Classical Electrodynamics

Lecture Notes

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Contents

Chapter 1

Introduction

1.1 Motivation

The main goal of this course is to take a first step into the High Energy Physics way of thinking about the world. Many tools used in more advanced theoretical classes will be introduced, along with explanations of important phenomena encountered in everyday life, such as electromagnetic waves (light), radiation (WiFi, telecommunications) and polarization of matter. The structure of these notes begins with a review of Maxwell's equations and techniques to solve them, followed by a powerful approximation method for solving Maxwell's equations: the multipole expansion. We will then study the propagation of waves in a macroscopic medium, introducing the concepts of polarization, the displacement field, and interface conditions. Finally, we will incorporate special relativity into our framework to achieve a complete description of classical electrodynamics (CED). Additional topics, such as the action principle in CED and the Kramers-Kronig relations, will also be introduced.

1.2 Vector and tensor calculus review

Before starting the dive deep into electromagnetism, we will review some important vector and tensor calculus results. A deeper introduction to covariant formalism is made in Chapter 7 but the main ideas will be briefly presented. The most important rule to remember is that we sum over repeated indices, unless we say not. Take for example the definition of the gradient, we will ommit the sum and just write :

$$\boldsymbol{\nabla} f = \sum_{i=1}^{3} \partial_i f \mathbf{e}_i \to \partial_i f \mathbf{e}_i$$

where $\partial_i = \partial/\partial x_i$ and $\mathbf{e}_i$ our basis. This way of writing will allow us to lighten the notation of some proofs or formulas. We can also define more usual operation using this convention. Here are a few examples :

- **Dot product :**

$$\mathbf{A} \cdot \mathbf{B} = A_i B_i$$

- **Divergence :**

$$\boldsymbol{\nabla} \cdot \mathbf{A} = \partial_i A_i$$

- **Laplacian :**

$$\nabla^2 f = \nabla \cdot \nabla f = \partial_i \partial_i f$$

- **Curl :**

$$\nabla \times \mathbf{A} = \epsilon_{ijk} \partial_j A_k \mathbf{e}_i$$

where we define the fully antisymmetric Levi-Civita tensor ϵ_{ijk} as :

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3), \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation of } (1,2,3), \\ 0 & \text{if any two indices are equal.} \end{cases}$$

It has the properties $\epsilon_{ijk} = -\epsilon_{jik}$. You can also deduce how to write component-wise :

$$(\nabla \times \mathbf{A})_i = \epsilon_{ijk} \partial_j A_k$$

Since the i index appears once, we don't sum on it anymore and it just represents the i-th term. take for example the component $i = 1$, the only non-zero terms appearing in the sum will be $\epsilon_{123} = 1$ and $\epsilon_{132} = -1$. This gives :

$$(\nabla \times \mathbf{A})_1 = \partial_2 A_3 - \partial_3 A_2 = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$

which is what you would expect for the x component of the curl.

Now, let's define some terms :

- 0-tensor : simply a scalar (divergence, dot product)
- 1-tensor : it is a vector (gradient, curl)
- 2-tensor : a matrix (N by M table of numbers)
- n-tensor : written as $T_{i_1 i_2 \ldots i_n}$. It is a multi dimensional table of numbers that transforms according to some rules.

For instance, ϵ_{ijk} is a rank 3 tensor.

Sometimes, we will say that the indices on which we sum are contracted indices. For example, we can define a 2-tensor from a 4-tensor by contracting it's indices : $P_{ij} = T_{ijkk}$. One can also remark that contracting indices is purely equivalent to multiplying by a Kronecker delta δ_{ij} : $P_{ij} = T_{ijkl}\delta_{kl} = T_{ijkk}$.

Let's explore some transformation rules of a tensor. Take a rank 1 tensor $\mathbf{A}$ and the rotation matrix $\hat{R}$. It has properties $\hat{R}^T \hat{R} = \hat{I}$ and $\det \hat{R} = 1$. Thus, the rotated vector will be $\mathbf{A}' = \hat{R}\mathbf{A} = R_{ij} A_j \mathbf{e}_i$. We can expand this definition to higher rank tensors :

$$T'_{j_1 j_2 j_3 j_4} = R_{j_1 i_1} R_{j_2 i_2} R_{j_3 i_3} R_{j_4 i_4} T_{i_1 i_2 i_3 i_4}$$

To finish this part, we will state two of the most important theorems of vector calculus :

- **Gauss theorem :**

$$\int_V \boldsymbol{\nabla} \cdot \mathbf{E}\, dV = \oint_{\partial V} \mathbf{E} \cdot d\boldsymbol{\sigma}$$

- **Stokes theorem :**

$$\int_S d\boldsymbol{\sigma} \cdot (\boldsymbol{\nabla} \times \mathbf{F}) = \oint_{\partial S} d\mathbf{l} \cdot \mathbf{F}$$

1.3 Fourier transform and residue theorem

To conclude with this chapter, we will review some theorems of complex calculus. First, we will define the Fourier transform $\tilde{f}(\mathbf{k})$ of a function $f(\mathbf{x})$ in n dimensions :

$$\tilde{f}(\mathbf{k}) = \int_{\mathbb{R}^n} \mathrm{d}^n x f(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}}$$

And the inverse Fourier transform :

$$f(\mathbf{x}) = \int_{\mathbb{R}^n} \frac{\mathrm{d}^n k}{(2\pi)^n} \tilde{f}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}$$

A normal convention taken in general is that the Fourier transform with respect to time will have opposite signs in the exponential. This allows to preserve the invariance under a Lorentz transformation. This leads to the definition of the Fourier transform of a function $f(\mathbf{x}, t)$:

$$\tilde{f}(\mathbf{k}, \omega) = \int_{\mathbb{R}^4} \mathrm{d}^3 x \ \mathrm{d}t f(\mathbf{x}, t) e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}}$$

And the inverse Fourier transform :

$$f(\mathbf{x}, t) = \int_{\mathbb{R}^4} \frac{\mathrm{d}^3 x}{(2\pi)^3} \frac{\mathrm{d}t}{2\pi} \tilde{f}(\mathbf{k}, \omega) e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}$$

Another important result from complex analysis will be stated : the residue theorem. Let D be a domain where $f(z)$ is holomorphic excepts at the points z_k. Then :

$$\oint_{\partial D} f(z) dz = 2\pi i \sum_k \mathrm{Res}(f, z_k)$$

where we define the residue of f at z_k as :

$$\mathrm{Res}\,(f, z_k) = \frac{1}{(n-1)!} \lim_{z\to z_k} \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} (z - z_k)^n f(z)$$

Equipped with these mathematical tools, we can move on to the core of classical electrodynamics : Maxwell's equations.

Chapter 2

Maxwell's Equations

2.1 Maxwell's equations and their consequences

First, let's recall Maxwell's equations in their differential and integral form as well as the Lorentz force :

Maxwell's Equations (Differential Form)

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{E} &= \frac{\rho}{\varepsilon_0} && \text{(Gauss' law)},\\
\boldsymbol{\nabla} \cdot \mathbf{B} &= 0 && \text{(Gauss' law for magnetism)},\\
\boldsymbol{\nabla} \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} && \text{(Faraday's law)},\\
\boldsymbol{\nabla} \times \mathbf{B} &= \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} && \text{(Ampère–Maxwell law)}.
\end{aligned}$$

Maxwell's Equations (Integral Form)

$$\begin{aligned}
&\oint_{\partial V} \mathbf{E} \cdot d\boldsymbol{\sigma} = \frac{1}{\varepsilon_0} \int_V \rho \, dV && \text{(Gauss' law)},\\
&\oint_{\partial V} \mathbf{B} \cdot d\boldsymbol{\sigma} = 0 && \text{(Gauss' law for magnetism)},\\
&\oint_{\partial S} \mathbf{E} \cdot d\boldsymbol{\ell} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\boldsymbol{\sigma} && \text{(Faraday's law)},\\
&\oint_{\partial S} \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 \int_S \mathbf{J} \cdot d\boldsymbol{\sigma} + \mu_0 \varepsilon_0 \frac{d}{dt} \int_S \mathbf{E} \cdot d\boldsymbol{\sigma} && \text{(Ampère–Maxwell law)}.
\end{aligned}$$

Lorentz Force

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

where we recall the magnetic permeability of vacuum $\mu_0 = 4\pi \cdot 10^{-7}$ NA^{-2} and the electric permittivity of vacuum $\varepsilon_0 = 8.8 \cdot 10^{-12}$ AsV^{-1}m^{-1}. These 11 equations fully describe the dynamics of the electromagnetic fields and the particles inside them. The problem is that they are unsolvable in practice for the vast majority of the systems. However, some simple systems can show an exact solution to these equations. Here are some of them :

Coulomb force : Consider a point charge Q at the origin. The charge density is thus simply given by $\rho(\mathbf{r}) = Q\delta^3(\mathbf{r})$. Spherical symmetry tells us that $\mathbf{E} = E(r)\mathbf{e}_r$. Using the integral form of Gauss's law with a sphere of radius r, we can find the electric field of this point-like charge :

$$\oint_{\partial V} \mathbf{E} \cdot d\boldsymbol{\sigma} = 4\pi r^2 E(r) = \frac{1}{\varepsilon_0}\int_V \rho \, dV = \frac{Q}{\varepsilon_0}$$

So it gives the usual electric field of a point charge :

$$\mathbf{E}(\mathbf{x}) = \frac{Q}{4\pi\varepsilon_0}\frac{\mathbf{r}}{r^3}$$

Since there is no free current or magnetic field, the force on another charge q due to the charge Q is simply given by the Coulomb force :

$$\mathbf{F} = \frac{qQ}{4\pi\varepsilon_0}\frac{\mathbf{r}}{r^3}$$

This reasoning can be expanded to many particles q_i at positions $\mathbf{r}_i$. The electric field at point $\mathbf{r}$ created by this cloud of particles is given by :

$$\mathbf{E}(\mathbf{x}) = \sum_i \frac{q_i}{4\pi\varepsilon_0}\frac{\mathbf{r} - \mathbf{r}_i}{|\mathbf{r} - \mathbf{r}_i|^3}$$

Ampere's force : Consider a wire in the $\mathbf{e}_z$ transporting a current I. Thus, we have $\mathbf{E}, \rho = 0$ and $\mathbf{J}(\mathbf{x}) = I\delta(x)\delta(y)\mathbf{e}_z$. Using Ampere-Maxwell's law along a circle at distance r around the wire :

$$\oint_{\partial S} \mathbf{B} \cdot d\boldsymbol{\ell} = 2\pi r B_\theta(r) = \mu_0 \int_S \mathbf{J} \cdot d\boldsymbol{\sigma} = \mu_0 I$$

The result can be expressed as :

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{2\pi}\frac{\mathbf{e}_z \times \mathbf{r}}{r^2}$$

Consider a second wire with current I' at distance d from the first one. One can find that the Lorentz force per unit length of wire is given by :

$$\frac{\delta F}{\delta l} = -\frac{\mu_0 I I'}{2\pi d}$$

where we replaced $qv = I'$ in the Lorentz formula. This is known as the Ampère's force.

Induction : Consider a time dependent magnetic field $\mathbf{B}(t)$ going through a surface S. Faraday's law tells us that a time varying magnetic flux generates an electromotive force $\mathcal{E}$ around the boundary. This is written :

$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$

where :

$$\mathcal{E} = \oint_{\partial S} \mathbf{E} \cdot d\boldsymbol{\ell} \quad \text{and} \quad \Phi_B = \int_S \mathbf{B} \cdot d\boldsymbol{\sigma}$$

This leads to the generation of a current inside the wire formed by the boundary ∂S. This current forms its own magnetic field that opposes the original one (due to minus sign). This is known as Lenz's law.

Charge conservation : We start from :

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}.$$

Now, take the divergence of both sides:

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{B}) = \boldsymbol{\nabla} \cdot (\mu_0 \mathbf{J}) + \mu_0 \varepsilon_0 \boldsymbol{\nabla} \cdot \frac{\partial \mathbf{E}}{\partial t}.$$

But, $\boldsymbol{\nabla} \cdot (\nabla \times \mathbf{B}) = 0$. Using Gauss' law $\boldsymbol{\nabla} \cdot \mathbf{E} = \rho/\varepsilon_0$:

$$0 = \mu_0 \boldsymbol{\nabla} \cdot \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial}{\partial t} (\boldsymbol{\nabla} \cdot \mathbf{E}).$$

$$0 = \mu_0 \boldsymbol{\nabla} \cdot \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \left(\frac{\rho}{\varepsilon_0} \right) = \mu_0 \left(\boldsymbol{\nabla} \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} \right).$$

We conclude with the continuity equation :

$$\boldsymbol{\nabla} \cdot \mathbf{J} + \partial_t \rho = 0.$$

To get some insights on what this equation implies, we integrate on both sides and use Gauss's law :

$$\oint_{\partial V} \mathbf{J} \cdot d\boldsymbol{\sigma} = -\partial_t \int_V \rho dV = -\partial_t Q$$

The rate of change of charge density in a volume is exactly balanced by the net current flowing out of that volume.

Electromagnetic waves : In vacuum, we start from the Ampere-Maxwell law and take the time derivative :

$$\boldsymbol{\nabla} \times \frac{\partial \mathbf{B}}{\partial t} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

Using Faraday's law of induction as well as the identity $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\nabla^2 \mathbf{E}$ (no free charges), we get :

$$\boldsymbol{\nabla} \times \frac{\partial \mathbf{B}}{\partial t} = \nabla^2 \mathbf{E} \implies \boxed{\nabla^2 \mathbf{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}}$$

Which is the wave equation with propagating velocity of $c^2 = 1/\mu_0\varepsilon_0$. The same equation can be found for the magnetic field :

$$\boxed{\nabla^2 \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}}$$

The components of **E** and **B** will be functions of the form $f(x - ct)$. These equations mean that the electric and magnetic fields propagate in free space with the speed of light c.

2.2 Electromagnetic potentials

2.2.1 Definitions

An equivalent way to describe CED is to use electromagnetic potentials instead of **E** and **B**. It can be more convenient in some scenarios. First, we can remark that **B** is divergent free $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$. This implies that (see exercise sheet 1) there exists a vector field **A** called the **Vector potential** such that we can write :

$$\boxed{\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}}$$

Now, let's put our new definition of **B** in Faraday's law :

$$\boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = \boldsymbol{\nabla} \times \mathbf{E}' = 0$$

where we have defined a new curl free electric field $\mathbf{E}'$. Thus, there exists a scalar field Φ called the **Scalar potential** such that $\mathbf{E}'$ can be written (see exercise sheet 1) :

$$\mathbf{E}' = -\boldsymbol{\nabla}\Phi \Leftrightarrow \boxed{\mathbf{E} = -\boldsymbol{\nabla}\Phi - \frac{\partial \mathbf{A}}{\partial t}}$$

It is possible to now inject these new set of variable Φ and **A** inside the other two Maxwell's equation :

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \rightarrow -\nabla^2 \Phi - \partial_t \boldsymbol{\nabla} \cdot \mathbf{A} = \frac{\rho}{\varepsilon_0} \tag{2.1}$$

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \rightarrow -\nabla^2 \mathbf{A} + \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) = \mu_0 \mathbf{J} - \mu_0 \varepsilon_0 \left(\frac{\partial}{\partial t} \boldsymbol{\nabla} \varphi + \frac{\partial^2 \mathbf{A}}{\partial t^2} \right) \tag{2.2}$$

2.2.2 Gauge invariance

From what we defined in the last section, Φ and **A** uniquely determines **E** and **B**. However, Φ and **A** are note observables, they cannot be measured (like the wavefunction of a particle for example, which is defined up to a overall phase factor $e^{i\phi}$), thus we have a certain freedom in the definition of Φ and $\mathbf{A}$ that can lead to the same $\mathbf{E}$ and $\mathbf{B}$. One can add a constant value to the potential, it will not matter to the physics of the situation (similarly to the definition of the mechanical energy of a system, a constant potential add nothing and can be neglected). This freedom in the choice of the potentials is called **Gauge invariance**. Let $\chi(\mathbf{x}, t)$ be an arbitrary scalar function. Thus, making the transformation :

$$\Phi(\mathbf{x}, t) \to \Phi(\mathbf{x}, t) + \partial_t \chi(\mathbf{x}, t) = \tilde{\Phi}(\mathbf{x}, t)$$
$$\mathbf{A}(\mathbf{x}, \mathbf{t}) \to \mathbf{A}(\mathbf{x}, \mathbf{t}) - \boldsymbol{\nabla} \chi(\mathbf{x}, t) = \tilde{\mathbf{A}}(\mathbf{x}, t)$$

will leave $\mathbf{E}$ and $\mathbf{B}$ invariant. In fact, we have :

$$\mathbf{E} \to -\boldsymbol{\nabla}\Phi - \boldsymbol{\nabla}\partial_t \chi - \frac{\partial \mathbf{A}}{\partial t} + \partial_t \boldsymbol{\nabla}\chi = -\boldsymbol{\nabla}\Phi - \frac{\partial \mathbf{A}}{\partial t} = \mathbf{E}$$
$$\mathbf{B} \to \boldsymbol{\nabla} \times \mathbf{A} - \boldsymbol{\nabla} \times (\boldsymbol{\nabla}\chi) = \boldsymbol{\nabla} \times \mathbf{A} = \mathbf{B}$$

What we did when we add this scalar function to the potentials is called a **Gauge transformation**. We will use a gauge transformation to simplify the new Maxwell's equations for the potentials. Since a gauge transformation doesn't affect our fields, we can impose an extra condition that remove this extra degree of freedom. This is what we call **fixing the gauge**. A condition that works rather well with our system is the **Lorenz gauge** :

$$\frac{1}{c^2}\frac{\partial \Phi}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{A} = 0$$

Now, our Maxwell's equations for the potentials simplify a lot. Taking eq.2.1 to which we add and subtract $\partial_t^2 \Phi / c^2$ give :

$$\frac{1}{c^2}\partial_t^2 \Phi - \nabla^2 \Phi = \frac{\rho}{\varepsilon_0} + \partial_t \underbrace{\left(\frac{1}{c^2}\partial_t \Phi + \boldsymbol{\nabla} \cdot \mathbf{A} \right)}_{=0 \text{ from gauge}}$$

By reshuffling eq.2.2, we manage to highlight a gauge condition :

$$\frac{1}{c^2}\partial_t^2 \mathbf{A} - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J} - \boldsymbol{\nabla} \underbrace{\left(\frac{1}{c^2}\partial_t \Phi + \boldsymbol{\nabla} \cdot \mathbf{A} \right)}_{=0 \text{ from gauge}}$$

This lead to the following set of decoupled partial differential equations :

$$\frac{1}{c^2}\partial_t^2 \Phi - \nabla^2 \Phi = \frac{\rho}{\varepsilon_0},$$

$$\frac{1}{c^2}\partial_t^2 \mathbf{A} - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}.$$

These forms of Maxwell's equations will be used in the following chapters to develop solving techniques.

