partial_t \vec{B} \, . \qquad (23)$$

We must show that the equations are Lorentz invariant, namely that after a Lorentz transformation the equation does not change its structure. We can write

$$\begin{aligned}
\partial'_\mu F'^{\mu\nu} = \eta_{\mu\rho} \partial'^\rho F'^{\mu\nu} &= \eta_{\mu\rho} \Lambda^\rho{}_\alpha \Lambda^\mu{}_\beta \Lambda^\nu{}_\gamma \partial^\alpha F^{\beta\gamma} && (24) \\
&= \eta_{\alpha\beta} \Lambda^\nu{}_\gamma \partial^\alpha F^{\beta\gamma} && (25) \\
&= \Lambda^\nu{}_\gamma \partial_\beta F^{\beta\gamma} && (26) \\
&= -\Lambda^\nu{}_\gamma \frac{1}{c\varepsilon_0} j^\gamma && (27) \\
&= -\frac{1}{c\varepsilon_0} j'^\nu \, . && (28)
\end{aligned}$$

This shows the first equation is Lorentz invariant. The second tells us

$$0 = \partial_\mu \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} = \Lambda^\gamma{}_\mu \partial'_\gamma \varepsilon^{\mu\nu\rho\sigma} \Lambda^\alpha{}_\rho \Lambda^\beta{}_\sigma F'_{\alpha\beta} \, . \qquad (29)$$

We can contract this with $\Lambda^\delta{}_\nu$ and use what we proved in exercise **2.a)**:

$$\begin{aligned}
0 &= \varepsilon^{\mu\nu\rho\sigma} \Lambda^\delta{}_\nu \Lambda^\gamma{}_\mu \Lambda^\alpha{}_\rho \Lambda^\beta{}_\sigma \partial'_\gamma F'_{\alpha\beta} && (30) \\
&= \det(\Lambda) \varepsilon^{\gamma\delta\alpha\beta} \partial'_\gamma F'_{\alpha\beta} \, . && (31)
\end{aligned}$$

Since $\det(\Lambda) = \pm 1$, we get

$$\partial'_\gamma \varepsilon^{\gamma\delta\alpha\beta} F'_{\alpha\beta} = 0 \, , \qquad (32)$$

which shows the second equation is Lorentz invariant.

Note: A previous version of this solution used the notation of $(\Lambda^{-T})_\mu{}^\nu$ as the matrix transforming covariant tensors. This year we avoided introducing it, but let us explain it here in case you encounter this notation in other references.

One can define $(\Lambda^{-T})_\mu{}^\nu$ by lowering and raising indices of our usual $\Lambda^\mu{}_\nu$:

$$(\Lambda^{-T})_\mu{}^\nu := \eta_{\mu\rho} \eta^{\nu\sigma} \Lambda^\rho{}_\sigma \, . \qquad (33)$$

Then this object is what transforms covariant vectors. Indeed, for V^μ a contravariant vector, the covariant V_μ defined by lowering index transforms as:

$$V_\mu = \eta_{\mu\nu} V^\nu \longrightarrow V'_\mu = \eta_{\mu\nu} V'^\nu = \eta_{\mu\nu} \Lambda^\nu{}_\rho V^\rho = \eta_{\mu\nu} \Lambda^\nu{}_\rho \eta^{\rho\sigma} V_\sigma = (\Lambda^{-T})_\mu{}^\sigma V_\sigma \, . \qquad (34)$$

Here is the proof that $(\Lambda^{-T})_\mu{}^\nu$ is the inverse transpose of $\Lambda^\mu{}_\nu$:

$$(\Lambda^{-T})_\mu{}^\nu \Lambda^\mu{}_\rho = \eta_{\mu\alpha}\eta^{\nu\beta}\Lambda^\alpha{}_\beta \Lambda^\mu{}_\rho = \eta^{\nu\beta}\eta_{\beta\rho} = \delta^\nu_\rho\,, \tag{35}$$

where we have used the usual property of Λ matrices in the second-to-last equality. As a consequence, the four following equations can be used, but in this course we use only the first line:

$$V'^\mu = \Lambda^\mu{}_\nu V^\nu \qquad \Lambda^\nu{}_\mu V'_\nu = V_\mu \tag{36}$$

$$V'_\mu = (\Lambda^{-T})_\mu{}^\nu V_\nu \qquad (\Lambda^{-T})_\nu{}^\mu V'^\nu = V^\mu \tag{37}$$

