Classical Electrodynamics

Week 9

1. Consider an infinite cylinder of radius a carrying the current I. The cylinder is surrounded by an insulator with magnetic permeability μ. A metallic cylindrical surface of radius $b > a$ conducts the current in the opposite direction.

Determine the magnetic field $\mathbf{H}$, the magnetic induction $\mathbf{B}$ and the magnetization $\mathbf{M}$ in every point in space. Find the free current density $\mathbf{J}$ and the average microscopic current density $\langle \mathbf{j} \rangle$.

2. In electrostatics, the n-th pole is given by

$$Q_{i_1 \ldots i_n} = \int d^3x \, \rho(\mathbf{x}) \, T_{i_1 \ldots i_n}(\mathbf{x}) \, , \tag{1}$$

where the totally symmetric tensor $T_{i_1 \ldots i_n}$ can be defined by

$$T_{i_1 \ldots i_n} = (2n-1)!! \; x_{i_1} \ldots x_{i_n} - A_{i_1 \ldots i_n} \, , \tag{2}$$

with the double factorial $(2n-1)!! = (2n-1)(2n-3)(2n-5)\ldots(5)(3)(1)$ and $(-1)!! = 1$. The tensor $A_{i_1 \ldots i_n}$ is an homogeneous polynomial of degree n in the components of $\mathbf{x}$ and it contains at least one Kronecker-δ so that the trace vanishes:

$$T_{i_1 \ldots i_n} \delta_{i_k i_l} = 0 \, , \qquad \forall \, k, l \in \{1, 2, \ldots, n\} \, , \qquad k \neq l \, . \tag{3}$$

It is convenient to introduce the notation $B_{(i_1 \ldots i_n)}$ for the total symmetrization of a tensor $B_{i_1 \ldots i_n}$. More precisely,

$$B_{(i_1 \ldots i_n)} \equiv \frac{1}{n!} \sum_{perm \; \sigma} B_{\sigma(i_1 \ldots i_n)} \, , \tag{4}$$

where the sum runs over all permutations σ of the n indices $i_1 \ldots i_n$. For example,

$$B_{(ij)} \equiv \frac{1}{2} \left(B_{ij} + B_{ji} \right) \, , \qquad v_{(i} w_{j)} \equiv \frac{1}{2} \left(v_i w_j + v_j w_i \right) \tag{5}$$

$$B_{(ijk)} \equiv \frac{1}{6} \left(B_{ijk} + B_{ikj} + B_{jik} + B_{jki} + B_{kij} + B_{kji} \right) \, . \tag{6}$$

a) Argue that the tensors $A_{i_1 \ldots i_n}$, for $n = 2, 3, 4$, must be of the form

$$A_{ij} = c_2 \, x^2 \delta_{ij} \tag{7}$$

$$A_{ijk} = c_3 \, x^2 \delta_{(ij} x_{k)} \tag{8}$$

$$A_{ijkl} = c_4 \, x^2 \delta_{(ij} x_k x_{l)} + c_4' \, x^4 \delta_{(ij} \delta_{kl)} \, , \tag{9}$$

where c_2, c_3, c_4 and c_4' are numerical constants.

b) Determine the constants c_2, c_3, c_4 and c'_4 imposing the trace condition (3).

c) How many independent components does the tensor $Q_{i_1 \ldots i_n}$ have? Start by working out the cases $n = 0, 1, 2$. Can you guess the formula for general n? **Hint:** Start by counting the number of independent components in a totally symmetric tensor with n indices and then impose the trace constraint.

***d)** Generalize the previous question to d space dimensions. Show that the number of independent components of a traceless symmetric tensor $Q_{i_1 \ldots i_n}$ in d dimensions (*i.e.*, each index can take the values $1, 2, \ldots, d$) is given by

$$(2n + d - 2)\frac{(n + d - 3)!}{n!(d - 2)!}\,. \tag{10}$$

3. *Velocity's transformations in Special Relativity*

In this exercise, you will study how velocities v_1 and v_2 transform from one reference frame $\mathcal{R}_1$ to another $\mathcal{R}_2$ in the context of Special Relativity, in different scenarios.

a) The reference frame $\mathcal{R}_1$ is moving along the x-axis with speed v_0 in the reference frame of the laboratory $\mathcal{R}_0$, and a particle is moving with speed v_1 along the x-axis in the reference frame $\mathcal{R}_1$. What is the velocity of the particle in the reference frame $\mathcal{R}_0$?

b) Two particles (with the same mass) are moving in the same direction with velocities v_1 and $v_2 > v_1$ in the reference frame $\mathcal{R}_1$. At what speed should the reference frame $\mathcal{R}_2$ move with respect to $\mathcal{R}_1$ so that the center of mass condition $v'_1 + v'_2 = 0$ is obeyed? What is the value of v'_1? Is your result compatible with your non-relativistic intuition?

c) The trajectory of a particle moving at constant velocity makes an angle θ with the x-axis of a reference frame $\mathcal{R}_1$. Compute the corresponding angle θ' in a reference frame $\mathcal{R}_2$ moving with speed v along the x-axis of $\mathcal{R}_1$.

d) Consider two particles (with the same mass) moving at the same speed v. The angle between their trajectories is θ. Find a reference frame in which $\mathbf{v}'_1 + \mathbf{v}'_2 = 0$.