Chapter 3

Solving Maxwell's Equations

In this chapter, we will study solving methods of Maxwell's equations. First, let's consider electrostatics, meaning we fix $\mathbf{J}, \mathbf{A} = 0$ and $\partial_t(\rho, \Phi) = 0$. Our problem reduces to the Poisson equation for the electrostatic potential $-\nabla^2\Phi(\mathbf{x}) = \rho(\mathbf{x})/\varepsilon_0$. One can find that if all charges are known, and $\Phi(\mathbf{x} \to \infty) \to 0$, we can simply express the potential as :

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \int d^3x \frac{\rho(\mathbf{x})}{|\mathbf{x} - \mathbf{x}'|}$$

We can see that in the case of a cloud of charges q_i located at positions $\mathbf{x}_i$, the charge density can be expressed as $\rho(\mathbf{x}) = \sum_i q_i \delta^3(\mathbf{x} - \mathbf{x}_i)$. This lead to the sum of point-like electrostatic potentials :

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \sum_i \frac{q_i}{|\mathbf{x} - \mathbf{x}_i|}$$

However, in some situations where the boundary of the space doesn't imply a vanishing potential or if not all charges are known, we must use more advanced tools.

3.1 Green's function method

3.1.1 Definition

Suppose we are given a set of charges inside some region of space V as well as some boundary conditions on Φ. To find Φ inside of V, we need to first solve Poisson's equation for the Green's function (G.F.) with given boundary conditions (b.c.) :

$$-\nabla^2 G(\mathbf{x}, \mathbf{x}') = \delta^3(\mathbf{x} - \mathbf{x}')$$

Two general type of b.c. exists for which the G.F. is unique:

1. **Dirichlet :** The potential has a definite value at the boundary : $\Phi|_{\partial V} = g(\mathbf{x})$.
2. **Neumann :** The normal derivative vanishes on the boundary : $-\mathbf{n} \cdot \mathbf{E} = \mathbf{n} \cdot \boldsymbol{\nabla}\Phi|_{\partial V} = \partial_{\mathbf{n}}\Phi|_{\partial V} = f(\mathbf{x})$. $\mathbf{n}$ is a unit vector directed outside of V. This condition mostly reflects the behavior of a conductor for example.

In general, the most useful G.F. are the one satisfying homogeneous b.c., meaning $f = g = 0$.

First, let's prove a master formula :

$$\Phi(\mathbf{x}) = \frac{1}{\varepsilon_0}\int_V d^3x' G\left(\mathbf{x}',\mathbf{x}\right)\rho\left(\mathbf{x}'\right) + \int_{\partial V} d\boldsymbol{\sigma}'\cdot\left[G\left(\mathbf{x}',\mathbf{x}\right)\boldsymbol{\nabla}_{\mathbf{x}'}\Phi\left(\mathbf{x}'\right) - \Phi\left(\mathbf{x}'\right)\boldsymbol{\nabla}_{\mathbf{x}'}G\left(\mathbf{x}',\mathbf{x}\right)\right]$$

To prove it, we define the quantity $F = \int d^3x'\boldsymbol{\nabla}_{\mathbf{x}'}\cdot\left[G\left(\mathbf{x}',\mathbf{x}\right)\boldsymbol{\nabla}_{\mathbf{x}'}\Phi\left(\mathbf{x}'\right) - \Phi\left(\mathbf{x}'\right)\boldsymbol{\nabla}_{\mathbf{x}'}G\left(\mathbf{x}',\mathbf{x}\right)\right]$. Then, we have :

$$\begin{aligned} F &= \int d^3x'\boldsymbol{\nabla}_{\mathbf{x}'}\cdot\left[G\left(\mathbf{x}',\mathbf{x}\right)\boldsymbol{\nabla}_{\mathbf{x}'}\Phi\left(\mathbf{x}'\right) - \Phi\left(\mathbf{x}'\right)\boldsymbol{\nabla}_{\mathbf{x}'}G\left(\mathbf{x}',\mathbf{x}\right)\right] \\ &= \int d^3x' G\left(\mathbf{x}',\mathbf{x}\right)\underbrace{\boldsymbol{\nabla}^{\mathbf{2}}_{\mathbf{x}'}\Phi\left(\mathbf{x}'\right)}_{=-\rho/\varepsilon_0} - \int d^3x'\Phi\left(\mathbf{x}'\right)\underbrace{\boldsymbol{\nabla}^{\mathbf{2}}_{\mathbf{x}'}G\left(\mathbf{x}',\mathbf{x}\right)}_{=-\delta^3(\mathbf{x}-\mathbf{x}')} \\ &= -\frac{1}{\varepsilon_0}\int d^3x' G\left(\mathbf{x}',\mathbf{x}\right)\rho(\mathbf{x}') + \Phi\left(\mathbf{x}\right) \end{aligned}$$

On the other side, Gauss's theorem gives :

$$F = \int_{\partial V} d\boldsymbol{\sigma}'\cdot\left[G\left(\mathbf{x}',\mathbf{x}\right)\boldsymbol{\nabla}_{\mathbf{x}'}\Phi\left(\mathbf{x}'\right) - \Phi\left(\mathbf{x}'\right)\boldsymbol{\nabla}_{\mathbf{x}'}G\left(\mathbf{x}',\mathbf{x}\right)\right]$$

Combining the last 2 steps proves the master formula.

If we have a Dirichlet b.c. with $\Phi|_{\partial V} = g(\mathbf{x})$, it is more convenient to choose green's function such that $G_D(\mathbf{x}',\mathbf{x})|_{\mathbf{x}'\in\partial V} = 0$. This lead to a simplified version of the master formula :

$$\Phi(\mathbf{x}) = \frac{1}{\varepsilon_0}\int_V d^3x' G_D\left(\mathbf{x}',\mathbf{x}\right)\rho\left(\mathbf{x}'\right) - \int_{\partial V} d\boldsymbol{\sigma}'\cdot g\left(\mathbf{x}'\right)\boldsymbol{\nabla}_{\mathbf{x}'}G_D\left(\mathbf{x}',\mathbf{x}\right)$$

Similarly for a Neuman b.c. with $\mathbf{n}\cdot\boldsymbol{\nabla}\Phi|_{\partial V} = f(\mathbf{x})$, we will choose green's function to respect $\mathbf{n}\cdot\boldsymbol{\nabla}G_N(\mathbf{x}',\mathbf{x})|_{\mathbf{x}'\in\partial V} = 0$. This will give :

$$\Phi(\mathbf{x}) = \frac{1}{\varepsilon_0}\int_V d^3x' G_N\left(\mathbf{x}',\mathbf{x}\right)\rho\left(\mathbf{x}'\right) + \int_{\partial V} d\boldsymbol{\sigma}'\cdot G_N\left(\mathbf{x}',\mathbf{x}\right)\boldsymbol{\nabla}_{\mathbf{x}'}\Phi\left(\mathbf{x}'\right)$$

One fundamental property of these green's functions are that they are **unique**. In fact, suppose there are two distinct solutions Φ_1, Φ_2 to the Poisson's equation $-\nabla^2\Phi_i = \rho/\varepsilon_0$, and $\Phi_i|_{\partial V} = g$ or $\partial_{\mathbf{n}}\Phi_i|_{\partial V} = f$, $i = 1, 2$.

Let $U = \Phi_1 - \Phi_2$, then $\nabla^2 U = 0$. Now, let's have a look at the quantity $\int_V(\boldsymbol{\nabla}U)^2$

$$\int_V(\boldsymbol{\nabla}U)^2 = \int_V \boldsymbol{\nabla}(U\boldsymbol{\nabla}U) = \int_{\partial V} d\boldsymbol{\sigma}(U\boldsymbol{\nabla}U)$$

But, the boudary conditions fixes $U|_{\partial V} = 0$ or $\boldsymbol{\nabla} U|_{\partial V} = 0$. So we have $\boldsymbol{\nabla} U = 0$, which means that Φ_1 and Φ_2 are equal (up to a constant). This proves the uniqueness of both Dirichlet and Neumann Green's function.

3.1.2 Green's function for entire space

Let's now derive explicitly the G.F. for the case where $V = \mathbb{R}^3$ and Dirichlet condition $G(\mathbf{x}', \mathbf{x} \to \infty) = 0$. We recall the equation satisfied by G :

$$-\nabla^2 G(\mathbf{x}, \mathbf{x}') = \delta^3(\mathbf{x} - \mathbf{x}')$$

Now, apply the Fourier transform on both sides $\int \mathrm{d}^3 x e^{-i\mathbf{k}\cdot\mathbf{x}}$. In the Fourier space, we have the following correspondence : $\boldsymbol{\nabla} \to -i\mathbf{k}$. This gives :

$$k^2 \tilde{G}(\mathbf{k}, \mathbf{x}') = e^{-i\mathbf{k}\cdot\mathbf{x}'}$$

We now apply the inverse Fourier transform :

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{(2\pi)^3} \int \mathrm{d}^3 k e^{i\mathbf{k}\cdot\mathbf{x}} \frac{1}{k^2} e^{-i\mathbf{k}\cdot\mathbf{x}'}$$

Let's now go to a spherical basis, where the angle θ is the angle between $\mathbf{x} - \mathbf{x}'$ and $\mathbf{k}$. This gives :

$$\begin{aligned} G(\mathbf{x}, \mathbf{x}') &= \frac{1}{(2\pi)^3} \cdot 2\pi \int_0^\infty dk \int_0^\pi d\theta k^2 \sin\theta \frac{1}{k^2} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \\ &= \frac{1}{(2\pi)^2} \int_0^\infty dk \int_{-1}^1 d(\cos\theta) e^{ik|\mathbf{x}-\mathbf{x}'|\cos\theta} \\ &= \frac{1}{(2\pi)^2} \int_0^\infty dk \frac{1}{ik|\mathbf{x}-\mathbf{x}'|} \left[e^{ik|\mathbf{x}-\mathbf{x}'|} - e^{-ik|\mathbf{x}-\mathbf{x}'|} \right] \end{aligned}$$

Defining $r = |\mathbf{x} - \mathbf{x}'|$ and using the parity of the previous integral, we have the following form :

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{2(2\pi)^2} \int_{-\infty}^{\infty} dk \frac{e^{ikr} - e^{-ikr}}{ikr} = \frac{1}{2(2\pi)^2} \frac{1}{ir} \left[\underbrace{\int_{-\infty}^{\infty} dk \frac{e^{ikr}}{k}}_{=I_1} - \underbrace{\int_{-\infty}^{\infty} dk \frac{e^{-ikr}}{k}}_{=I_2} \right]$$

We will first look at I_1. Let's go to the complex plane to solve the more general integral :

$$\hat{I}_1 = \int_{\mathcal{C}} dk \frac{e^{ikr}}{k} = \int_{-R}^{R} dk \frac{e^{ikr}}{k} + \int_\gamma dk \frac{e^{ikr}}{k}$$

Where γ is the circling part. This part can be parametrized by $k = Re^{i\theta} = R(\cos\theta + i\sin\theta)$ with $\theta \in [0, \pi]$ and $dk = iRe^{i\theta} d\theta = ik d\theta$. It gives :

$$\int_\gamma dk \frac{e^{ikr}}{k} = i \int_0^\pi d\theta e^{irR\cos\theta} e^{-rR\sin\theta}$$

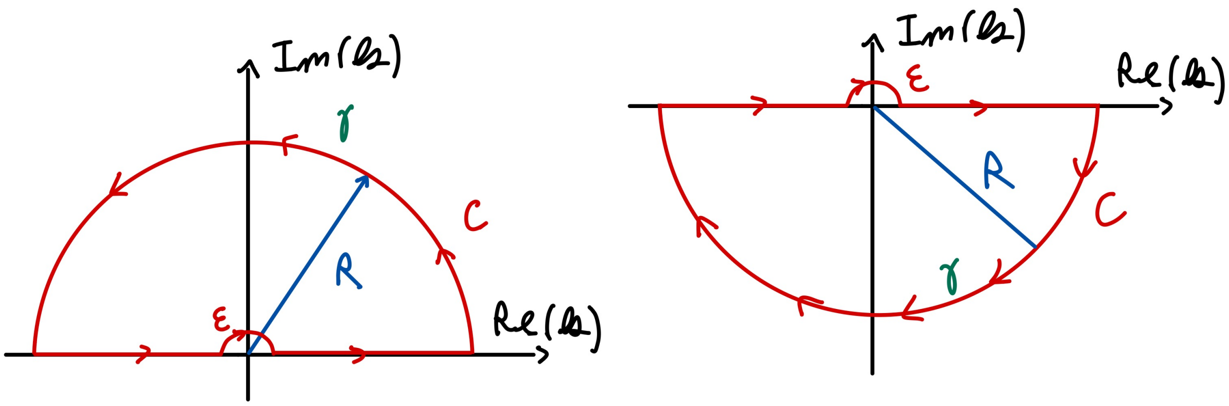


Figure 3.1: Left : contour for I_1 ; Right : contour for I_2.

But, when $R \to \infty$, the integrand goes to 0 so the integral on γ is 0. The singular part is on the axis $\text{Im}(z) = 0$. We thus slightly deform the contour to avoid the singular part. Since the integrand is well defined inside the part circle, we have $\int_\mathcal{C} dk \frac{e^{ikr}}{k} = 0$ from Cauchy's theorem. Thus, $I_1 = 0$.

For I_2, the treatment is nearly the same, with a parametrization $k = Re^{-i\theta}$ to still have a vanishing exponential when $R \to \infty$. However, the singularity now has to be treated. Taking a contour small enough won't affect the integral from $-\infty$ to ∞ but we now have to compute the residue in $k = 0$, which is straightforward :

$$\text{Res}\left(\frac{e^{-ikr}}{k}, 0\right) = e^{-ir\cdot 0} = 1$$

Finally, this gives $I_2 = 2\pi i$.
Putting everything together gives the final form of our green's function :

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{2(2\pi)^2}\frac{1}{ir} \cdot 2\pi i = \frac{1}{4\pi r}$$

Replacing by the definition of r :

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|}$$

3.2 Image charge method

We have derived the method to find G.F. for the entire space. But what about other geometries of space ? The answer relies on uniqueness and linearity of the G.F. If we find one G.F. that works well with our boundary conditions, it is unique. For instance, what is the G.F. for the space with $z > 0$ and Dirichlet conditions, i.e. $G(\mathbf{x} = (x, y, 0), \mathbf{x}') = 0$? It is possible to show that the Green's function given by :

$$G(\mathbf{x}, \mathbf{x}') - G(\mathbf{x}, \tilde{\mathbf{x}}') = \frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|} - \frac{1}{4\pi|\mathbf{x} - \tilde{\mathbf{x}}'|}$$

with $\tilde{\mathbf{x}}' = (x', y', -z')$, respects the boundary condition at $z = 0$:

$$[G(\mathbf{x}, \mathbf{x}') - G(\mathbf{x}, \tilde{\mathbf{x}}')]_{z=0} = \frac{1}{4\pi}\left(\frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + z'^2}} - \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + z'^2}}\right) = 0$$

We can see this trick as placing an imaginary charge outside of the volume considered, at a mirrored position with respect to the boundary as you can see on Fig.3.2.

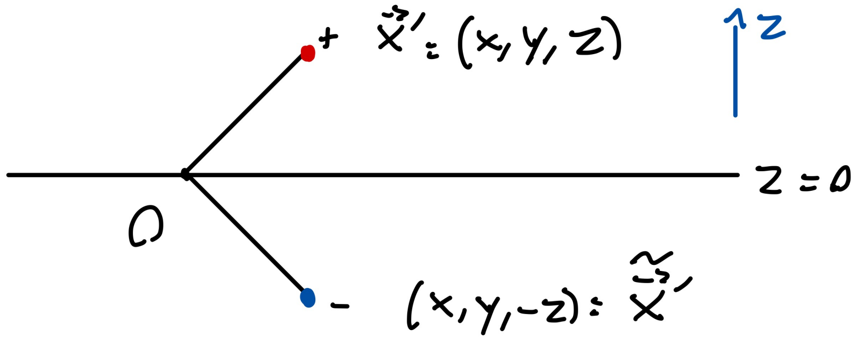


Figure 3.2: Imaginary charge outside of V.

In a similar way of Neumann boundary condition but this time, we would add an identical image charge (same charge), which would lead to the G.F. :

$$G(\mathbf{x}, \mathbf{x}') + G(\mathbf{x}, \tilde{\mathbf{x}}') = \frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|} + \frac{1}{4\pi|\mathbf{x} - \tilde{\mathbf{x}}'|}$$

Last example, if we consider a quarter space with $x > 0$ and $y > 0$, we would have opposites image charges at positions $\mathbf{x_1'} = (-x', y', z')$, $\mathbf{x_2'} = (x', -y', z')$ and a same image charge at position $\mathbf{x_3'} = (-x', -y', z')$, giving the G.F. :

$$G(\mathbf{x}, \mathbf{x}') - G(\mathbf{x}, \mathbf{x_1'}) - G(\mathbf{x}, \mathbf{x_2'}) + G(\mathbf{x}, \mathbf{x_3'})$$

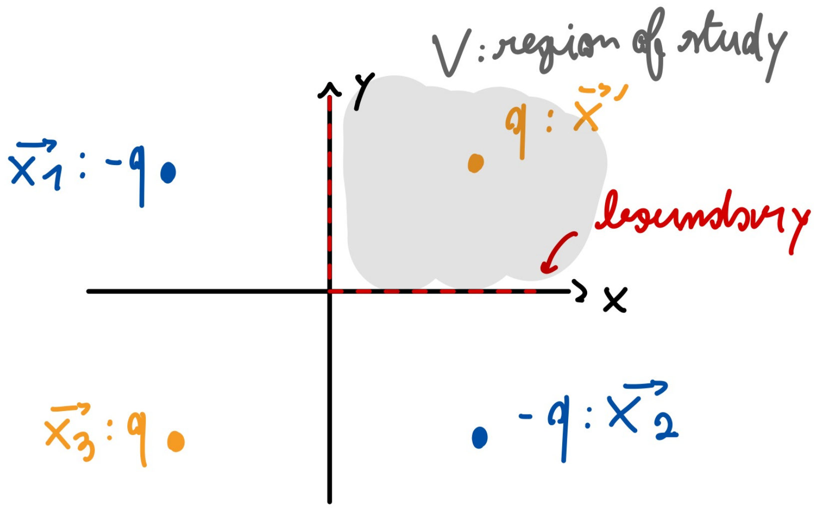


Figure 3.3: Quarter plane

3.3 Expansion in eigenfunctions

Another method to find the G.F. for a certain region is the expansion in eigenfunctions, which is very similar to a Fourier series expansion. Start with an eigenvalue problem :

$$-\nabla^2\psi_n(\mathbf{x}) = \lambda_n\psi_n(\mathbf{x})$$

with ψ_n satisfying the boundary conditions. We will also impose orthonomalization :

$$\int_V d^3x\, \psi_n^*(\mathbf{x})\psi_m(\mathbf{x}) = \delta_{nm}$$

Then, by completeness, we can write :

$$\delta^3(\mathbf{x}-\mathbf{x}') = \sum_n \psi_n^*(\mathbf{x}')\psi_n(\mathbf{x})$$

We can now expand our G.F. in this basis :

$$G(\mathbf{x},\mathbf{x}') = \sum_n c_n(\mathbf{x}')\psi_n(\mathbf{x})$$

To determine the coefficients c_n, we inject the expansion inside the equation satisfied by the G.F. which has to be satisfied for each index n independently :

$$\begin{aligned}
-\nabla^2_{\mathbf{x}}(c_n(\mathbf{x}')\psi_n(\mathbf{x})) &= \psi_n^*(\mathbf{x}')\psi_n(\mathbf{x}) \\
c_n(\mathbf{x}')\lambda_n\psi_n(\mathbf{x}) &= \psi_n^*(\mathbf{x}')\psi_n(\mathbf{x}) \\
c_n(\mathbf{x}') &= \frac{\psi_n^*(\mathbf{x}')}{\lambda_n}
\end{aligned}$$

This gives :

$$G(\mathbf{x},\mathbf{x}') = \sum_n \frac{\psi_n^*(\mathbf{x}')\psi_n(\mathbf{x})}{\lambda_n}$$

As a example in 1D, with $x \in [0,L]$ we can take our basis as $\psi_n(x) = A\cos(\sqrt{\lambda_n}x) + B\sin(\sqrt{\lambda_n}x)$. D.b.c imposes that $A = 0$ and $B\sin(\sqrt{\lambda_n}L) = 0$ so we have for $n > 0$:

$$\psi_n(x) = B_n\sin(\sqrt{\lambda_n}x) \quad , \quad \lambda_n = \frac{\pi^2 n^2}{L^2} \quad , \quad B_n = \sqrt{2/L}$$

3.4 Magnetostatics

In the case where $\rho, \Phi = 0$, and $\partial_t(\mathbf{J},\mathbf{A}) = 0$, Maxwell's equation for the vector potential reduces to :

$$-\nabla^2\mathbf{A}(\mathbf{x}) = \mu_0\mathbf{J}$$

which is the same problem as in electrostatics for each components. We can apply the same methods stated before.