Using the inverse transformation, we can write

$$\eta_{\mu\alpha}\partial^\alpha F^{\mu\nu} = \eta_{\mu\alpha}(\Lambda^{-T})^\alpha_\gamma \partial'^\gamma (\Lambda^{-T})^\mu_\delta (\Lambda^{-T})^\nu_\sigma F'^{\delta\sigma}\,. \tag{38}$$

Now we can use

$$\eta_{\mu\alpha}(\Lambda^{-T})^\alpha_\gamma = \eta_{\gamma\alpha}\Lambda^\alpha_\mu \tag{39}$$

and write

$$\begin{aligned}\eta_{\mu\alpha}(\Lambda^{-T})^\alpha_\gamma \partial'^\gamma (\Lambda^{-T})^\mu_\delta (\Lambda^{-T})^\nu_\sigma F'^{\delta\sigma} &= \eta_{\gamma\alpha}\Lambda^\alpha_\mu \partial'^\gamma (\Lambda^{-T})^\mu_\delta (\Lambda^{-T})^\nu_\sigma F'^{\delta\sigma} = \\ &= (\Lambda^{-T})^\nu_\sigma \partial'_\alpha F'^{\alpha\sigma}\,.\end{aligned} \tag{40}$$

As we know, the four-current transform as any other four vector:

$$(\Lambda^{-T})^\nu_\sigma \partial'_\alpha F'^{\alpha\sigma} = -\frac{1}{c\epsilon_0}(\Lambda^{-T})^\nu_\sigma j'^\sigma\,. \tag{41}$$

If we now multiply on both sides by $\Lambda^\tau{}_\nu$ we conclude our demonstration:

$$\partial'_\alpha F'^{\alpha\tau} = -\frac{1}{c\epsilon_0} j'^\tau\,. \tag{42}$$

For the second equation we will use

$$\varepsilon^{\mu\nu\rho\sigma} = \frac{1}{\det(\Lambda)}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \Lambda^\rho{}_\gamma \Lambda^\sigma{}_\delta \varepsilon^{\alpha\beta\gamma\delta}\,, \tag{43}$$

and

$$\Lambda^\theta{}_\zeta (\Lambda^{-T})^\zeta_\pi = \delta^\theta_\pi\,. \tag{44}$$

We can proceed with the derivation:

$$\begin{aligned}&\varepsilon^{\mu\nu\rho\sigma}\partial'_\nu F'_{\rho\sigma} = \\ =&\frac{1}{\det(\Lambda)}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \Lambda^\rho{}_\gamma \Lambda^\sigma{}_\delta \varepsilon^{\alpha\beta\gamma\delta}(\Lambda^{-T})_\nu^{\ \tau}\partial_\tau (\Lambda^{-T})_\rho^{\ \phi}(\Lambda^{-T})_\sigma^{\ \eta} F_{\phi\eta} = \\ =&\frac{1}{\det(\Lambda)}\Lambda^\mu{}_\alpha \varepsilon^{\alpha\beta\gamma\delta}\delta^\tau_\beta\,\delta^\phi_\gamma\,\delta^\eta_\delta\,\partial_\tau F_{\phi\eta} = \\ =&\frac{1}{\det(\Lambda)}\Lambda^\mu{}_\alpha \varepsilon^{\alpha\tau\phi\eta}\partial_\tau F_{\phi\eta} = 0\,.\end{aligned} \tag{45}$$

2. *Lorentz invariants*

a) Using the field strength $F_{\mu\nu}$ build two Lorentz invariants quadratic in the fields $\mathbf{E}$ and $\mathbf{B}$.

b) Given a tensor $T^{\mu\nu}$ build a Lorentz invariant linear in this tensor.

