

# Classical Electrodynamics

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## Week 10

**Note:** Exercise **0.** marked by (\*) is optional and is similar to exercise **1.** of week 2. We suggest that you start with exercise **1.** We left it there as it provide a good exercise to practice the tensor notation and the formula can be useful for the rest of the problem sheet.

We use the metric  $\eta_{\mu\nu} = \text{diag}(-, +, +, +)$ .

### 0. (\*) *Lévi-Civita symbol*

The Lévi-Civita symbol is defined by

$$\varepsilon^{\mu\nu\rho\sigma} = \begin{cases} 1 & \text{if } (\mu\nu\rho\sigma) \text{ is an even permutation of } (0123) \\ -1 & \text{if } (\mu\nu\rho\sigma) \text{ is an odd permutation of } (0123) \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

a) Defining the Lorentz transformation of  $\varepsilon^{\mu\nu\rho\sigma}$  by

$$\varepsilon'^{\mu\nu\rho\sigma} = \Lambda^\mu_\alpha \Lambda^\nu_\beta \Lambda^\rho_\gamma \Lambda^\sigma_\delta \varepsilon^{\alpha\beta\gamma\delta}, \quad (2)$$

show that  $\varepsilon'^{\mu\nu\rho\sigma} = \det(\Lambda)\varepsilon^{\mu\nu\rho\sigma}$  and that  $\det(\Lambda) = \pm 1$ .

b) Verify the following identities

$$\varepsilon^{\mu\nu\rho\sigma} \varepsilon_{\mu\nu\rho\sigma} = -4! \quad (3)$$

$$\varepsilon^{\mu\nu\rho\sigma} \varepsilon_{\alpha\nu\rho\sigma} = -6 \delta_\alpha^\mu \quad (4)$$

$$\varepsilon^{\mu\nu\rho\sigma} \varepsilon_{\alpha\beta\rho\sigma} = -2 (\delta_\alpha^\mu \delta_\beta^\nu - \delta_\beta^\mu \delta_\alpha^\nu) \quad (5)$$

### 1. *Maxwell equations*

Using the definition of the field strength  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ , where  $A^\mu = (\Phi, c\mathbf{A})^T$ , verify that the equations

$$\partial_\mu F^{\mu\nu} = -\frac{1}{c\varepsilon_0} j^\nu \quad (6)$$

$$\partial_\mu \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} = 0 \quad (7)$$

describe Maxwell equations. Show that equations (6) and (7) are Lorentz invariant.

## 2. Lorentz invariants

- a) Using the field strength  $F_{\mu\nu}$  build two Lorentz invariants quadratic in the fields  $\mathbf{E}$  and  $\mathbf{B}$ .
- b) Given a tensor  $T^{\mu\nu}$  build a Lorentz invariant linear in this tensor.

## 3. Stress-energy tensor

The electromagnetic stress-energy tensor is given by

$$T^{\mu\nu} = \varepsilon_0 \left( F^\mu_\alpha F^{\nu\alpha} - \frac{1}{4} \eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right), \quad (8)$$

where  $F_{\mu\nu}$  is the field-strength tensor and  $\eta_{\mu\nu}$  is the Minkowski metric.

- a) Compute the trace  $\eta_{\mu\nu} T^{\mu\nu}$ .
- b) Verify that

$$T^{00} = u = \frac{\varepsilon_0}{2} \mathbf{E}^2 + \frac{1}{2\mu_0} \mathbf{B}^2 \quad (9)$$

is the electromagnetic energy density.

- c) Verify that

$$T^{0i} = \frac{1}{c} (\mathbf{S})_i, \quad (10)$$

where  $\mathbf{S}$  is the Poynting vector. Notice that  $\frac{1}{c} T^{0i}$  is also the momentum density (component  $i$ ) in the electromagnetic field.

- d) Using Maxwell equations (6) and (7) show that

$$\partial_\mu T^{\mu\nu} = \frac{1}{c} j_\alpha F^{\alpha\nu}. \quad (11)$$

Rewrite the  $\nu = 0$  component of this equation using the relations (9) and (10). What is the physical meaning of this equation? What about the spatial components  $\nu = 1, 2, 3$ ?