3.5 Dynamics

3.5.1 Retarded and Advanced Green's functions

Recall Maxwell's equations for the potentials in the general case :

$$\frac{1}{c^2}\partial_t^2\Phi(\mathbf{x},t) - \nabla^2\Phi(\mathbf{x},t) = \frac{\rho(\mathbf{x},t)}{\varepsilon_0},$$

$$\frac{1}{c^2}\partial_t^2\mathbf{A}(\mathbf{x},t) - \nabla^2\mathbf{A}(\mathbf{x},t) = \mu_0\mathbf{J}(\mathbf{x},t).$$

We can define the box operator $\Box = \frac{1}{c^2}\partial_t^2 - \nabla^2$ in order to reduce our equations :

$$\Box\Phi(\mathbf{x},t) = \rho(\mathbf{x},t)/\varepsilon_0 \quad , \quad \Box\mathbf{A}(\mathbf{x},t) = \mu_0\mathbf{J}(\mathbf{x},t)$$

Let's only focus on the scalar potential Φ for now. We can look again for G.F. that solves a more general equation with respect to the box operator :

$$\Box_{\mathbf{x},t}G(\mathbf{x},t,\mathbf{x}',t') = \delta^3(\mathbf{x}-\mathbf{x}')\delta(t-t')$$

The solution for the electrostatic potential is given by :

$$\Phi(\mathbf{x}) = \frac{1}{\varepsilon_0}\int d^3x'dt'\,G(\mathbf{x},t,\mathbf{x}',t')\rho(\mathbf{x}',t') + \text{b.c.}$$

In this course, we won't discuss boundaries on the dynamical case. We will fix $G \to 0$ when $|\mathbf{x}| \to \infty$. We can now consider two types of G.F. :

1. Retarded Green's function : $G_R(t < t') = 0$
2. Advanced Green's function : $G_A(t > t') = 0$

The most commonly used G.F. is the retarded one. $G_R(t < t') = 0$ means that an event at time $t' > t$ cannot influence what happens at time t in the past. Let's proceed as before and go into Fourier space by applying $\int \mathrm{d}^3x\ \mathrm{d}t\, e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}}$ on Green's equation. Using that in this space, $\partial_t \to i\omega$,the box operator transforms as $\Box \to k^2 - \omega^2/c^2$, giving :

$$\tilde{G}(\mathbf{k},\omega,\mathbf{x}',t') = \frac{e^{-i\mathbf{k}\cdot\mathbf{x}'+i\omega t'}}{k^2-\omega^2/c^2}$$

Back to real space, the G.F. reads :

$$G(\mathbf{x},t,\mathbf{x}',t') = \frac{1}{(2\pi)^4}\int d^3k\,d\omega\,\frac{e^{-i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})-i\omega(t-t')}}{k^2-\omega^2/c^2}$$

We will first perform the integral over ω. There are two singularities on the real axis of ω at $\omega = \pm ck$. Let $\mathcal{I}$ be the integral over ω :

$$\mathcal{I} = \int_{-\infty}^{\infty} d\omega\,\frac{e^{-i\omega(t-t')}}{k^2-\omega^2/c^2} = -\frac{c}{2k}\int_{-\infty}^{\infty} d\omega\,e^{-i\omega\Delta t}\left(\frac{1}{\omega-ck} - \frac{1}{\omega+ck}\right).$$

where $\Delta t = t - t'$. We will focus on the retarded G.F., meaning that $G = 0$ for $\Delta t < 0$. We will now use this condition to define our integration contour. For $\Delta t < 0$, and $\omega = a + ib$, the integrand takes the form :

$$e^{-i\omega\Delta t} = e^{ia|\Delta t|}e^{-b|\Delta t|}$$

We want this part to vanish as the contour goes to ∞ so we need $b > 0$, meaning that we close the contour above the real axis. In addition to that, $G = 0$ when $\Delta t < 0$ so we will take a little deviation of our contour above the singular point so that they do not belong to the interior of the contour, leading to an integral giving 0 by Cauchy's theorem. This corresponds to the standard prescription $\omega \to \omega + i\epsilon$ ensuring causality in the definition of the retarded Green's function. A similar reasoning can be applied to find the contour for $\Delta t > 0$. We want a parametrization $\omega = a + ib$ such that :

$$e^{-i\omega\Delta t} = e^{-ia|\Delta t|}e^{b|\Delta t|}$$

vanishes as the contour goes to ∞. We thus need $b < 0$, i.e. we close the contour from below the real axis. For $\Delta t > 0$, the causality condition requires that the poles contribute to the integral ($G \neq 0$), hence we include them inside the contour (with the small shift above the real axis). The contour chosen are summarized on Fig.4.1.

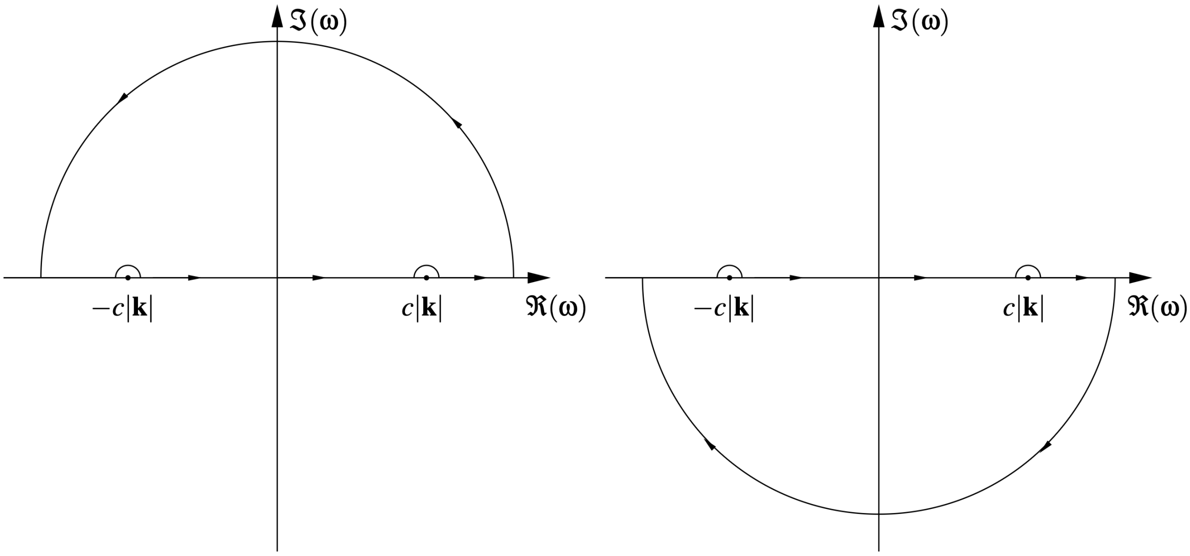


Figure 3.4: Left : contour for $\Delta t < 0$; Right : contour for $\Delta t > 0$. Note that for the advanced G.F., the cases $\Delta t < 0$ and $\Delta t > 0$ are reversed.

Now, let's proceed to the computation of the residues :

$$\text{Res}\left(e^{-i\omega\Delta t}\left(\frac{1}{\omega - ck} - \frac{1}{\omega + ck}\right), ck\right) = e^{-ick\Delta t}$$
$$\text{Res}\left(e^{-i\omega\Delta t}\left(\frac{1}{\omega - ck} - \frac{1}{\omega + ck}\right), -ck\right) = -e^{ick\Delta t}$$

By the residue theorem, we have :

$$\mathcal{I} = -\frac{c}{2k}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega\Delta t}\left(\frac{1}{\omega - ck} - \frac{1}{\omega + ck}\right). = -\frac{c}{2k}\cdot(-2\pi i)\cdot\left(e^{-ick\Delta t} - e^{ick\Delta t}\right)$$

where the minus sign comes from the fact that the contour is taken in the anticlockwise direction. Define $\mathbf{r} = \mathbf{x} - \mathbf{x}'$, $r = |\mathbf{r}|$, we have :

$$G(\mathbf{x}, t, \mathbf{x}', t') = -\frac{\pi i c}{(2\pi)^4}\int d^3k\, \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{k}\left(e^{ick\Delta t} - e^{-ick\Delta t}\right)$$

Define now spherical coordinates such that $\mathbf{k}\cdot\mathbf{r} = kr\cos\theta$, we have :

$$\begin{aligned} G(\mathbf{x}, t, \mathbf{x}', t') &= -\frac{2\pi^2 ic}{(2\pi)^4}\int_0^\infty dk \int_{-1}^1 d(\cos\theta)\, ke^{ikr\cos\theta}\left(e^{ick\Delta t} - e^{-ick\Delta t}\right) \\ &= -\frac{2\pi^2 ic}{(2\pi)^4}\int_0^\infty dk\, \frac{e^{ikr} - e^{-ikr}}{ir}\left(e^{ick\Delta t} - e^{-ick\Delta t}\right) \\ &= -\frac{2\pi^2 c}{(2\pi)^4}\frac{1}{r}\int_0^\infty dk\, e^{ik[r+c\Delta t]} + e^{-ik[r+c\Delta t]} - e^{ik[r-c\Delta t]} - e^{-ik[r-c\Delta t]} \end{aligned}$$

Note that the integrals with $-k$ in the exponential can be changed as $-k \to k$ and $\int_0^\infty dk \to \int_{-\infty}^0 dk$ so that we can make use of the Dirac delta definition $\delta(x) = \int \frac{dk}{2\pi}e^{ikx}$:

$$G(\mathbf{x}, t, \mathbf{x}', t') = -\frac{2\pi^2 c}{(2\pi)^3}\frac{1}{r}\int_{-\infty}^\infty \frac{dk}{2\pi}\, e^{ik[r+c\Delta t]} - e^{ik[r-c\Delta t]} = -\frac{c}{4\pi r}\left[\delta\left(r + c[t-t']\right) - \delta\left(r - c[t-t']\right)\right]$$

Now, remember that $t - t' > 0$ so we will never have $r + c[t-t'] = 0$ so the first Dirac will always be 0. Recall that $\delta(ax) = \delta(x)/|a|$, this will lead to our final expression of the Retarded Green's function :

$$G_R(\mathbf{x}, t, \mathbf{x}', t') = \frac{\delta\left(t - t' - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)}{4\pi|\mathbf{x}-\mathbf{x}'|} = \frac{\delta\left(t - t' - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)}{4\pi|\mathbf{x}-\mathbf{x}'|}\Theta(t-t')$$

where we explicitly wrote the condition $t > t'$ with the Heaviside function :

$$\Theta(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0. \end{cases}$$

A similar computation gives the Advanced Green's function :

$$G_A(\mathbf{x}, t, \mathbf{x}', t') = \frac{\delta\left(t - t' + \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)}{4\pi|\mathbf{x}-\mathbf{x}'|} = \frac{\delta\left(t - t' + \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)}{4\pi|\mathbf{x}-\mathbf{x}'|}\Theta(t'-t)$$

Remark : the Heaviside function is redundant with the Dirac delta before. For the retarded G.F., we can remark that if $t < t'$, the argument inside the Delta will never be 0 : $t - t' - \frac{|\mathbf{x}-\mathbf{x}'|}{c} < 0$ in this case. The same goes for the advanced G.F.

3.5.2 Explicit form of the potentials

Now that we found the general green's function, we can write the form of the scalar and vector potential :

$$\Phi(\mathbf{x},t) = \frac{1}{4\pi\varepsilon_0}\int d^3x'\,\frac{\rho\left(\mathbf{x}',\, t-\frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)}{|\mathbf{x}-\mathbf{x}'|},$$

$$\mathbf{A}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int d^3x'\,\frac{\mathbf{J}\left(\mathbf{x}',\, t-\frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)}{|\mathbf{x}-\mathbf{x}'|}.$$

The sources ρ and $\mathbf{J}$ are evaluated at the retarded time $t_{\mathrm{ret}} = t - |\mathbf{x}-\mathbf{x}'|/c$, which means that the electromagnetic effects propagates at the finite speed of time c.

Chapter 4

Properties of Electromagnetic Fields

4.1 Electrostatic energy

Let us recall the potential energy of a system of point charges :

$$\mathcal{U} = \sum_{\langle ij \rangle} \frac{q_i q_j}{4\pi\varepsilon_0} \frac{1}{|\mathbf{r}_i - \mathbf{r}_j|}$$

Where $\langle ij \rangle$ denotes every pair of charges. In continuum, it would read as :

$$\mathcal{U} = \frac{1}{8\pi\varepsilon_0} \int d^3x\, d^3y\, \frac{\rho(\mathbf{x})\rho(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|}$$

where the 1/2 factor takes into account the redundancy of the charges. We can rewrite this expression using the electrostatic potential :

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \int d^3x'\, \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|},$$

This gives :

$$\mathcal{U} = \frac{1}{2} \int d^3x\, \rho(\mathbf{x})\Phi(\mathbf{x})$$

But, Poisson's equation directly links the charge density to the potential $\nabla^2\Phi = -\rho/\varepsilon_0$:

$$\mathcal{U} = -\frac{\varepsilon_0}{2} \int d^3x\, \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}\Phi(\mathbf{x}))\Phi(\mathbf{x}) = \frac{\varepsilon_0}{2} \int d^3x\, \boldsymbol{\nabla}\Phi(\mathbf{x}) \cdot \boldsymbol{\nabla}\Phi(\mathbf{x}) = \frac{\varepsilon_0}{2} \int d^3x\, |\mathbf{E}|^2$$

Thus, the energy of the charges is just the energy of the fields. We can rewrite in terms of energy density :

$$\mathcal{U} = \int d^3x\, u \quad , \quad u = \frac{1}{2}\varepsilon_0 |\mathbf{E}|^2$$

4.2 Electromagnetic energy

Recall :

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{4.1}$$

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{4.2}$$

Now, take the dot product between **B** and eq.4.1 and **E** and eq.4.2, and take the difference of the two equations :

$$\mu_0 \mathbf{J} \cdot \mathbf{E} + \mu_0 \varepsilon_0 \frac{1}{2} \frac{\partial |\mathbf{E}|^2}{\partial t} + \frac{1}{2} \frac{\partial |\mathbf{B}|^2}{\partial t} = \mathbf{E} \cdot \boldsymbol{\nabla} \times \mathbf{B} - \mathbf{B} \cdot \boldsymbol{\nabla} \times \mathbf{E} = -\boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{B}) \tag{4.3}$$

Now, define two quantities. First, the Poynting vector :

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}$$

It represents the directional energy flux, i.e. the energy of an EM field is directed along **S**. The units are $[\mathbf{S}] = \text{W/m}^2 = \text{Jm}^{-2}\text{s}^{-1}$. Define now the electromagnetic energy density :

$$u = \frac{1}{2} \varepsilon_0 |\mathbf{E}|^2 + \frac{1}{2\mu_0} |\mathbf{B}|^2$$

Inserting these definition in eq.4.3, we find a conservation law :

$$\mathbf{J} \cdot \mathbf{E} + \frac{\partial u}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{S} = 0 \tag{4.4}$$

Integrating eq.4.4 over a volume V, we find :

$$\int_V d^3x \, \mathbf{J} \cdot \mathbf{E} + \frac{\partial}{\partial t} \int_V d^3x \, u + \int_{\partial V} d\boldsymbol{\sigma} \cdot \mathbf{S} = 0 \tag{4.5}$$

Eq. (4.5) contains three terms, each with an important physical meaning. The first one can be interpreted as the work done on the charges inside the volume, since $\mathbf{J} \cdot \mathbf{E}$ has the dimensions of power ($\mathbf{J} = \rho \mathbf{v}$ and thus $\mathbf{J} \cdot \mathbf{E} \sim \mathbf{v} \cdot \mathbf{F}$). The second term represents the time variation of the electromagnetic energy stored in V, $\partial_t \mathcal{U}$. The third term is the energy flux across the boundary ∂V, given by the Poynting vector **S**.
Rearranging Eq. (4.5), we obtain

$$\frac{\partial}{\partial t} \int_V d^3x \, u = -\int_V d^3x \, \mathbf{J} \cdot \mathbf{E} - \int_{\partial V} d\boldsymbol{\sigma} \cdot \mathbf{S}.$$

In other words, the time variation of the electromagnetic energy stored in V is exactly equal to the opposite of (i) the work delivered to the charges inside V and (ii) the net flux of electromagnetic energy leaving the volume through its boundary.
In summary: **any decrease of electromagnetic energy in the volume is compensated by the energy transferred to matter (work on the charges) and by the energy flowing out through the Poynting vector.**

4.3 Electromagnetic waves

Let's consider plane waves in vacuum, meaning $\rho, \mathbf{J} = 0$ and $\Box\Phi = 0, \Box\mathbf{A} = 0$. The general solutions for the wave equation are of the form :

$$\begin{aligned}\Phi(\mathbf{x},t) &= \hat{\Phi}(\mathbf{k})\, e^{\,i(\mathbf{k}\cdot\mathbf{x}-\omega_{\mathbf{k}}t)} + \text{c.c.}\\ \mathbf{A}(\mathbf{x},t) &= \hat{\mathbf{A}}(\mathbf{k})\, e^{\,i(\mathbf{k}\cdot\mathbf{x}-\omega_{\mathbf{k}}t)} + \text{c.c.}\end{aligned}$$

where c.c. denotes the complex conjugate[1]. Plugin this Ansatz inside our wave equation gives the dispersion relation :

$$\omega_{\mathbf{k}}^2 - c^2|\mathbf{k}|^2 = 0 \Leftrightarrow \omega_{\mathbf{k}} = \pm c|\mathbf{k}|$$