Solution

a) The only two Lorentz scalars quadratic in $\mathbf{E}$ and $\mathbf{B}$ that we can build from the field-strength tensor $F_{\mu\nu}$, the Levi-Civita symbol $\varepsilon^{\mu\nu\rho\sigma}$ and the metric $\eta_{\mu\nu}$ are:

i.
$$F_{\mu\nu}F^{\mu\nu} = 2F_{0i}F^{0i} + F_{ij}F^{ij} = 2(c^2B^2 - E^2)$$

This quantity is Lorentz invariant since:

$$F'_{\mu\nu}F'^{\mu\nu} = F'_{\mu\nu}\Lambda^{\mu}_{\alpha}\Lambda^{\nu}_{\beta}F^{\alpha\beta} = F_{\alpha\beta}F^{\alpha\beta}$$

ii.
$$F_{\mu\nu}\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = 2\varepsilon^{ijk}F_{0i}F_{jk} = -4c\mathbf{E}\cdot\mathbf{B}$$

The Lorentz transform of this quantity is:

$$\begin{aligned}
2F'_{\mu\nu}\tilde{F}'^{\mu\nu} &= \varepsilon^{\mu\nu\rho\sigma}F'_{\mu\nu}F'_{\rho\sigma} \\
&= \frac{1}{\det(\Lambda)}\varepsilon'^{\mu\nu\rho\sigma}F'_{\mu\nu}F'_{\rho\sigma} \\
&= \frac{1}{\det(\Lambda)}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\Lambda^{\rho}{}_{\gamma}\Lambda^{\sigma}{}_{\delta}\varepsilon^{\alpha\beta\gamma\delta}F'_{\mu\nu}F'_{\rho\sigma} \\
&= \frac{1}{\det(\Lambda)}\varepsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta} \\
&= \frac{1}{\det(\Lambda)}2F_{\alpha\beta}\tilde{F}^{\gamma\delta}.
\end{aligned} \tag{46}$$

Then the quantity $F_{\mu\nu}\tilde{F}^{\mu\nu}$ is invariant under Lorentz transformations such that $\det(\Lambda) = +1$.

b) The only Lorentz invariant linear in $T^{\mu\nu}$ is the trace $\eta_{\mu\nu}T^{\mu\nu}$:

$$\eta'_{\mu\nu}T'^{\mu\nu} = \eta_{\mu\nu}T'^{\mu\nu} = \eta_{\mu\nu}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}T^{\alpha\beta} = \eta_{\alpha\beta}T^{\alpha\beta}. \tag{47}$$

3. *Stress-energy tensor*
The electromagnetic stress-energy tensor is given by

$$T^{\mu\nu} = \varepsilon_0\left(F^{\mu}_{\ \alpha}F^{\nu\alpha} - \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right), \tag{48}$$

where $F_{\mu\nu}$ is the field-strength tensor and $\eta_{\mu\nu}$ is the Minkowski metric.

a) Compute the trace $\eta_{\mu\nu}T^{\mu\nu}$.

b) Verify that

$$T^{00} = u = \frac{\varepsilon_0}{2}\mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2 \tag{49}$$

is the electromagnetic energy density.

c) Verify that

$$T^{0i} = \frac{1}{c}(\mathbf{S})_i\,, \tag{50}$$

where $\mathbf{S}$ is the Poynting vector. Notice that $\frac{1}{c}T^{0i}$ is also the momentum density (component i) in the electromagnetic field.

d) Using Maxwell equations (11) and (12) show that

$$\partial_\mu T^{\mu\nu} = \frac{1}{c} j_\alpha F^{\alpha\nu}\,. \tag{51}$$

Rewrite the $\nu = 0$ component of this equation using the relations (49) and (50). What is the physical meaning of this equation? What about the spatial components $\nu = 1, 2, 3$?

Solution

a) The trace of the electromagnetic stress-energy tensor is:

$$\begin{aligned}\eta_{\mu\nu}T^{\mu\nu} &= \varepsilon_0\left(F^{\mu}{}_{\alpha}F^{\nu\alpha}\eta_{\mu\nu} - \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\eta_{\mu\nu}\right)\\ &= \varepsilon_0\left(F_{\nu\alpha}F^{\nu\alpha} - F_{\alpha\beta}F^{\alpha\beta}\right)\\ &= 0.\end{aligned} \tag{52}$$

b) Let us calculate :

$$T^{00} = \varepsilon_0\left(F^{0}{}_{\alpha}F^{0\alpha} - \frac{1}{4}\eta^{00}F_{\alpha\beta}F^{\alpha\beta}\right) \tag{53}$$

$$= \varepsilon_0(\vec{E}^2 + \frac{1}{4}2(c^2\vec{B}^2 - \vec{E}^2)) \tag{54}$$

$$= \frac{\varepsilon_0}{2}\vec{E}^2 + \frac{1}{2\mu_0}\vec{B}^2 = u\,, \tag{55}$$

where we used $F_{0i} = -E_i$ to obtain the first term and the result of ex.2 for the second. In the last line we used the relation $\varepsilon_0\mu_0c^2 = 1$ to obtain the usual energy.