The group velocity (in norm) is simply the speed of light $v_g = |\partial_{\mathbf{k}}\omega| = c$. Next, we impose Lorentz gauge, which translates in the reciprocal space to :

$$\frac{1}{c^2}\omega_{\mathbf{k}}\hat{\Phi} - \mathbf{k}\cdot\hat{\mathbf{A}} = 0 \Rightarrow \hat{\Phi} = \frac{c\mathbf{k}\cdot\mathbf{A}}{|\mathbf{k}|}$$

Next, we want to find the fields $\mathbf{E}$ and $\mathbf{B}$. We search expressions of the same form as the potentials, i.e, as plane waves :

$$\begin{aligned}\mathbf{E}(\mathbf{x},t) &= \hat{\mathbf{E}}(\mathbf{k})\, e^{\,i(\mathbf{k}\cdot\mathbf{x}-\omega_{\mathbf{k}}t)} + \text{c.c.}\\ \mathbf{B}(\mathbf{x},t) &= \hat{\mathbf{B}}(\mathbf{k})\, e^{\,i(\mathbf{k}\cdot\mathbf{x}-\omega_{\mathbf{k}}t)} + \text{c.c.}\end{aligned}$$

From the definition of the fields from the potentials $\mathbf{E} = -\boldsymbol{\nabla}\Phi - \partial_t\mathbf{A}$ and $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$, we find for the electric field :

$$\mathbf{E}(\mathbf{x},t) = (-i\mathbf{k}\hat{\Phi} + i\omega_{\mathbf{k}}\hat{\mathbf{A}})\, e^{\,i(\mathbf{k}\cdot\mathbf{x}-\omega_{\mathbf{k}}t)} + \text{c.c.} \implies \hat{\mathbf{E}}(\mathbf{k}) = -ic\frac{\mathbf{k}}{|\mathbf{k}|}\mathbf{k}\cdot\hat{\mathbf{A}} + ic|\mathbf{k}|\hat{\mathbf{A}}$$

Let's define the perpendicular component (with respect to $\mathbf{k}$) of the potential vector : $\hat{\mathbf{A}}_\perp(\mathbf{k}) = \hat{\mathbf{A}} - \dfrac{\mathbf{k}\cdot\mathbf{k}\cdot\hat{\mathbf{A}}}{|\mathbf{k}|^2}$ such that we can write :

$$\hat{\mathbf{E}}(\mathbf{k}) = ic|\mathbf{k}|\hat{\mathbf{A}}_\perp(\mathbf{k})$$

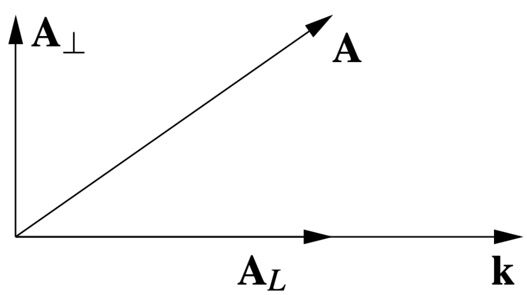


Figure 4.1: Decomposition of $\hat{\mathbf{A}}$ into 2 orthogonal components.

[1]One can remark that this ansatz works for a particular type of wave, which has only one Fourier component. The most general form would take an infinite sum over possible wave vectors $\sum_{\mathbf{k}}$.

Thus, the **E** field must be transverse to the wave vector **k**.
For the **B** field, we can similarly find :

$$\hat{\mathbf{B}} = i\mathbf{k} \times \hat{\mathbf{A}} = i\mathbf{k} \times \hat{\mathbf{A}}_\perp$$

So we found that the same goes for **B**, it is transverse to **k**. We can thus directly link **E** to **B** :

$$\mathbf{B} = \frac{1}{c}\mathbf{n} \times \mathbf{E}$$

with $\mathbf{n} = \mathbf{k}/|\mathbf{k}|$ the unit vector in the direction of propagation of the wave. We found that **E**, **B** and **k** are 3 orthogonal vectors.

4.4 Energy of Electromagnetic waves

Let's first compute the mean values of the fields :

$$\langle \mathbf{E}^2 \rangle = \left\langle \left(\hat{\mathbf{E}}(\mathbf{k})\, e^{\,i(\mathbf{k}\cdot\mathbf{x} - \omega_{\mathbf{k}} t)} + \hat{\mathbf{E}}^*(\mathbf{k})\, e^{\,-i(\mathbf{k}\cdot\mathbf{x} - \omega_{\mathbf{k}} t)} \right)^2 \right\rangle = 2\hat{\mathbf{E}} \cdot \hat{\mathbf{E}}^*$$
$$\langle \mathbf{B}^2 \rangle = 2\hat{\mathbf{B}} \cdot \hat{\mathbf{B}}^* = \frac{1}{c^2}\langle \mathbf{E}^2 \rangle$$

Where the mean value is defined as $\langle f(t) \rangle = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(t)dt$. This allows us to compute the average energy density :

$$\langle u \rangle = \frac{\varepsilon_0}{2}\left(\langle \mathbf{E}^2 \rangle + c^2 \langle \mathbf{B}^2 \rangle\right) = 2\varepsilon_0 \hat{\mathbf{E}} \cdot \hat{\mathbf{E}}^*$$

We can also compute the average energy flux defined by the Poynting vector :

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B} = \frac{1}{c\mu_0}\mathbf{E}\times(\mathbf{n} \times \mathbf{E}) = c\,\varepsilon_0 |\mathbf{E}|^2 \mathbf{n} = cu\mathbf{n}$$

Which yields :

$$\langle \mathbf{S} \rangle = c\langle u \rangle \mathbf{n} = 2\varepsilon_0 c |\hat{\mathbf{E}}|^2 \mathbf{n}$$

Note that in the norms, it does not matter if we use $\mathbf{E}$ or $\hat{\mathbf{E}}$ since the complex phase have norm 1.

Chapter 5

Liénard-Wiechert Potentials

Now that we established some fundamental formulas in the previous chapters, we can apply it to the special case of a moving particle of charge q along a known trajectory $\mathbf{x}_0(t)$. Let's write the charge and current density :

$$\rho(\mathbf{x}, t) = q\delta^3(\mathbf{x} - \mathbf{x}_0(t)) \tag{5.1}$$

$$\mathbf{J}(\mathbf{x}, t) = q\mathbf{v}(t)\delta^3(\mathbf{x} - \mathbf{x}_0(t)) \tag{5.2}$$

where $\mathbf{v}(t) = \dot{\mathbf{x}}_0(t)$. Let's check the charge conservation :

$$\dot{\rho} + \boldsymbol{\nabla} \cdot \mathbf{J} = -q\dot{\mathbf{x}}_0(t) \cdot \boldsymbol{\nabla}\delta^3(\mathbf{x} - \mathbf{x}_0(t)) + q\dot{\mathbf{x}}_0(t) \cdot \boldsymbol{\nabla}\delta^3(\mathbf{x} - \mathbf{x}_0(t)) = 0$$

We will first derive the fields Φ and $\mathbf{A}$ using our previously introduced methods. Since we are in the case of a moving charge, we need to use the retarded Green's function for the box operator $\Box$. Using our general formula derived in the Chapter 3, we have for the scalar potential :

$$\begin{aligned}
\Phi(\mathbf{x}) &= \frac{1}{\varepsilon_0}\int_V d^3x' dt'\, G_R(\mathbf{x}, t, \mathbf{x}', t')\rho(\mathbf{x}', t') \\
&= \frac{q}{4\pi\varepsilon_0}\int d^3x'\, dt'\, \frac{\delta(t - t' - |\mathbf{x} - \mathbf{x}'|/c)}{|\mathbf{x} - \mathbf{x}'|}\delta^3(\mathbf{x}' - \mathbf{x}_0(t')) \\
&= \frac{q}{4\pi\varepsilon_0}\int dt'\, \frac{\delta(t - t' - |\mathbf{x} - \mathbf{x}_0(t')|/c)}{|\mathbf{x} - \mathbf{x}_0(t')|},
\end{aligned}$$

Where we have used the expression of the retarded GF and integrated over $\mathbf{x}'$ first. A similar expression can be derived for $\mathbf{A}$.

A legitimate question is why using the retarded G.F. ? We want to consider only the field at point $(\mathbf{x}, t)$ created by the charge at point $(\mathbf{x}_0(t^*), t^*)$ in the past, where t^* denote the moment the field that will be seen at $(\mathbf{x}, t)$ is created by the moving charge (this consideration is due to the finite velocity of the fields propagation c). Indeed, we don't want to consider a field created in the "future", but only the past influence of the moving charge on the fields. This consideration is equivalent to saying that the Green's function is localized on the light cone of the moving charge, see Fig.5.1.

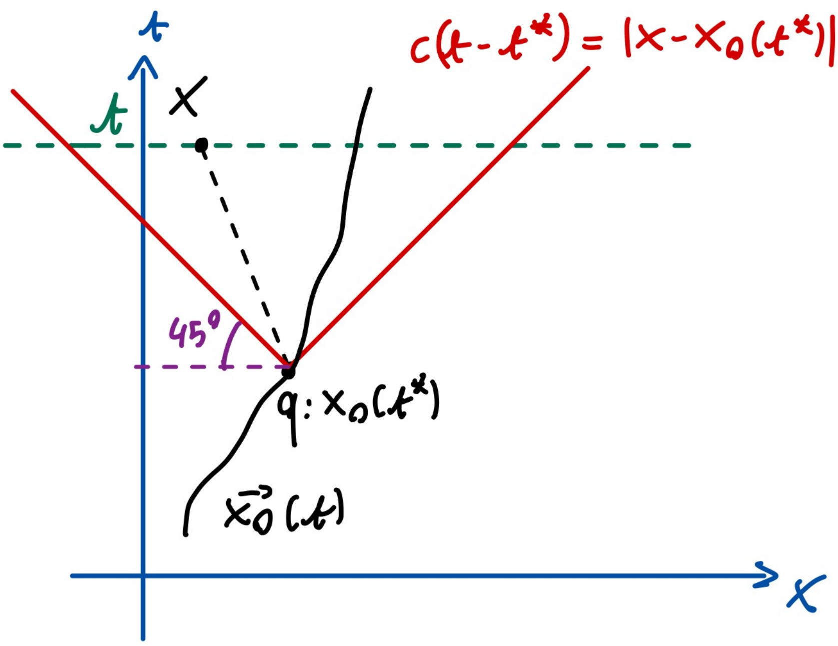


Figure 5.1: Trajectory of the moving charge in space-time. In red the light cone of the particle. Only points inside can be influenced by the moving charge. In green the position of the fields we want to determine. t^* is the moment the field at $(\mathbf{x}, t)$ is "created".

Now, introduce $\mathbf{R}(t') = \mathbf{x} - \mathbf{x}_0(t')$ and $\mathbf{n} = \mathbf{R}/R$. We can compute the time derivative of the norm :

$$\frac{\partial R}{\partial t'} = \partial_{t'}|\mathbf{x} - \mathbf{x}_0(t')| = -\mathbf{v}(t') \cdot \frac{\mathbf{x} - \mathbf{x}_0(t')}{|\mathbf{x} - \mathbf{x}_0(t')|} = -\mathbf{v}(t') \cdot \mathbf{n}$$

Back to the integral, let $f(t') = t - t' - R/c$ the argument of the Delta function. We need to find the zeros of f.
Let's compute f' :

$$\frac{df(t')}{dt'} = -1 - \frac{1}{c}\frac{\partial R}{\partial t'} = -1 + \frac{1}{c}\mathbf{v}(t') \cdot \mathbf{n} < 0$$

So f is a continuously decreasing function of time. This shows that there is no more than 1 solution t^* to the equation $f(t^*) = 0$ as long as $|\mathbf{v}| < c$, which is always the case .

We can now discuss the solutions in different regimes. At $t = t'$, one finds $f(t') = -R/c \leq 0$, with equality only if $R = 0$, i.e. if the source and the observation point coincide. For $t' \ll t$, one has $f(t') \to +\infty$. Since $f(t')$ is continuous in t', the intermediate value theorem ensures that there exists a unique retarded time t^* such that $f(t^*) = 0$. Geometrically, this corresponds to the intersection of the source worldline $(\mathbf{x}_0(t'), t')$ with the past light cone of the observation event $(\mathbf{x}, t)$.

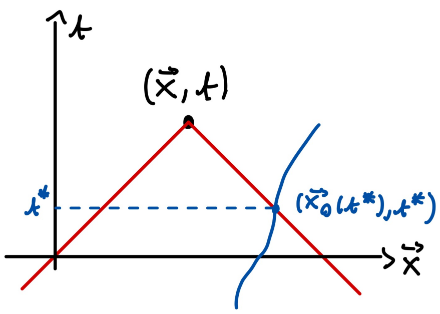


Figure 5.2: Light cone

Take the retarded time that satisfies $f(t') = 0$: $t^*(\mathbf{x}, t)$. We will not explicitly find the expression for this time but we will still be able to extract the fields. We will make use of the formula :

$$\int dz\ G(z)\delta(f(z)) = \frac{G(z^*)}{|f'(z^*)|}$$

Let's apply it to the expressions for the potential, i.e. $f(t')$ and $G(t') = 1/R(t')$, we get :

$$\Phi(\mathbf{x}, t) = \frac{q}{4\pi\varepsilon_0}\frac{1}{R(t^*)}\frac{1}{\Big(1 - \boldsymbol{\beta}(t^*)\cdot\mathbf{n}(t^*)\Big)} = \frac{q}{4\pi\varepsilon_0}\frac{1}{\Big(R - \mathbf{R}\cdot\boldsymbol{\beta}\Big)} \tag{5.3}$$

$$\mathbf{A}(\mathbf{x}, t) = \frac{q}{4\pi\varepsilon_0 c}\frac{1}{R(t^*)}\frac{\boldsymbol{\beta}(t^*)}{\Big(1 - \boldsymbol{\beta}(t^*)\cdot\mathbf{n}(t^*)\Big)} = \frac{q}{4\pi\varepsilon_0 c}\frac{\boldsymbol{\beta}}{\Big(R - \mathbf{R}\cdot\boldsymbol{\beta}\Big)} \tag{5.4}$$

where we define the velocity in units of c, $\boldsymbol{\beta} = \mathbf{v}/c$. We will now omit the explicit dependence on t^* in the expressions only to lighten the notation. We now have to compute the fields, given by $\mathbf{E} = -\boldsymbol{\nabla}\Phi - \partial_t\mathbf{A}$ and $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$. To do so, we will need to find the derivatives of t^*. First, take the time derivative ∂_t of $f(t^*) = 0$:

$$\partial_t\big(t - t^* - R/c\big) = 1 - \frac{\partial t^*}{\partial t} - \frac{1}{c}\frac{\partial R}{\partial t} = 1 - \frac{\partial t^*}{\partial t} - \frac{1}{c}\frac{\partial R}{\partial t^*}\frac{\partial t^*}{\partial t} = 0$$

But $\partial_{t^*}R = -c\boldsymbol{\beta}\cdot\mathbf{n}$, with the variables all evaluated at t^*. This gives :

$$\frac{\partial t^*}{\partial t} = \frac{1}{1 - \boldsymbol{\beta}\cdot\mathbf{n}}$$

Now for $\boldsymbol{\nabla}_{\mathbf{x}}t^*$, start by differentiating $f(t^*) = 0$ with respect to $\mathbf{x}$, and noting that $\nabla_{\mathbf{x}}t = 0$, we obtain :

$$\nabla t^* = -\frac{1}{c}\boldsymbol{\nabla}R(t^*)$$

The gradient of R has an explicit dependence on $\mathbf{x}$ and an implicit one through $t^*(\mathbf{x}, t)$:

$$\boldsymbol{\nabla}R(t^*) = \left.\frac{\partial R}{\partial\mathbf{x}}\right|_{t^*} + \frac{\partial R}{\partial t^*}\boldsymbol{\nabla}t^* = \frac{\mathbf{x} - \mathbf{x}_0(t^*)}{R(t^*)} - \mathbf{n}\cdot\mathbf{v}(t^*) = \mathbf{n} - (\mathbf{n}\cdot\mathbf{v})\,\nabla t^*$$

Hence :

$$\boldsymbol{\nabla}t^* = -\frac{1}{c}\Big[\mathbf{n} - (\mathbf{n}\cdot\mathbf{v})\,\nabla t^*\Big].$$

so that finally :

$$\boldsymbol{\nabla}t^*(\mathbf{x}, t) = -\frac{\mathbf{n}}{c}\frac{1}{1 - \mathbf{n}\cdot\boldsymbol{\beta}}$$

Let's now compute the derivatives of the potentials. First, let's compute the gradient of the scalar potential term. We have to compute the spatial gradient :

$$\boldsymbol{\nabla}\left[\frac{1}{R - \mathbf{R}\cdot\boldsymbol{\beta}}\right]$$

where we recall :

$$\mathbf{R} = \mathbf{x} - \mathbf{x}_0(t^*), \qquad R = |\mathbf{R}|, \qquad \mathbf{n} = \frac{\mathbf{R}}{R}, \qquad \boldsymbol{\beta} = \frac{\mathbf{v}(t^*)}{c}.$$

With all quantities evaluated at the retarded time $t^* = t^*(t, \mathbf{x})$.
Let's also recall the important derivatives :

$$\boldsymbol{\nabla} t^* = -\frac{\mathbf{n}}{c(1-\boldsymbol{\beta}\cdot\mathbf{n})} \quad , \quad \boldsymbol{\nabla} R = \frac{\mathbf{n}}{1-\boldsymbol{\beta}\cdot\mathbf{n}}$$

Now, let's compute:

$$\partial_i(\mathbf{R}\cdot\boldsymbol{\beta}) = \partial_i(R_j\beta_j).$$

Applying the product rule:

$$\partial_i(R_j\beta_j) = (\partial_i R_j)\beta_j + R_j(\partial_i\beta_j).$$

Since $R_j = x_j - x_{0j}(t^*)$, we have:

$$\partial_i R_j = \frac{\partial(x_j - x_{0j}(t^*))}{\partial x_i} = \delta_{ij} - \frac{\partial x_{0j}}{\partial t^*}\frac{\partial t^*}{\partial x_i}.$$

Because $\frac{\partial x_{0j}}{\partial t^*} = v_j$ and

$$\frac{\partial t^*}{\partial x_i} = -\frac{n_i}{c(1-\boldsymbol{\beta}\cdot\mathbf{n})},$$

we get:

$$\partial_i R_j = \delta_{ij} + \frac{\beta_j n_i}{1-\boldsymbol{\beta}\cdot\mathbf{n}}.$$

Next, since $\boldsymbol{\beta} = \boldsymbol{\beta}(t^*)$,

$$\partial_i\beta_j = \frac{d\beta_j}{dt^*}\frac{\partial t^*}{\partial x_i} = -\frac{\dot{\beta}_j n_i}{c(1-\boldsymbol{\beta}\cdot\mathbf{n})}.$$

Combining both results gives:

$$\partial_i(R_j\beta_j) = (\delta_{ij} + \frac{\beta_j n_i}{1-\boldsymbol{\beta}\cdot\mathbf{n}})\beta_j - R_j\frac{\dot{\beta}_j n_i}{c(1-\boldsymbol{\beta}\cdot\mathbf{n})}.$$

After summing over j, we obtain:

$$\partial_i(R_j\beta_j) = \beta_i + \frac{\beta^2 n_i}{1-\boldsymbol{\beta}\cdot\mathbf{n}} - \frac{(\mathbf{R}\cdot\dot{\boldsymbol{\beta}})\, n_i}{c(1-\boldsymbol{\beta}\cdot\mathbf{n})}.$$

Which gives in vectorial form :

$$\boldsymbol{\nabla}(\mathbf{R}\cdot\boldsymbol{\beta}) = \boldsymbol{\beta} + \frac{\mathbf{v}(\boldsymbol{\beta}\cdot\mathbf{n})}{c(1-\boldsymbol{\beta}\cdot\mathbf{n})} - \frac{R\dot{\boldsymbol{\beta}}}{c(1-\boldsymbol{\beta}\cdot\mathbf{n})}.$$

Furthermore, let $D = R - \mathbf{R}\cdot\boldsymbol{\beta}$, we have

$$\boldsymbol{\nabla} D = \boldsymbol{\nabla} R - \boldsymbol{\nabla}(\mathbf{R}\cdot\boldsymbol{\beta}) = \frac{\mathbf{n}}{1-\boldsymbol{\beta}\cdot\mathbf{n}} - \boldsymbol{\beta} - \frac{\mathbf{v}(\boldsymbol{\beta}\cdot\mathbf{n})}{c(1-\boldsymbol{\beta}\cdot\mathbf{n})} + \frac{R\dot{\boldsymbol{\beta}}}{c(1-\boldsymbol{\beta}\cdot\mathbf{n})}.$$

Finally,

$$\boldsymbol{\nabla}\left(\frac{1}{R-\mathbf{R}\cdot\boldsymbol{\beta}}\right)=-\frac{\boldsymbol{\nabla}D}{D^2}=-\frac{1}{(R-\mathbf{R}\cdot\boldsymbol{\beta})^2}\left[\frac{\mathbf{n}}{1-\boldsymbol{\beta}\cdot\mathbf{n}}-\boldsymbol{\beta}-\frac{\mathbf{v}(\boldsymbol{\beta}\cdot\mathbf{n})}{c(1-\boldsymbol{\beta}\cdot\mathbf{n})}+\frac{R\dot{\boldsymbol{\beta}}}{c(1-\boldsymbol{\beta}\cdot\mathbf{n})}\right].$$

The expression for the gradient of the scalar potential is then :

$$\begin{aligned}-\boldsymbol{\nabla}\Phi &= \frac{-q}{4\pi\varepsilon_0}\frac{-1}{|R-\boldsymbol{\beta}\cdot\mathbf{R}|^2}\left(\frac{\mathbf{n}}{1-\boldsymbol{\beta}\cdot\mathbf{n}}-\left(\boldsymbol{\beta}+\frac{\boldsymbol{\beta}^2\mathbf{n}}{1-\boldsymbol{\beta}\cdot\mathbf{n}}\right)+\boldsymbol{R}\cdot\dot{\boldsymbol{\beta}}\frac{\mathbf{n}}{c(1-\boldsymbol{\beta}\cdot\mathbf{n})}\right)\\ &= \frac{q}{4\pi\varepsilon_0}\frac{1}{R^2}\frac{1}{|1-\boldsymbol{\beta}\cdot\mathbf{n}|^3}\left(\mathbf{n}-\boldsymbol{\beta}(1-\boldsymbol{\beta}\cdot\mathbf{n})-\beta^2\mathbf{n}+\frac{\mathbf{R}\cdot\dot{\boldsymbol{\beta}}}{c}\mathbf{n}\right)\end{aligned}$$

Next, a similar procedure can lead to the time derivative of the vector potential :

$$-\frac{\partial\mathbf{A}}{\partial t}=\frac{q}{4\pi\varepsilon_0}\left[\frac{\boldsymbol{\beta}}{c}\frac{1}{R^2}\frac{1}{|1-\boldsymbol{\beta}\cdot\mathbf{n}|^3}\left(\beta^2c-\boldsymbol{\beta}\cdot\mathbf{n}c-\dot{\boldsymbol{\beta}}\cdot\mathbf{R}\right)-\frac{\dot{\boldsymbol{\beta}}}{c}\frac{1}{R}\frac{1}{|1-\boldsymbol{\beta}\cdot\mathbf{n}|^2}\right]$$

This allows to find, after rearranging some terms, the expression for the electric field :

$$\mathbf{E}=\frac{q}{4\pi\varepsilon_0}\frac{1}{R^2}\frac{1}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}\left((\mathbf{n}-\boldsymbol{\beta})\left(1-\beta^2\right)+\frac{R}{c}\mathbf{n}\times((\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}})\right)$$

Let's look deeper in this formula. Two terms can be extracted from it :

$$\mathbf{E}_{1/R^2}=\frac{q}{4\pi\varepsilon_0}\frac{1}{R^2}\frac{(\mathbf{n}-\boldsymbol{\beta})\left(1-\beta^2\right)}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3} \tag{5.5}$$

$$\mathbf{E}_{1/R}=\frac{q}{4\pi\varepsilon_0 c}\frac{1}{R}\frac{\mathbf{n}\times((\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}})}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3} \tag{5.6}$$

Eq.(5.5) is a Coulomb-like term, independent of $\dot{\boldsymbol{\beta}}$. It scales as $1/R^2$, meaning it reflects the near field of the moving particle, which falls off rapidly with distance. We can see that in the limit $|\boldsymbol{\beta}| \ll 1$, this term reduces to the familiar Coulomb field. This term is a generalization of the Coulomb field of a moving charge with constant velocity.

Eq.(5.6) scales as $1/R$ and arises from the acceleration of the particle $\dot{\boldsymbol{\beta}}$. It is responsible for the emission of **radiation**. It decays slower with distance than the Coulomb term, allowing to travels longer distance and is transverse to $\mathbf{n}$, which means it describes a propagating wave.
Can we now estimate the energy carried away with this radiation term ? First, we could derive analogously the magnetic field :

$$\mathbf{B}=\frac{1}{c}\mathbf{n}\times\mathbf{E}$$

This allows to find the energy flux[1] along $\mathbf{n}$:

$$\frac{d\mathcal{E}}{d\Omega dt} = \lim_{R\to\infty} R^2 \mathbf{S}\cdot\mathbf{n}$$

Using the definition of $\mathbf{S}$, we can show that $\mathbf{S}\cdot\mathbf{n} = \varepsilon_0 c E^2$. At large distances, $\mathbf{E} = \mathbf{E}_{1/R}$. Now let's evaluate in two regimes :

$$\frac{d\mathcal{E}}{d\Omega dt} = \frac{q^2}{16\pi^2\varepsilon_0 c}\frac{(\mathbf{n}\times((\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}))^2}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^6} \tag{5.7}$$

Non relativistic limit Take $|\boldsymbol{\beta}| \ll 1$ such that $1-\boldsymbol{\beta}\cdot\mathbf{n} \approx \mathbf{1}$, eq.(5.7) reduces to :

$$\frac{d\mathcal{E}}{d\Omega dt} = \frac{q^2}{16\pi^2\varepsilon_0 c}|\dot{\boldsymbol{\beta}}\times\mathbf{n}|^2$$

Let's now integrate over the complete solid angle[2] $d\Omega = \sin\theta\, d\theta\, d\varphi$ to have the total power radiated[3], defined as :

$$P = \frac{d\mathcal{E}}{dt} = \int d\Omega\, \frac{d\mathcal{E}}{d\Omega dt} = \oint d\boldsymbol{\sigma}\cdot\mathbf{S}$$

We parametrize θ as the angle between $\dot{\boldsymbol{\beta}}$ and $\mathbf{n}$, leading to the so-called Larmor formula :

$$P = \frac{q^2}{16\pi^2\varepsilon_0 c}\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta\, \sin\theta \sin^2\theta \cdot |\dot{\boldsymbol{\beta}}|^2 = \frac{q^2|\dot{\boldsymbol{\beta}}|^2}{6\pi\varepsilon_0 c}$$

Ultra relativistic limit Now consider $|\boldsymbol{\beta}| \approx 1$. We need to make a distinction between the lab frame and the charge frame. What we are interested in is the power radiated in it's frame. We want the energy radiated during it's history. Thus, we need to change from t to t'. We saw earlier that :

$$\frac{\partial t'}{\partial t} = \frac{1}{1-\boldsymbol{\beta}\cdot\mathbf{n}}$$

The energy flux in the moving charge frame is thus :

$$\frac{d\mathcal{E}}{d\Omega dt'} = \frac{d\mathcal{E}}{d\Omega dt}\frac{dt}{dt'} = \frac{q^2}{16\pi^2\varepsilon_0 c}\frac{(\mathbf{n}\times((\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}))^2}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^5}$$

We can't do much more to simplify this expression. Let's apply it to a real case of a rectilinear motion $\boldsymbol{\beta} \parallel \dot{\boldsymbol{\beta}}$. The radiative part of the electric field is given by :

$$\begin{aligned}
\mathbf{E}_{\text{rad}} &= \frac{q}{4\pi\varepsilon_0}\frac{1}{Rc}\frac{1}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}}) \\
&= \frac{q}{4\pi\varepsilon_0 c}\frac{1}{R}\frac{1}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}((\mathbf{n}\cdot\dot{\boldsymbol{\beta}})\mathbf{n}-\dot{\boldsymbol{\beta}}) \\
&= \frac{q}{4\pi\varepsilon_0 c}\frac{1}{R}\frac{1}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}\left(-\dot{\boldsymbol{\beta}}_\perp\right)
\end{aligned}$$

[1] Also analog to the amount of energy emitted in a solid angle $d\Omega$ during a time dt.

[2] Same as integrating over a closed sphere of radius R

[3] If $\mathbf{S}$ is time dependent, we first need to time average before computing the power : $P = \oint d\boldsymbol{\sigma}\cdot\langle\mathbf{S}\rangle_t$

The power per unit of solid angle is thus :

$$\frac{dP'}{d\Omega} = \frac{d\mathcal{E}}{d\Omega dt'} = \frac{q^2|\dot{\boldsymbol{\beta}}|^2}{16\pi^2\varepsilon_0 c}\frac{\sin^2(\theta)}{(1-|\boldsymbol{\beta}|\cos(\theta))^5}$$

We see that taking the non-relativistic limit gives back the Larmor formula. In the ultra relativistic limit, we see that the power is mostly directed in the direction $\theta \sim 0$ since $1-\beta\cos\theta \to 0$ (see next paragraph). Let's now compute the total power radiated :

$$P' = \frac{d\mathcal{E}}{dt'} = \frac{q^2\dot{\beta}^2}{8\pi\varepsilon_0 c}\int_0^\pi d\theta \, \frac{\sin^3(\theta)}{(1-\beta\cos(\theta))^5}$$

Let $x = \cos\theta$, $dx = \sin\theta\, d\theta$, $-1 < x < 1$, which gives :

$$P' = \frac{q^2\dot{\beta}^2}{8\pi\varepsilon_0 c}\int_{-1}^1 dx\, \frac{1-x^2}{(1-\beta x)^5} = \frac{q^2\dot{\beta}^2}{8\pi\varepsilon_0 c}\cdot\frac{4}{3}\frac{1}{(1-\beta^2)^3}$$

Defining the Lorentz factor $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$, we find :

$$P' = \frac{q^2\dot{\beta}^2\gamma^6}{6\pi\varepsilon_0 c}$$

This is the relativistic generalization of the Larmor formula.

Angular distribution of the radiation To conclude this chapter, we will elaborate on the remark made previously on the concentration of radiation around $\theta \sim 0$. Going back to the power per unit of sold angle, we find by expanding around $\theta \sim 0$:

$$\frac{dP'}{d\Omega} = \frac{q^2\dot{\beta}^2}{16\pi^2\varepsilon_0 c}\frac{\sin^2(\theta)}{(1-\beta\cos(\theta))^5} \approx \frac{q^2\dot{\beta}^2}{16\pi^2\varepsilon_0 c}\frac{\theta^2}{(1-\beta(1-\theta^2/2))^5}$$

Using that $\gamma^2 = \dfrac{1}{(1-\beta)(1+\beta)} \approx \dfrac{1}{2(1-\beta)}$ when $\beta \approx 1$, we get :

$$\frac{dP'}{d\Omega} \approx \frac{q^2\dot{\beta}^2}{16\pi^2\varepsilon_0 c}\frac{\theta^2}{\left(\frac{1}{2\gamma^2}+\frac{\theta^2}{2}\right)^5} = \frac{q^2\dot{\beta}^2}{16\pi^2\varepsilon_0 c}\frac{2^5\theta^2\gamma^{10}}{\left(1+\gamma^2\theta^2\right)^5}$$

This expression is maximal when $\theta^* = \pm\dfrac{1}{2\gamma}$, meaning that all radiation is concentrated on a cone of aperture θ^* in front of the moving charge. A plot of the angular distribution of the power as a function of the angle is shown in Fig.5.3.

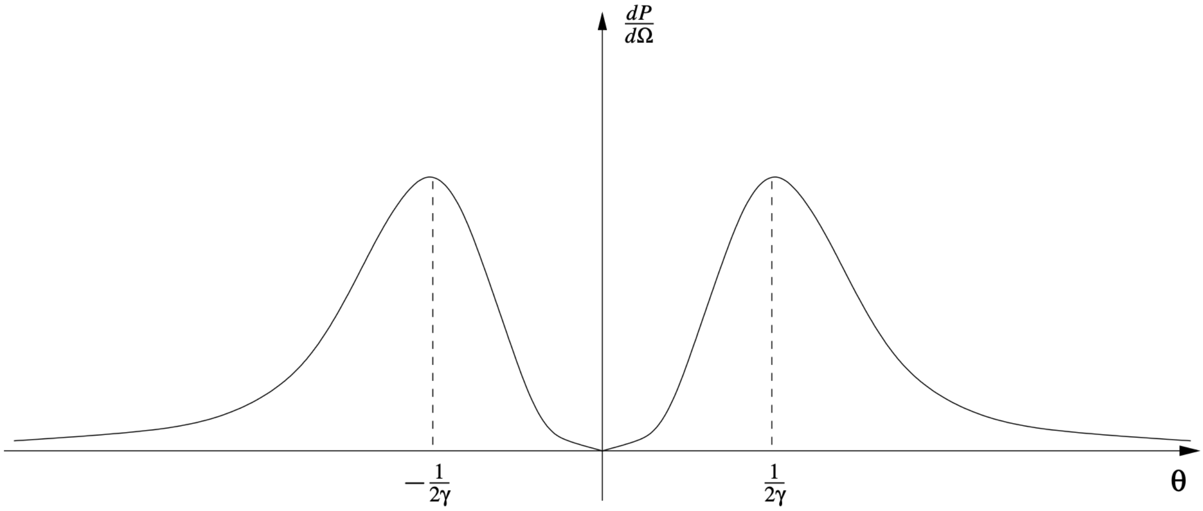


Figure 5.3: Angular distribution of the radiative power per unit of solid angle.

Chapter 6

Multipole Expansion

Let's move on to one of the most well known approximation technique in classical electrodynamics. The idea is that very far away from the source, the latter can be treated as a point-like object, leading to very common behavior, even for very complex systems. First, let's make a reminder of the Taylor expansion for functions of many variables. In this chapter, we will use **Einstein summation convention**.

6.1 Taylor formula for multi-variable functions

First, recall the usual Taylor expansion of a function $f(x)$:

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \frac{d^n f}{dx^n}\bigg|_{x=0}$$

Now, let f be a function of 2 variables $f(x, y)$. In a similar way :

$$f(x,y) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{x^n}{n!} \frac{y^k}{k!} \, \partial_x^n \partial_y^k f \big|_{x,y=0} .$$

This double sum runs over all pairs of indices $(n, k) \in \mathbb{N}^2$:

k	$\cdots$			
3	$\bullet$	$\bullet$	$\bullet$	$\cdots$
2	$\bullet$	$\bullet$	$\bullet$	$\cdots$
1	$\bullet$	$\bullet$	$\bullet$	$\cdots$
0	$\bullet$	$\bullet$	$\bullet$	$\cdots$
	0	1	2	n

Each point (n, k) corresponds to a term $\frac{x^n}{n!} \frac{y^k}{k!} \, \partial_x^n \partial_y^k f \big|_0$.

We can group the terms **along diagonals** corresponding to the same total order of differentiation,

$$m = n + k.$$

On each diagonal, $n = m - k$, and since $n \geq 0$, we must have $k \leq m$. Hence, the double sum can be rewritten as

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_{n,k} \;=\; \sum_{m=0}^{\infty}\sum_{k=0}^{m} a_{m-k,k},$$

which means that we now sum *diagonal by diagonal* instead of line by line.
Applying this change of indices to our function f gives:

$$f(x,y) = \sum_{m=0}^{\infty}\sum_{k=0}^{m} \frac{x^{m-k}}{(m-k)!}\frac{y^k}{k!}\,\partial_x^{m-k}\partial_y^k f\big|_{x,y=0}\,.$$

where $x_1 = x$ and $x_2 = y$.
Applying the change of indices to the double Taylor expansion, we have

$$f(x,y) = \sum_{m=0}^{\infty}\sum_{k=0}^{m} \frac{x^{m-k}}{(m-k)!}\frac{y^k}{k!}\,\partial_x^{m-k}\partial_y^k f\big|_{x,y=0}\,.$$

Here m represents the total order of differentiation (and of the polynomial term). For each fixed m, the inner sum over k collects all the derivatives of total order m, containing $m-k$ derivatives with respect to x and k derivatives with respect to y.
To understand how this can be written in a more symmetric way, note that a term of order m with k derivatives with respect to y corresponds to all possible sequences of m derivative operators, each being either ∂_x or ∂_y. There are exactly

$$\binom{m}{k} = \frac{m!}{k!(m-k)!}.$$

such sequences, corresponding to the number of ways to choose which k of the m positions are associated with ∂_y (and the remaining $m-k$ with ∂_x).
Therefore, instead of summing over k and introducing factorial factors $(m-k)!k!$ explicitly, we can equivalently sum over all sequences of m indices $(i_1, i_2, \ldots, i_m)$, each index taking one of the two possible values:

$$i_j \in \{1,2\}, \qquad x_1 = x, \quad x_2 = y.$$

Each sequence $(i_1, i_2, \ldots, i_m)$ represents one possible ordering of partial derivatives, and all such orderings contribute equally because partial derivatives commute. Hence, we divide by $m!$ to compensate for the overcounting of identical terms.
With this in mind, both the powers of x and the derivatives can be expressed as sums over these sequences:

$$\frac{x^{m-k}}{(m-k)!}\frac{y^k}{k!}\,\partial_x^{m-k}\partial_y^k \quad\longleftrightarrow\quad \frac{1}{m!}\sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\cdots\sum_{i_m=1}^{2} x_{i_1}x_{i_2}\ldots x_{i_m}\,\partial_{x_{i_1}}\partial_{x_{i_2}}\ldots\partial_{x_{i_m}}.$$

Finally, substituting this correspondence into the expression for $f(x,y)$ gives the compact and fully symmetric form:

$$f(x,y) = \sum_{m=0}^{\infty}\sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\cdots\sum_{i_m=1}^{2} \frac{x_{i_1}x_{i_2}\ldots x_{i_m}}{m!}\,\partial_{x_{i_1}}\partial_{x_{i_2}}\ldots\partial_{x_{i_m}} f\big|_{x,y=0}\,.$$

This form emphasizes the symmetry between the variables x and y and makes the connection with the general d-dimensional Taylor expansion in multi-index notation more transparent :

$$f(\mathbf{x}) = \sum_{m=0}^{\infty} \frac{x_{i_1} x_{i_2} \dots x_{i_m}}{m!} \, \partial_{x_{i_1}} \partial_{x_{i_2}} \dots \partial_{x_{i_m}} f \Big|_{\mathbf{x}=0} .$$

where the summation indices runs over all variables $i_j = 1, ..., d$. We can remark that two rank m tensors appears :

$$V_{i_1, i_2 \dots i_m} = x_{i_1} x_{i_2} \dots x_{i_m} \quad U_{i_1, i_2 \dots i_m} = \partial_{x_{i_1}} \partial_{x_{i_2}} \dots \partial_{x_{i_n}} f \Big|_{\mathbf{x}=0} .$$

6.2 Multipole expansion in electrostatics

Let's consider the case in Fig.6.1. We are interested in a region of space far from the source. The latter is in general a localized system of charge, with density $\rho(\mathbf{x}')$. What does the potential or electric field at $\mathbf{x}$ which satisfies $|\mathbf{x}| \gg |\mathbf{x}'|$ looks like ?