c) Now :

$$\begin{aligned}T^{0i} &= \varepsilon_0\left(F^{0}{}_{\alpha}F^{i\alpha} - \frac{1}{4}\eta^{0i}F_{\alpha\beta}F^{\alpha\beta}\right) = \varepsilon_0 F^{0}{}_{\alpha}F^{i\alpha}\\ &= \varepsilon_0 F^{0}{}_{j}F^{ij} = \varepsilon_0 E_j c\epsilon^{ijk}B_k = \varepsilon_0 c(\vec{E}\times\vec{B})^i\\ &= \frac{1}{c}(\vec{S})^i\end{aligned} \tag{56}$$

where we have used that $F_{ij} = c\epsilon_{ijk}B^k$.

d) We have :

$$\begin{aligned}\partial_\mu T^{\mu\nu} &= \varepsilon_0 \left(\partial_\mu \left(F^\mu_{\ \alpha} F^{\nu\alpha} \right) - \frac{1}{4}\eta^{\mu\nu}\partial_\mu \left(F_{\alpha\beta}F^{\alpha\beta} \right) \right) \\ &= \varepsilon_0 \left((\partial_\mu F^\mu_{\ \alpha}) F^{\nu\alpha} + F^\mu_{\ \alpha}\partial_\mu F^{\nu\alpha} - \frac{1}{2}\partial^\nu F_{\alpha\beta}F^{\alpha\beta} \right) \\ &= \varepsilon_0 \left(-\frac{1}{c\varepsilon_0} j_\alpha F^{\nu\alpha} + F^\mu_{\ \alpha}\partial_\mu F^{\nu\alpha} - \frac{1}{2}\partial^\nu F_{\alpha\beta}F^{\alpha\beta} \right). \end{aligned} \tag{57}$$

In these manipulations, it is important to specify on which term the derivative acts. When we write $\partial_\mu F^\mu_{\ \alpha} F^{\alpha\nu}$, we mean $(\partial_\mu F^\mu_{\ \alpha}) F^{\alpha\nu}$ i.e. the derivative only acts on the first term and not on the second one.

We now want to show that the extra terms that we got are zero. To do so, we need to express terms of the form $\partial_\mu F^{\nu\alpha}$. Since we already used the first Maxwell equation, we should use the second equation $\partial_\mu \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} = 0$. We will show later that this equation implies the Bianchi identity :

$$\partial^\nu F^{\alpha\beta} + \partial^\alpha F^{\beta\nu} + \partial^\beta F^{\nu\alpha} = 0 \tag{58}$$

which allows us to express $\partial^\nu F^{\alpha\beta} = -\partial^\alpha F^{\beta\nu} - \partial^\beta F^{\nu\alpha}$. Using this:

$$\begin{aligned} F^\mu_{\ \alpha}\partial_\mu F^{\nu\alpha} - \frac{1}{2}\partial^\nu F_{\alpha\beta}F^{\alpha\beta} &= F^\mu_{\ \alpha}\partial_\mu F^{\nu\alpha} + \frac{1}{2}\partial^\alpha F^{\beta\nu} F_{\alpha\beta} + \frac{1}{2}\partial^\beta F^{\nu\alpha} F_{\alpha\beta} \\ &= (\partial^\mu F^{\nu\alpha}) F_{\mu\alpha} + \frac{1}{2}(\partial^\alpha F^{\beta\nu}) F_{\alpha\beta} + \frac{1}{2}(\partial^\beta F^{\alpha\nu}) F_{\beta\alpha} \\ &= 0. \end{aligned} \tag{59}$$

Remember that contracted indices are dummy variables : one can rename them without changing the value of the expression. You can see that the second line is the sum of three identical terms with different dummy indices. Now we can conclude :

$$\partial_\mu T^{\mu\nu} = -\frac{1}{c} j_\alpha F^{\nu\alpha} = \frac{1}{c} j_\alpha F^{\alpha\nu}. \tag{60}$$