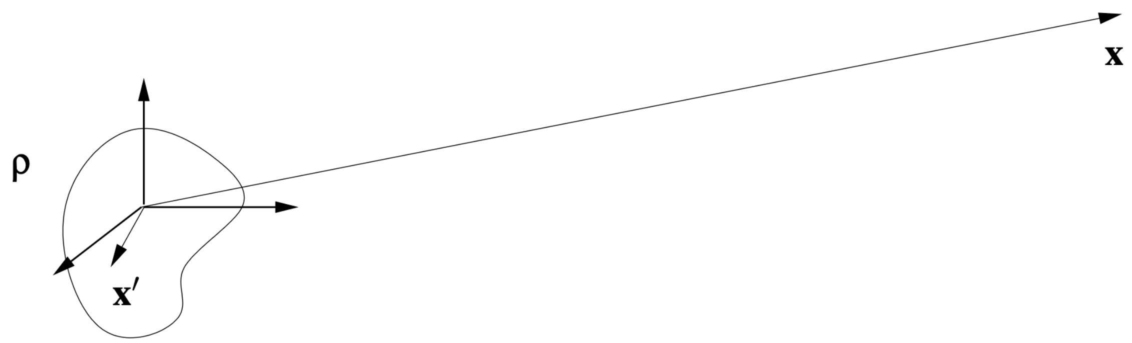


Figure 6.1: A localized system of charge far from the region of interest.

At leading order, we could only consider the total charge $Q = \int d^3x' \rho(\mathbf{x}')$ of the system to be relevant. The multipole expansion will make this intuition systematic at any order of precision wanted. To do that, let's Taylor expand our Green's function $\frac{1}{|\mathbf{x} - \mathbf{x}'|}$ around $\mathbf{x}' = 0$:

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \sum_{n=0}^{\infty} \frac{1}{n!} x'_{i_1} x'_{i_2} \dots x'_{i_n} \, \partial_{x'_{i_1}} \partial_{x'_{i_2}} \dots \partial_{x'_{i_n}} \frac{1}{|\mathbf{x} - \mathbf{x}'|} \Big|_{\mathbf{x}'=0} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{x'_{i_1} x'_{i_2} \dots x'_{i_n}}{|\mathbf{x}|^{2n+1}} T_{i_1 i_2 \dots i_n}(\mathbf{x})$$

Where we defined the tensor T as :

$$T_{i_1 i_2 \dots i_n}(\mathbf{x}) = |\mathbf{x}|^{2n+1} \, \partial'_{i_1} \partial'_{i_2} \dots \partial'_{i_n} \frac{1}{|\mathbf{x} - \mathbf{x}'|} \Big|_{\mathbf{x}'=0}$$

with the more compact notation $\partial'_{i_j} = \partial_{x'_{i_j}}$, and the indices runs from 1 to 3 here. Let's look at a few terms :

$$T_i(\mathbf{x}) = |\mathbf{x}|^3 \partial'_i \frac{1}{|\mathbf{x}-\mathbf{x}'|} = |\mathbf{x}|^3 \frac{x_i - x'_i}{|\mathbf{x}-\mathbf{x}'|^3} \xrightarrow{\mathbf{x}'=0} x_i$$

$$\begin{aligned} T_{ij}(\mathbf{x}) &= |\mathbf{x}|^5 \partial'_i \partial'_j \frac{1}{|\mathbf{x}-\mathbf{x}'|} \\ &= |\mathbf{x}|^5 \partial'_i \left(\frac{x_j - x'_j}{|\mathbf{x}-\mathbf{x}'|^3} \right) \\ &= |\mathbf{x}|^5 \frac{3(x_i - x'_i)(x_j - x'_j) - \delta_{ij} |\mathbf{x}-\mathbf{x}'|^2}{|\mathbf{x}-\mathbf{x}'|^5} \\ &\xrightarrow{\mathbf{x}'=0} 3x_i x_j - \delta_{ij} |\mathbf{x}|^2 \end{aligned}$$

We can find recursively a general form of this tensor :

$$T_{i_1 i_2 \dots i_n}(\mathbf{x}) = x_{i_1} \dots x_{i_n} (2n-1)!! - A_{i_1 i_2 \dots i_n}(\mathbf{x})$$

Where the n-tensor A contains Kronecker deltas and the double factorial defined as :

$$n!! = \begin{cases} n(n-2)(n-4) \cdot \ldots \cdot 5 \cdot 3 \cdot 1 & \text{if } n \text{ odd} \\ n(n-2)(n-4) \cdot \ldots \cdot 6 \cdot 4 \cdot 2 & \text{if } n \text{ even} \\ 1 & \text{if } n = -1, 0 \end{cases}$$

Let's focus now on some important properties of the tensor $T_{i_1 i_2 \dots i_n}(\mathbf{x})$.

T is symmetric The order in which we take the derivatives doesn't change so we have $T_{..i...j..} = T_{..j...i..}$.

T is traceless Indeed, let's compute the trace :

$$\mathrm{Tr}(T) = T_{i\alpha jk\dots}\delta_{i\alpha} = T_{iijk\dots} = |\mathbf{x}|^{2n+1} \underbrace{\partial'_i \partial'_i}_{=\nabla^2_{\mathbf{x}'}} \partial'_j \partial'_k \dots \frac{1}{|\mathbf{x}-\mathbf{x}'|} \Bigg|_{\mathbf{x}'=0} = |\mathbf{x}|^{2n+1} \partial'_j \partial'_k \dots \delta^3(\mathbf{x}-\mathbf{x}')$$

Where we commuted the nabla operator with all the other derivatives to obtain the differential equation satisfied by the Green's function. Since $|\mathbf{x}| \gg |\mathbf{x}'|$, the delta will never be satisfied and give always 0.

One last property is the following :

$$x'_{i_1} x'_{i_2} \dots x'_{i_n} T_{i_1 i_2 \dots i_n}(\mathbf{x}) = x_{i_1} x_{i_2} \dots x_{i_n} T_{i_1 i_2 \dots i_n}(\mathbf{x}')$$

To show this relation, first convince yourself that :

$$T_{i_1 i_2 \dots i_n}(\mathbf{x}) A_{i_1 i_2 \dots i_n}(\mathbf{x}') = 0$$

This is due to the fact that $A_{i_1 i_2 \dots i_n}(\mathbf{x}')$ contains only terms of the form $\delta_{i_j i_k}$ which will make appear the trace of T, thus giving 0. Then, we can write :

$$x'_{i_1} x'_{i_2} \dots x'_{i_n} T_{i_1 i_2 \dots i_n}(\mathbf{x}) = \left(x'_{i_1} x'_{i_2} \dots x'_{i_n} - \frac{A_{i_1 i_2 \dots i_n}(\mathbf{x}')}{(2n-1)!!} \right) T_{i_1 i_2 \dots i_n}(\mathbf{x})$$

But since $T_{i_1 i_2 \dots i_n}(\mathbf{x}') = x'_{i_1} \dots x'_{i_n}(2n-1)!! - A_{i_1 i_2 \dots i_n}(\mathbf{x}')$, we have :

$$\begin{aligned} x'_{i_1} x'_{i_2} \dots x'_{i_n} T_{i_1 i_2 \dots i_n}(\mathbf{x}) &= \frac{T_{i_1 i_2 \dots i_n}(\mathbf{x}')}{(2n-1)!!} T_{i_1 i_2 \dots i_n}(\mathbf{x}) \\ &= \frac{T_{i_1 i_2 \dots i_n}(\mathbf{x}')}{(2n-1)!!} \left(x_{i_1} \dots x_{i_n}(2n-1)!! - A_{i_1 i_2 \dots i_n}(\mathbf{x}) \right) \\ &= x_{i_1} x_{i_2} \dots x_{i_n} T_{i_1 i_2 \dots i_n}(\mathbf{x}') \end{aligned}$$

We can now proceed to Taylor expand the general form of the scalar potential :

$$\begin{aligned} \Phi(\mathbf{x}) &= \frac{1}{4\pi\varepsilon_0} \int d^3x' \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \\ &= \frac{1}{4\pi\varepsilon_0} \int d^3x' \rho(\mathbf{x}') \sum_{n=0}^{\infty} \frac{1}{n!} \frac{x'_{i_1} x'_{i_2} \dots x'_{i_n}}{|\mathbf{x}|^{2n+1}} T_{i_1 i_2 \dots i_n}(\mathbf{x}) \\ &= \frac{1}{4\pi\varepsilon_0} \sum_{n=0}^{\infty} \frac{1}{n!} \frac{x_{i_1} x_{i_2} \dots x_{i_n}}{|\mathbf{x}|^{2n+1}} \int d^3x' \rho(\mathbf{x}') T_{i_1 i_2 \dots i_n}(\mathbf{x}') \end{aligned}$$

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \sum_{n=0}^{\infty} \frac{1}{n!} \frac{x_{i_1} x_{i_2} \dots x_{i_n}}{|\mathbf{x}|^{2n+1}} Q_{i_1 i_2 \dots i_n} = \sum_{n=0}^{\infty} \Phi^{(n)}(\mathbf{x})$$

Where we defined the 2^n multipole moment :

$$Q_{i_1 i_2 \dots i_n} = \int d^3x' \rho(\mathbf{x}') T_{i_1 i_2 \dots i_n}(\mathbf{x}')$$

We can write a few of them :
$n = 0$: monopole moment $Q = \int d^3x' \rho(\mathbf{x}')$
$n = 1$: dipole moment $Q_i = \int d^3x' \rho(\mathbf{x}') x'_i$
$n = 2$: quadrupole moment $Q_{ij} = \int d^3x' \rho(\mathbf{x}') (3x'_i x'_j - |\mathbf{x}'|^2 \delta_{ij})$
$n = 3$: octopole moment $Q_{ijk} = \int d^3x' \rho(\mathbf{x}') (15x'_i x'_j x'_k - 3|\mathbf{x}'|^2 [\delta_{ij} x'_k + \delta_{ik} x'_j + \delta_{jk} x'_i])$

Let's make a few remarks on this central formula.

- We can see that the $n = 0$ term is $\Phi^{(0)}(\mathbf{x}) \sim Q/|\mathbf{x}|$, which is expected from a point charge (monopole term).

- Let a denote the typical size of the charge distribution ρ, i.e. $|\mathbf{x}'| \sim a$, $\rho \sim a^{-3}$. We can see that the n-th moment goes as a^n, meaning that the potential goes as :

$$\Phi(\mathbf{x}) \sim \frac{Q}{\varepsilon_0} \sum_{n=0}^{\infty} \frac{1}{x} \left(\frac{a}{x}\right)^n$$

If the total charge is $Q = 0$, the leading behavior will be the one of a electric dipole. If the dipole moment $\mathbf{d} = (Q_x, Q_y, Q_z) = 0$, the leading contribution will be the one of a quadrupole and so on.

Let's find the corresponding electric fields :
0-th order, monopole q :

$$\Phi^{(0)}(\mathbf{x}) = \frac{q}{4\pi\varepsilon_0 |\mathbf{x}|} \implies \mathbf{E}^{(0)}(\mathbf{x}) = \frac{q}{4\pi\varepsilon_0} \frac{\mathbf{x}}{|\mathbf{x}|^3}$$

1-st order, dipole moment $\mathbf{Q}^{(1)}$:

$$\Phi^{(1)}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \frac{\mathbf{Q}^{(1)} \cdot \mathbf{x}}{|\mathbf{x}|^3} \implies \mathbf{E}^{(1)}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \left[-\frac{\mathbf{Q}^{(1)}}{|\mathbf{x}|^3} + \frac{3(\mathbf{Q}^{(1)} \cdot \mathbf{x})\mathbf{x}}{|\mathbf{x}|^5} \right]$$

6.3 Multipole expansion in magnetostatics (leading non trivial order)

To study the magnetic multipoles, we can proceed in a similar way since the expression of the potential vector use the same Green's function. In this section, we will restrain ourselves to the leading orders :

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int d^3x' \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}$$

So in terms of components :

$$A_i(\mathbf{x}) = \frac{\mu_0}{4\pi} \frac{1}{|\mathbf{x}|} \int d^3x' J_i(\mathbf{x}') + \frac{\mu_0}{4\pi} \frac{x_j}{|\mathbf{x}|^3} \int d^3x' J_i(\mathbf{x}') x'_j + \mathcal{O}(|\mathbf{x}|^{-3})$$

Recall the charge conservation $\boldsymbol{\nabla} \cdot \mathbf{J} + \dot{\rho} = 0 \implies \partial_i J_i = 0$, so we have :

$$0 = \int d^3x \; x_i \partial_j J_j = \int d^3x \; \partial_j (x_i J_j) - \int d^3x \; J_j \underbrace{\partial_j x_i}_{\delta_{ij}} = \underbrace{\oint d\sigma_j \; x_i J_j}_{=0} - \int d^3x \; J_i$$

Where we used Gauss's theorem and the fact that $\mathbf{J}$ vanishes far from the source (localized charges). We just showed that no magnetic monopole exists. In a similar way, we can show that :

$$\int d^3x \; (x_i J_j + x_j J_i) = 0$$

Let's now compute the dipole term :

$$\begin{aligned}
x_j \int d^3x' J_i(\mathbf{x}') x'_j &= \frac{1}{2}\int d^3x' \left(J_i(\mathbf{x}') x'_j x_j - x_j J_j(\mathbf{x}') x'_i\right) \\
&= \frac{1}{2}\int d^3x' J_i(\mathbf{x}\cdot\mathbf{x}') - x'_i(\mathbf{J}\cdot\mathbf{x}) \\
&= \frac{1}{2}\left(\int d^3x' \mathbf{J}(\mathbf{x}\cdot\mathbf{x}') - \mathbf{x}'(\mathbf{J}\cdot\mathbf{x})\right)_i \\
&= \frac{1}{2}\left(\int d^3\, x'\mathbf{x}\times(\mathbf{x}'\times\mathbf{J})\right)_i \\
&= \frac{1}{2}\left(\mathbf{x}\times\int d^3x'\mathbf{x}'\times\mathbf{J}(\mathbf{x}')\right)_i
\end{aligned}$$

We can define two quantities. First, the magnetization or magnetic moment density $\mathbf{M}$:

$$\mathbf{M}(\mathbf{x}') = \frac{1}{2}\mathbf{x}'\times\mathbf{J}(\mathbf{x}')$$

Secondly, the magnetic moment $\mathbf{m}$:

$$\mathbf{m} = \frac{1}{2}\int d^3x'\mathbf{x}'\times\mathbf{J}(\mathbf{x}') = \int d^3x'\,\mathbf{M}(\mathbf{x}')$$

This allows us to write the vector potential at leading order in the compact form :

$$\mathbf{A}^{(1)}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{x}}{|\mathbf{x}|^3}$$

In analogy to the electrostatic term, the next orders will yield quadrupole, octopole behavior. The leading magnetic field term is given by :

$$\mathbf{B}^{(1)}(\mathbf{x}) = \boldsymbol{\nabla}\times\mathbf{A}^{(1)} = \frac{\mu_0}{4\pi}\left[-\frac{\mathbf{m}}{|\mathbf{x}|^3} + \frac{3(\mathbf{m}\cdot\mathbf{x})\mathbf{x}}{|\mathbf{x}|^5}\right]$$

Which is very similar to the form found for the dipole term of the electric field.

6.4 Multipole expansion for radiation

Let us now consider a localized distribution of charges forming a cloud of characteristic size a, with typical charge velocities v. We can associate with the internal dynamics of this system a characteristic timescale $T = a/v$. The radiation emitted by the moving charges will then have a typical frequency $\omega \sim 1/T$, corresponding to a wavelength $\lambda = c/\omega \sim (c/v)\,a$.
In the non-relativistic regime, where $v \ll c$, we therefore obtain the condition $\lambda \gg a$: the wavelength of the emitted radiation is much larger than the size of the source. This separation of scales justifies the use of a multipole expansion to describe the radiation field. Since we are interested in the fields far from the charges, we also consider $|\mathbf{x}| \gg a$. We can extract 2 different regimes :

1. The radiation region, where $a \ll \lambda \ll |\mathbf{x}|$. The region where we are very far away from the source.

2. the quasi-static region, where $a \ll |\mathbf{x}| \ll \lambda$. The region where the fields vary very slowly with time.

A summary of the system is shown on Fig.6.2

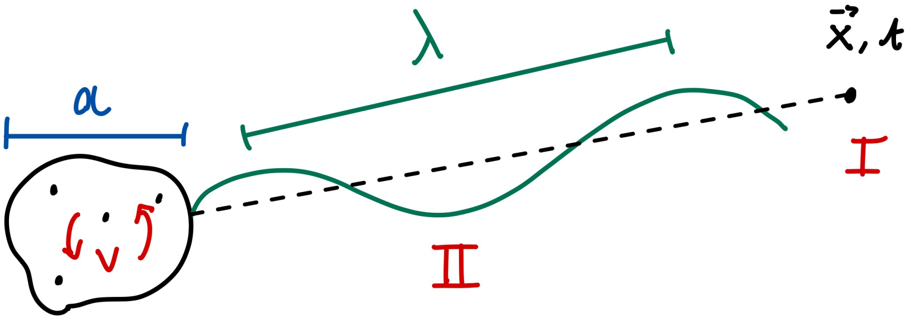


Figure 6.2: Study case. The two regions are shown.

6.4.1 Radiation region

We first start by the study of the radiation region : $a \ll \lambda \ll |\mathbf{x}|$. It will be enough to study the vector potential. Recall the general formula for the vector potential in the dynamical case :

$$\mathbf{A}(\mathbf{x}, t) = \frac{\mu_0}{4\pi} \int d^3x' \frac{\mathbf{J}\left(\mathbf{x}', t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)}{|\mathbf{x} - \mathbf{x}'|}$$

We would like to Taylor expand this expression in terms of :

$$\frac{|\mathbf{x}'|}{|\mathbf{x}|} \sim \frac{a}{|\mathbf{x}|} \ll 1 \quad \text{and} \quad \frac{|\mathbf{x}'|}{\lambda} \sim \frac{a}{\lambda} \ll 1$$

To do so, we make use of the first terms in the expansion of the Green's function :

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \frac{1}{|\mathbf{x}|} + \frac{x_i x_i'}{|\mathbf{x}|^3} + \frac{1}{|\mathbf{x}|}\mathcal{O}\left(\frac{|\mathbf{x}'|^2}{|\mathbf{x}|^2}\right) = \frac{1}{|\mathbf{x}|} + \frac{\mathbf{x} \cdot \mathbf{x}'}{|\mathbf{x}|^3} + \frac{1}{|\mathbf{x}|}\mathcal{O}\left(\frac{|\mathbf{x}'|^2}{|\mathbf{x}|^2}\right)$$

We can also expand the argument of the current density :

$$\frac{|\mathbf{x} - \mathbf{x}'|}{c} = \frac{|\mathbf{x}|}{c} - \frac{\mathbf{x} \cdot \mathbf{x}'}{c|\mathbf{x}|} + \mathcal{O}\left(\frac{1}{|\mathbf{x}|}\right) = \frac{|\mathbf{x}|}{c} - \frac{\mathbf{n} \cdot \mathbf{x}'}{c} + \mathcal{O}\left(\frac{1}{|\mathbf{x}|}\right)$$

Where we used that $\frac{|\mathbf{x}-\mathbf{x}'|}{c} \sim \frac{|\mathbf{x}-\mathbf{x}'|}{\lambda}$ to expand in terms of $|\mathbf{x}'|/c$. At first order in $1/|\mathbf{x}|$, we find the radiation term of the vector potential :

$$\mathbf{A}(\mathbf{x}, t) = \frac{\mu_0}{4\pi} \frac{1}{|\mathbf{x}|} \int d^3x' \mathbf{J}\left(\mathbf{x}', t - \frac{|\mathbf{x}|}{c} + \frac{\mathbf{n} \cdot \mathbf{x}'}{c}\right) \left[1 + \mathcal{O}\left(\frac{a}{|\mathbf{x}|}\right)\right]$$

Expanding in terms of $\mathbf{n}\cdot\mathbf{x}'/c \sim a/\lambda$ the current density yields the multipole expansion of the radiation :

$$\mathbf{A}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\frac{1}{|\mathbf{x}|}\sum_{n=0}^{\infty}\frac{1}{n!}\int d^3x' \left(\frac{\mathbf{n}\cdot\mathbf{x}'}{c}\right)^n \partial_t^n \mathbf{J}\left(\mathbf{x}', t-\frac{|\mathbf{x}|}{c}\right) = \frac{\mu_0}{4\pi}\frac{1}{|\mathbf{x}|}\sum_{n=0}^{\infty}\frac{1}{n!}\int d^3x' \left(\frac{\mathbf{n}\cdot\mathbf{x}'}{c}\right)^n \mathbf{J}^{(n)}$$

We have that $\partial_t \mathbf{J} \sim 1/T \sim \omega$, thus :

$$\int d^3x' \left(\frac{\omega|\mathbf{x}'|}{c}\right)^n \sim \left(\frac{\omega a}{c}\right)^n \sim \left(\frac{a}{\lambda}\right)^n$$

Now, let's concentrate on the fields. We will consider the leading order in the a/λ expansion of $\mathbf{A}$:

$$\mathbf{A}^{(0)}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\frac{1}{|\mathbf{x}|}\int d^3x' \mathbf{J}\left(\mathbf{x}', t-\frac{|\mathbf{x}|}{c}\right)$$

Let's find the $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$ field. First, notice that the $\boldsymbol{\nabla}$ operator will be applied once on the $1/|\mathbf{x}|$ term, giving a $1/|\mathbf{x}|^2$ term, and once on the $\mathbf{J}$ term. The leading term is thus :

$$\begin{aligned}
\mathbf{B}(\mathbf{x},t) &= \frac{\mu_0}{4\pi}\frac{1}{|\mathbf{x}|}\int d^3x' \boldsymbol{\nabla}_{\mathbf{x}}\times\mathbf{J}\left(\mathbf{x}', t-\frac{|\mathbf{x}|}{c}\right)\\
&= \frac{\mu_0}{4\pi}\frac{1}{|\mathbf{x}|}\int d^3x' -\frac{\mathbf{n}}{c}\times\dot{\mathbf{J}}\left(\mathbf{x}', t-\frac{|\mathbf{x}|}{c}\right)\\
&= -\frac{\mu_0}{4\pi|\mathbf{x}|c}\mathbf{n}\times\int d^3x' \dot{\mathbf{J}}\left(\mathbf{x}', t-\frac{|\mathbf{x}|}{c}\right)\\
&= \frac{1}{c}\dot{\mathbf{A}}\times\mathbf{n}
\end{aligned}$$

This shows that at $|\mathbf{x}| \gg a$, the radiation is a plane wave in the direction of $\mathbf{n}$ (away from the source) such that :

$$\mathbf{B} = \frac{1}{c}\mathbf{n}\times\mathbf{E} \implies \mathbf{E} = -c\mathbf{n}\times\mathbf{B} = \mathbf{n}\times(\mathbf{n}\times\dot{\mathbf{A}})$$

So the electric field is :

$$\mathbf{E}(\mathbf{x},t) = \frac{\mu_0}{4\pi|\mathbf{x}|}\mathbf{n}\times\mathbf{n}\times\int d^3x' \dot{\mathbf{J}}\left(\mathbf{x}', t-\frac{|\mathbf{x}|}{c}\right)$$

Now, to make sens of these formulas, recall the definition of the electric dipole moment :

$$\mathbf{d}(t) = \int d^3x'\, \mathbf{x}'\, \rho(\mathbf{x}',t)$$

Its time derivative is

$$\dot{\mathbf{d}}(t) = \int d^3x'\, \mathbf{x}'\, \frac{\partial\rho}{\partial t}$$

Using the continuity equation $\partial_t\rho + \boldsymbol{\nabla}\cdot\mathbf{J} = 0$, we get :

$$\dot{\mathbf{d}} = -\int d^3x'\, \mathbf{x}'\, (\boldsymbol{\nabla}_{\mathbf{x}'}\cdot\mathbf{J})$$

Applying the vector identity, which is easier to see in the index form:

$$\nabla_j(x_i J_j) = x_i(\nabla_j J_j) + J_i$$

we find

$$\dot{\mathbf{d}} = -\int d^3x'\, \nabla'_j(\mathbf{x}' J_j) + \int d^3x'\, \mathbf{J}$$

The first term vanishes as a surface integral at infinity, leaving

$$\dot{\mathbf{d}} = \int d^3x'\, \mathbf{J}(\mathbf{x}', t)$$

Thus, we can express in a very compact form the leading terms of the magnetic field for radiation :

$$\mathbf{B}(\mathbf{x}, t) = \frac{\mu_0}{4\pi c|\mathbf{x}|}\ddot{\mathbf{d}}(t - |\mathbf{x}|/c) \times \mathbf{n}$$

The Poynting vector is then given by :

$$\mathbf{S} = \frac{\mathbf{E} \times \mathbf{B}}{\mu_0} = \frac{c}{\mu_0}|\mathbf{B}|^2\mathbf{n} = \frac{\mu_0|\ddot{\mathbf{d}}|^2 \sin^2\theta}{16\pi^2 c|\mathbf{x}|^2}\mathbf{n}$$

Where θ is the angle between $\mathbf{d}$ and $\mathbf{n}$. We can then write the radiated energy per unit of times by integrating over the solid angle:

$$\frac{d\mathcal{E}}{dt} = \frac{|\ddot{\mathbf{d}}|^2}{6\pi\varepsilon_0 c^3}$$

Which is our usual non-relativistic Larmor formula for a general distribution of charges that gives rise to an electric dipole moment $\mathbf{d}$. In fact, in the previous chapter, for a single charge, we had $\mathbf{d} = q\mathbf{x}_0(t)$ such that :

$$\frac{d\mathcal{E}}{dt} = \frac{|\ddot{\mathbf{d}}|^2}{6\pi\varepsilon_0 c^3} = \frac{q^2|\ddot{\mathbf{x}}_0|^2}{6\pi\varepsilon_0 c^3} = \frac{q^2|\dot{\boldsymbol{\beta}}|^2}{6\pi\varepsilon_0 c}$$

Keep in mind that $\mathbf{d}$ is always evaluated at the retarded time $t - |\mathbf{x}|/c$.

6.4.2 Quasi-static region

Now, let's briefly talk about the the quasi-static regime : $a \ll |\mathbf{x}| \ll \lambda$. In this case, the sub-leading term in $a/|\mathbf{x}|$ are more important than those in a/λ. We have :

$$\mathbf{A}(\mathbf{x}, t) = \frac{\mu_0}{4\pi}\int d^3x'\, \frac{1}{|\mathbf{x} - \mathbf{x}'|}\mathbf{J}\left(\mathbf{x}',\, t - \underbrace{\frac{|\mathbf{x} - \mathbf{x}'|}{c}}_{\sim a/\lambda}\right) \simeq \frac{\mu_0}{4\pi}\int d^3x'\, \frac{\mathbf{J}(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|}\left[1 + \mathcal{O}\left(\frac{a}{\lambda}\right)\right]$$

This is the same case as for the magnetostatic expansion for a given time t. The same can be performed for the scalar potential :

$$\begin{aligned}
\Phi(\mathbf{x},t) &\simeq \frac{1}{4\pi\varepsilon_0}\int d^3x'\,\frac{\rho(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}\left[1+\mathcal{O}\left(\frac{a}{\lambda}\right)\right] \\
&= \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{n!}\frac{x_{i_1}x_{i_2}\ldots x_{i_n}}{|\mathbf{x}|^{2n+1}}Q_{i_1 i_2\ldots i_n}\left[1+\mathcal{O}\left(\frac{a}{\lambda}\right)\right] \\
&\sim \sum_{n=0}^{\infty}\frac{a^n}{|\mathbf{x}|^{n+1}}\left[1+\mathcal{O}\left(\frac{a}{\lambda}\right)\right]
\end{aligned}$$

Remark : The transition region between $|\mathbf{x}| \gg \lambda$ and $|\mathbf{x}| \ll \lambda$ occurring at $|\mathbf{x}| \sim \lambda$ is difficult to describe even if both are far larger than a

Chapter 7

Electromagnetic Fields in a Macroscopic Medium

7.1 Microscopic and macroscopic fields

In reality, electric charges are localized within atoms and molecules. In any material medium, electromagnetic fields therefore never propagate in a true vacuum: they constantly interact with the microscopic charges of matter. These interactions give rise to polarization and magnetization effects, and in some cases to electronic transitions responsible for absorption or amplification. As a result, our previous description must be slightly refined. However, we will see that Maxwell's equations retain their familiar form, with only minor modifications.

Let's define the **microscopic fields e** and **b**. These quantities represent the *exact* electric and magnetic fields, which are highly complex functions of space and time. We also introduce the **microscopic charge and current densities**, η and $\mathbf{j}$, which provide an exact description of the charge and current distributions within the system. For instance, η is strongly localized around atomic nuclei and decays extremely rapidly away from them. These densities vary significantly over length scales of the order of the Bohr radius, $a_0 = 0.529 \times 10^{-10}$ m. The microscopic fields **e** and **b** satisfy the *exact* Maxwell equations:

$$\nabla \cdot \mathbf{e} = \frac{\eta}{\varepsilon_0}, \tag{7.1}$$

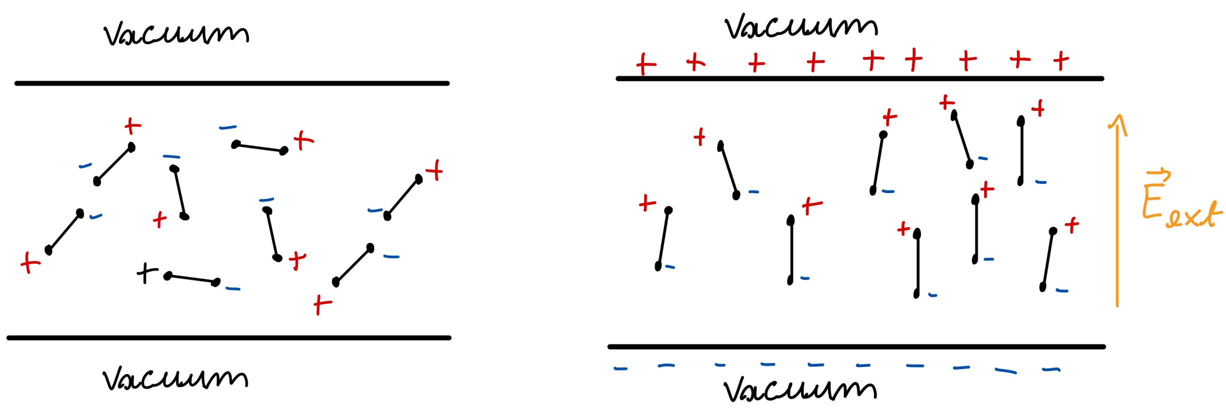
$$\nabla \cdot \mathbf{b} = 0, \tag{7.2}$$

$$\nabla \times \mathbf{e} = -\frac{\partial \mathbf{b}}{\partial t}, \tag{7.3}$$

$$\nabla \times \mathbf{b} = \mu_0 \mathbf{j} \; + \; \mu_0 \varepsilon_0 \frac{\partial \mathbf{e}}{\partial t}. \tag{7.4}$$

From now on, we will denote by **E**, **B**, **J** and ρ the corresponding **macroscopic quantities**. These represent the spatially averaged electric and magnetic fields, current density, and charge density. They are the **macroscopic fields**, which are the relevant quantities when describing electromagnetic phenomena at scales much larger than the atomic one, that is, for the vast majority of physical situations encountered in practice.

Let's first develop some physical intuition about how a medium can influence electromagnetic fields. Consider a *dielectric medium* placed between two vacuum regions, and suppose that an electric field passes through it, oriented along the $\mathbf{z}$ direction. We may model the medium as an ensemble of dipoles that can move freely. How does the electric field $\mathbf{E}_{\text{ext}} = E_z\mathbf{z} = \langle \mathbf{e}_{\text{ext}} \rangle$ affect the motion of these dipoles? Since each dipole moment tends to align with the electric field[1], the medium as a whole develops a partial ordering of dipoles along the field direction. This collective alignment of dipoles is illustrated in Fig. 7.1.



(a) $\mathbf{E} = \mathbf{0}$ (b) $\mathbf{E} \neq \mathbf{0}$

Figure 7.1: Alignment of the dipoles in the dielectric medium.

This alignment leads to a slight accumulation of bound charges at the surfaces of the dielectric, which in turn creates an internal electric field opposed to the applied one, $\mathbf{E}_{\text{ext}}$. As a result, the total microscopic field inside the material, $\mathbf{e}_{\text{tot}} = \mathbf{e}_{\text{ext}} + \mathbf{e}_{\text{dip}}$, is reduced compared to the external field.[2]

We will now make this intuition more rigorous.

Define an *averaging window function* $f(\mathbf{x})$ that has the following properties :

1. $\displaystyle\int d^3x'\, f(\mathbf{x}' - \mathbf{x}) = 1$
2. $f(\mathbf{x}' - \mathbf{x}) = 0$ if $|\mathbf{x}' - \mathbf{x}| > R$, with R large compared to atomic sizes a_0 but small compared to observation distances L : $a_0 \ll R \ll L$.

Let's give some example of possible choices of f :

$$f(\mathbf{x}) = \begin{cases} 1, & \text{if } -\dfrac{R}{2} < |\mathbf{x}| < \dfrac{R}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

$$f(\mathbf{x}) = \begin{cases} \dfrac{3}{4\pi R^3}, & \text{if } r < R, \\ 0, & \text{if } r > R. \end{cases}$$

[1] This follows from the interaction energy between a dipole and an electric field, $\mathcal{E} = -\mathbf{d} \cdot \mathbf{E}$, which is minimized when $\mathbf{d} \parallel \mathbf{E}$.