Let us write the zeroth component of the equation above:

$$\begin{cases} \partial_\mu T^{\mu 0} = \partial_0 T^{00} + \partial_i T^{i0} = \frac{1}{c}\frac{\partial u}{\partial t} + \frac{1}{c}\nabla \cdot \mathbf{S} \\ \frac{1}{c} j_\alpha F^{\alpha 0} = \frac{1}{c} j_0 F^{00} + \frac{1}{c} j_i F^{i0} = -\frac{1}{c}\mathbf{j}\cdot\mathbf{E} \end{cases} \tag{61}$$

(You can check that $T^{\mu\nu}$ is symmetric, $j_i = (\vec{j})_i$ and $F^{i0} = -(\vec{E})_i$). So:

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{j}\cdot\mathbf{E} \tag{62}$$

expressing the fact that the variation of electromagnectic energy density (u) is due to the flux of energy density (Poynting vector) and the dissipation of energy via Joule effect (right-hand side).

The $\nu = 0$ component expresses the variation of the electromagnetic energy. The spatial components $\nu = 1, 2, 3$ will express the time variation of electromagnetic momentum density $\frac{1}{c}\frac{\partial \mathbf{S}}{\partial t}$ in terms of the flux of electromagnetic momentum (stress components T^{ij}) and an external force term (right-hand side).

Proof of the Bianchi identity Starting from the Maxwell equation (12), we prove the Bianchi identity (58). The LHS of Maxwell equation is a four-vector whereas the one of the Bianchi identity is a rank 3 tensor. In order to pass from one to the other we can do:

$$0 = \varepsilon_{\mu\alpha\beta\gamma}\varepsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} \tag{63}$$

We can use a method similar to exercise 2 to show that

$$\varepsilon_{\mu\alpha\beta\gamma}\varepsilon^{\mu\nu\rho\sigma} \propto \delta^\nu_\alpha\delta^\rho_\beta\delta^\sigma_\gamma - \delta^\nu_\beta\delta^\rho_\alpha\delta^\sigma_\gamma - \delta^\nu_\gamma\delta^\rho_\beta\delta^\sigma_\alpha - \delta^\nu_\alpha\delta^\rho_\gamma\delta^\sigma_\beta + \delta^\nu_\beta\delta^\rho_\gamma\delta^\sigma_\alpha + \delta^\nu_\gamma\delta^\rho_\alpha\delta^\sigma_\beta \tag{64}$$

When the above is contracted with $\partial_\mu F_{\rho\sigma}$, half of the terms simplify (permutations of ρ and σ just give a minus sign because of $F_{\rho\sigma}$). We get that:

$$\begin{aligned} &(\delta^\nu_\alpha\delta^\rho_\beta\delta^\sigma_\gamma - \delta^\nu_\beta\delta^\rho_\alpha\delta^\sigma_\gamma - \delta^\nu_\gamma\delta^\rho_\beta\delta^\sigma_\alpha - \delta^\nu_\alpha\delta^\rho_\gamma\delta^\sigma_\beta + \delta^\nu_\beta\delta^\rho_\gamma\delta^\sigma_\alpha + \delta^\nu_\gamma\delta^\rho_\alpha\delta^\sigma_\beta)\partial_\nu F_{\rho\sigma} = 0 \\ \Leftrightarrow\ &\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0. \end{aligned} \tag{65}$$

Another way to prove the identity is to write Maxwell's equation $\varepsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0$ in components. For $\mu = 0$ it is:

$$\partial_1 F_{23} - \partial_2 F_{13} - \partial_3 F_{21} - \partial_1 F_{32} + \partial_2 F_{31} + \partial_3 F_{12} = 2(\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12}) = 0$$

This is exactly the Bianchi identity (58) for $\nu = 1$, $\alpha = 2$, $\beta = 3$. We can do the same with $\mu = 1, 2, 3$ and see that the expression $\partial_\nu F_{\alpha\beta} + \partial_\alpha F_{\beta\nu} + \partial_\beta F_{\nu\alpha}$ is zero when two indices are the same, because of the antisymmetry of the $F_{\mu\nu}$ tensor. Thus we have proven the Bianchi identity component by components.