[2] For conductors, there are enough free moving charges to fully screen the external field.

$$f(\mathbf{x}) = \left(\pi R^2\right)^{-3/2} e^{-r^2/R^2}$$

Note that all of these definition tend to a Dirac in the limit $R \to 0$.
We can now define rigorously our averaged/macroscopic fields :

$$\begin{cases} \mathbf{E}(\mathbf{x},t) = \displaystyle\int d^3x'\, f(\mathbf{x}'-\mathbf{x})\,\mathbf{e}(\mathbf{x}',t) = \langle \mathbf{e}(\mathbf{x},t)\rangle \\ \mathbf{B}(\mathbf{x},t) = \displaystyle\int d^3x'\, f(\mathbf{x}'-\mathbf{x})\,\mathbf{b}(\mathbf{x}',t) = \langle \mathbf{b}(\mathbf{x},t)\rangle \end{cases}$$

One key property from this definition is that the derivatives commutes with the averaging :

$$\partial_{x_i}\mathbf{E} = \langle \partial_{x_i}\mathbf{e}\rangle, \qquad \partial_t\mathbf{E} = \langle \partial_t\mathbf{e}\rangle$$

The second relation is straightforward. Let's show the first one. We have :

$$\partial_{x_i}\mathbf{E}(\mathbf{x},t) = \int \partial_{x_i}\big[f(\mathbf{x}'-\mathbf{x})\big]\;\mathbf{e}(\mathbf{x}',t)\,d^3x'$$

Since $\partial_{x_i} f(\mathbf{x}'-\mathbf{x}) = -\,\partial_{x_i'} f(\mathbf{x}'-\mathbf{x})$, we have

$$\partial_{x_i}\mathbf{E}(\mathbf{x},t) = -\int \partial_{x_i'} f(\mathbf{x}'-\mathbf{x})\;\mathbf{e}(\mathbf{x}',t)\,d^3x'$$

Integrating by parts in $\mathbf{x}'$ (the boundary term vanishes by the decay/support assumptions on f),

$$\partial_{x_i}\mathbf{E}(\mathbf{x},t) = \int f(\mathbf{x}'-\mathbf{x})\,\partial_{x_i'}\mathbf{e}(\mathbf{x}',t)\,d^3x' = \big\langle\, \partial_{x_i}\mathbf{e}(\mathbf{x},t)\,\big\rangle$$

Hence, spatial derivatives commute with the averaging operator. We can derive the macroscopic/averaged Maxwell's equations :

$$\begin{aligned} \nabla\cdot\mathbf{E} &= \frac{\langle\eta\rangle}{\varepsilon_0}, \\ \nabla\times\mathbf{B} &= \mu_0\langle\mathbf{j}\rangle + \mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}, \\ \nabla\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t}, \\ \nabla\cdot\mathbf{B} &= 0. \end{aligned}$$

Let's now try to compute $\langle\eta\rangle$. The microscopic charge density η can be seen as a sum over all particles, each with a specific microscopic charge density η_n localized on atomic scales :

$$\eta(\mathbf{x}) = \sum_n \eta_n(\mathbf{x}-\mathbf{x}_n)$$

The averaged microscopic charge density is given by :

$$\langle\eta(\mathbf{x})\rangle = \sum_n \int d^3x'\,\eta_n(\mathbf{x}'-\mathbf{x}_n)f(\mathbf{x}'-\mathbf{x})$$

Since f is smooth on atomic scales, we can Taylor expand f around $\mathbf{x} = \mathbf{x}_n$, $\mathbf{x}_n$ being the center of the particle n :

$$f(\mathbf{x}' - \mathbf{x}) = f\left(\mathbf{x}_n - \mathbf{x}\right) + (\mathbf{x}' - \mathbf{x}_n) \cdot \boldsymbol{\nabla} f(\mathbf{x_n} - \mathbf{x}) + \dots$$

Where we neglected quadrupole terms. This gives :

$$\begin{aligned}\langle \eta(\mathbf{x}) \rangle &= \sum_n \int d^3x'\, \eta_n(\mathbf{x}' - \mathbf{x}_n) f(\mathbf{x}_n - \mathbf{x}) + \sum_n \int d^3x'\, \eta_n(\mathbf{x}' - \mathbf{x}_n)(\mathbf{x}' - \mathbf{x}_n) \cdot \boldsymbol{\nabla} f(\mathbf{x_n} - \mathbf{x}) \\ &= \sum_n f(\mathbf{x}_n - \mathbf{x}) \int d^3x'\, \eta_n(\mathbf{x}' - \mathbf{x}_n) + \sum_n \boldsymbol{\nabla} f(\mathbf{x_n} - \mathbf{x}) \cdot \int d^3x'\, \eta_n(\mathbf{x}' - \mathbf{x}_n)(\mathbf{x}' - \mathbf{x}_n)\end{aligned}$$

But, recall the definition of the monopole and dipole moment of particle n:

$$q_n = \int d^3x'\, \eta_n(\mathbf{x}' - \mathbf{x}_n), \qquad \mathbf{d}_n = \int d^3x'\, \eta_n(\mathbf{x}' - \mathbf{x}_n)(\mathbf{x}' - \mathbf{x}_n)$$

This yields :

$$\langle \eta(\mathbf{x}) \rangle = \sum_n q_n f(\mathbf{x}_n - \mathbf{x}) + \sum_n \mathbf{d}_n \cdot \boldsymbol{\nabla} f(\mathbf{x_n} - \mathbf{x})$$

We can now define the *macroscopic charge density* or free charge density :

$$\rho(\mathbf{x}) = \sum_n q_n f(\mathbf{x}_n - \mathbf{x})$$

Note that although its explicit dependence on f, the detailed shape of f does not matter. It can be seen by integrating ρ :

$$\int d^3x\, \rho(\mathbf{x}) = \sum_n q_n \int d^3x\, f(\mathbf{x}_n - \mathbf{x}) = \sum_n q_n = Q$$

With Q being the total charge of the system. We are still consistent with the definition of ρ we used to have. We can then define the *electric polarization of the medium* or dipole density:

$$\mathbf{P}(\mathbf{x}) = \sum_n \mathbf{d}_n f(\mathbf{x_n} - \mathbf{x})$$

This gives :

$$\langle \eta(\mathbf{x}) \rangle = \rho(\mathbf{x}) - \boldsymbol{\nabla}_\mathbf{x} \cdot \mathbf{P}(\mathbf{x})$$

It is common to have a density of free carriers $\rho = 0$ but the polarization is generally non-zero so we can ignore the quadrupole terms. A similar derivation can be carried out for the averaged current density :

$$\langle \mathbf{j} \rangle = \mathbf{J} + \frac{\partial \mathbf{P}}{\partial t} + \boldsymbol{\nabla} \times \mathbf{M}$$

Where $\mathbf{J}$ is the *macroscopic current density* :

$$\mathbf{J}(\mathbf{x}) = \sum_n \mathbf{j}_n f(\mathbf{x}_n - \mathbf{x})$$

with $\mathbf{j}_n$ the total current density at atomic site n, and $\mathbf{M}$ is the *magnetization*, defined through the magnetic moments at each atomic sites $\mathbf{m}_n$ as :

$$\mathbf{M}(\mathbf{x}) = \sum_n \mathbf{m}_n f(\mathbf{x}_n - \mathbf{x}), \qquad \mathbf{m}_n = \frac{1}{2}\int d^3x'\mathbf{x}' \times \mathbf{j}_n(\mathbf{x}')$$

From these definitions, we can find back our usual conservation law :

$$\partial_t\langle\eta\rangle + \boldsymbol{\nabla}\cdot\langle\mathbf{j}\rangle = \partial_t\rho + \boldsymbol{\nabla}\cdot\mathbf{J} = 0$$

7.2 Maxwell's Equations in a Macroscopic Medium

Starting from the quantities derived in the previous section, the **macroscopic Maxwell equations** can be written as:

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0} - \boldsymbol{\nabla}\cdot\mathbf{P}(\mathbf{x}),$$

$$\nabla\times\mathbf{B} = \mu_0\left(\mathbf{J} + \frac{\partial\mathbf{P}}{\partial t} + \boldsymbol{\nabla}\times\mathbf{M}\right) + \mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}.$$

where we recall that $\mathbf{P}$ is the *polarization* field, describing the density of electric dipole moments within the medium, and $\mathbf{M}$ is the *magnetization*, representing the magnetic dipole density.

To simplify these equations and separate the contributions of bound and free sources, we introduce two auxiliary fields. The first one is the *electric displacement field* $\mathbf{D}$, defined as:

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}.$$

The second one is the *magnetic field* $\mathbf{H}$, defined by:

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M},$$

where $\mathbf{B}$ is the *magnetic induction field*[3] (or magnetic flux density).

In terms of these macroscopic fields, Maxwell's equations take their familiar form:

$$\nabla\cdot\mathbf{D} = \rho,$$

$$\nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t},$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t},$$

$$\nabla\cdot\mathbf{B} = 0.$$

[3]Note the change in terminology

These are the **macroscopic Maxwell equations**, where ρ and $\mathbf{J}$ now represent the *free* charge and current densities, while the effects of bound charges and currents are absorbed into the fields $\mathbf{D}$ and $\mathbf{H}$.

Integral form. Using Gauss's and Stokes' theorems, the macroscopic Maxwell equations read:

$$\oint_{\partial V} \mathbf{D}\cdot d\mathbf{S} = \int_V \rho\, dV$$

$$\oint_{\partial S} \mathbf{H}\cdot d\boldsymbol{\ell} = \int_S \mathbf{J}\cdot d\mathbf{S} + \frac{\partial}{\partial t}\int_S \mathbf{D}\cdot d\mathbf{S},$$

$$\oint_{\partial S} \mathbf{E}\cdot d\boldsymbol{\ell} = -\frac{\partial}{\partial t}\int_S \mathbf{B}\cdot d\mathbf{S},$$

$$\oint_{\partial V} \mathbf{B}\cdot d\mathbf{S} = 0.$$

To determine the electromagnetic fields in a material, one needs to specify the **constitutive relations** linking the pairs $(\mathbf{E}, \mathbf{D})$ and $(\mathbf{B}, \mathbf{H})$. These relations characterize the electromagnetic response of the medium.

In many materials, this relation is both *local* and *linear*, such that

$$D_i = \varepsilon_{ij} E_j, \qquad H_i = (\mu^{-1})_{ij} B_j,$$

where ε_{ij} and μ_{ij} are the components of the permittivity and permeability tensors, respectively.

For **isotropic materials**, these tensors reduce to scalars:

$$\varepsilon_{ij} = \varepsilon\, \delta_{ij}, \qquad \mu_{ij} = \mu\, \delta_{ij},$$

which leads to the simple relations

$$\mathbf{D} = \varepsilon\, \mathbf{E}, \qquad \mathbf{H} = \frac{1}{\mu}\mathbf{B}.$$

Here, ε is the *electric permittivity* and μ the *magnetic permeability* of the medium.

Usually, $\varepsilon > \varepsilon_0$. For **paramagnetic** materials, one has $\mu > \mu_0$, whereas for **diamagnetic** materials, $0 < \mu < \mu_0$.

In general, the permittivity may depend on the field amplitude, $\varepsilon = \varepsilon(\mathbf{E})$. Expanding around $\mathbf{E} = 0$, one obtains

$$\varepsilon(\mathbf{E}) = \varepsilon(0) + \mathbf{E}\cdot \left.\frac{\partial \varepsilon}{\partial \mathbf{E}}\right|_{\mathbf{E}=0} + \dots$$

However, in most cases, the dependence is weak and ε can be treated as constant.

Finally, comparing the electric field in vacuum and in a dielectric medium for the same free charge density ρ, we have

$$\nabla\cdot \mathbf{E}_{\text{vac}} = \frac{\rho}{\varepsilon_0}, \qquad \nabla\cdot \mathbf{E}_{\text{med}} = \frac{\rho}{\varepsilon}.$$

Since typically $\varepsilon > \varepsilon_0$, it follows that

$$E_{\text{vac}} > E_{\text{med}},$$

meaning that the presence of matter partially **screens** the electric field due to the polarization of bound charges.

7.3 Matching conditions

We now derive the **boundary conditions** that the electromagnetic fields must satisfy at the interface between two media characterized by (ε_1, μ_1) and (ε_2, μ_2). Let the interface be defined by the plane $z = z_b$, and let $\hat{\mathbf{n}} = \hat{\mathbf{z}}$ be the unit normal pointing from medium 1 to medium 2.

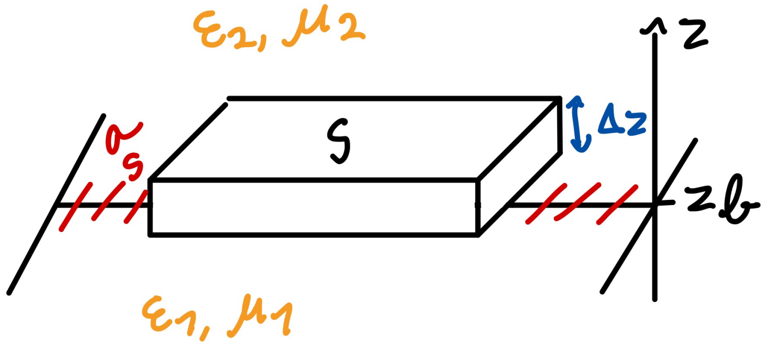


Figure 7.2: Interface between medium 1 and 2. We consider a volume $V = S\Delta z$.

Discontinuity of D Consider a small Gaussian volume of height Δz and base surface S crossing the boundary, such that $\Delta z \ll \sqrt{S}$. Applying Gauss's theorem to $\nabla \cdot \mathbf{D} = \rho$, we have :

$$\oint_{\partial V} \mathbf{D} \cdot d\mathbf{S} = \int_V \rho \, dV.$$

If a surface charge density σ_s is present at $z = z_b$, i.e. $\rho = \sigma_s \, \delta(z - z_b)$ and decomposing the displacement field as $\mathbf{D} = \mathbf{D}^{\perp} + \mathbf{D}^{\parallel}$, this becomes

$$S \left(\mathbf{D}_2^{\perp} - \mathbf{D}_1^{\perp} \right) = S\rho_s,$$

which gives the first boundary condition:

$$\boxed{\mathbf{D}_2^{\perp} - \mathbf{D}_1^{\perp} = \rho_s.}$$

Discontinuity of B From $\nabla \cdot \mathbf{B} = 0$, applying the same reasoning with the same gaussian volume crossing the boundary yields

$$\boxed{\mathbf{B}_2^{\perp} - \mathbf{B}_1^{\perp} = 0.}$$

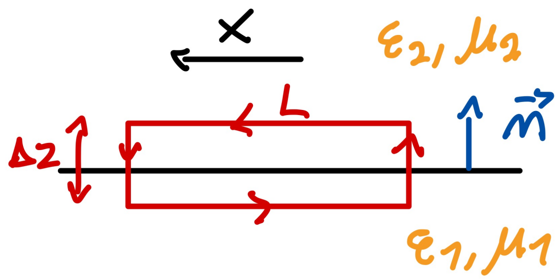


Figure 7.3: 2D view of the Gaussian volume.

Continuity of E Next, we apply Faraday's law in integral form to the area crossing the boundary depicted in Fig.7.3 :

$$\oint_{\partial S} \mathbf{E}\cdot d\boldsymbol{\ell} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S}.$$

Letting $\Delta z \to 0$, the magnetic flux term vanishes, leading to

$$L\left(\mathbf{E}_2^{\parallel} - \mathbf{E}_1^{\parallel}\right) = 0,$$

or equivalently,

$$\boxed{\mathbf{E}_2^{\parallel} - \mathbf{E}_1^{\parallel} = 0.}$$

Discontinuity of H Let's consider the same Gaussian domain as Fig.7.3 for the Maxwell Ampère's law :

$$\oint_{\partial S} \mathbf{H}\cdot d\boldsymbol{\ell} = \int_S \mathbf{J}\cdot d\mathbf{S} + \frac{d}{dt}\int_S \mathbf{D}\cdot d\mathbf{S}$$

Let $\hat{\mathbf{n}}$ be the unit normal to the interface (pointing from medium 1 to medium 2), and $\hat{\mathbf{t}}$ the tangent direction along the surface of the boundary (along $\hat{\mathbf{x}}$ on Fig.7.3). The vector perpendicular to the surface, following the right-hand rule is given by $\hat{\mathbf{u}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}}$. Along the upper and lower horizontal sides of the loop, the field is approximately constant, giving

$$\oint_{\partial S} \mathbf{H}\cdot d\boldsymbol{\ell} = (\mathbf{H}_2^{\parallel}\cdot\hat{\mathbf{t}})L - (\mathbf{H}_1^{\parallel}\cdot\hat{\mathbf{t}})L = L\,(\mathbf{H}_2^{\parallel} - \mathbf{H}_1^{\parallel})\cdot\hat{\mathbf{t}}$$

If a surface current density $\mathbf{J}_s$ flows along the interface, we can write

$$\mathbf{J}(\mathbf{r}) = \mathbf{J}_s\,\delta(z - z_b)$$

The total current crossing the loop is then

$$I_{\text{enc}} = \int_S \mathbf{J}\cdot\hat{\mathbf{u}}\,dS = L\,\mathbf{J}_s\cdot\hat{\mathbf{u}} = L\,\mathbf{J}_s\cdot(\hat{\mathbf{t}}\times\hat{\mathbf{n}}) = L\,(\mathbf{J}_s\times\hat{\mathbf{n}})\cdot\hat{\mathbf{t}}.$$

The displacement current term scales as $L\,\Delta z$ and vanishes in the limit $\Delta z \to 0$, since $\mathbf{D}$ remains finite.

The final boundary condition arises :

$$L\left(\mathbf{H}_2^{\parallel} - \mathbf{H}_1^{\parallel}\right)\cdot\hat{\mathbf{t}} = L\left(\mathbf{J}_s \times \hat{\mathbf{n}}\right)\cdot\hat{\mathbf{t}}$$

Noting that the equality must hold for any tangent direction $\hat{\mathbf{t}}$, we obtain the last matching condition :

$$\boxed{\mathbf{H}_2^{\parallel} - \mathbf{H}_1^{\parallel} = \mathbf{J}_s \times \hat{\mathbf{n}}}$$

Summary of boundary conditions

$$\begin{aligned}
\mathbf{D}_2^{\perp} - \mathbf{D}_1^{\perp} &= \sigma_s,\\
\mathbf{B}_2^{\perp} - \mathbf{B}_1^{\perp} &= 0,\\
\mathbf{E}_2^{\parallel} - \mathbf{E}_1^{\parallel} &= 0,\\
\mathbf{H}_2^{\parallel} - \mathbf{H}_1^{\parallel} &= \mathbf{J}_s \times \hat{\mathbf{n}}.
\end{aligned}$$

These relations are known as the **electromagnetic boundary conditions**. They express the continuity or discontinuity of the fields across an interface in terms of the free surface charge density σ_s and surface current density $\mathbf{J}_s$ at the interface.

7.4 Electromagnetic Energy in a Medium

We now derive the expression for the electromagnetic energy density, flux, and conservation law in a medium characterized by permittivity ε and permeability μ.

Starting from Maxwell's equations in vacuum, recall the conservation equation we found in chapter 4 :

$$\frac{\partial u}{\partial t} + \mathbf{E}\cdot\mathbf{J} + \boldsymbol{\nabla}\cdot\mathbf{S} = 0,$$

where

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}, \qquad u = \frac{\varepsilon_0}{2}|\mathbf{E}|^2 + \frac{1}{2\mu_0}|\mathbf{B}|^2.$$

Now, in a material with polarization $\mathbf{P}$ and magnetization $\mathbf{M}$, the averaged current density is :

$$\langle\mathbf{j}\rangle = \mathbf{J} + \frac{\partial\mathbf{P}}{\partial t} + \boldsymbol{\nabla}\times\mathbf{M}.$$

Thus, the energy equation becomes

$$\frac{\partial u}{\partial t} + \mathbf{E}\cdot\langle\mathbf{j}\rangle + \boldsymbol{\nabla}\cdot\mathbf{S} = 0.$$

The only difference from the vacuum case lies in the additional terms involving $\mathbf{P}$ and $\mathbf{M}$. We have term by term :

$$\frac{\partial u}{\partial t} = \varepsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} + \frac{1}{\mu_0} \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t}$$

$$\mathbf{E} \cdot \langle \mathbf{j} \rangle = \mathbf{E} \cdot \mathbf{J} + \mathbf{E} \cdot \frac{\partial \mathbf{P}}{\partial t} + \mathbf{E} \cdot (\boldsymbol{\nabla} \times \mathbf{M}).$$

$$\boldsymbol{\nabla} \cdot \mathbf{S} = \frac{1}{\mu_0} \boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{B})$$

Using the identity

$$\mathbf{E} \cdot \left(\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \frac{\partial \mathbf{P}}{\partial t} \right) = \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t} (\mathbf{E} \cdot \mathbf{D}),$$

and combining

$$\mathbf{E} \cdot (\boldsymbol{\nabla} \times \mathbf{M}) = \boldsymbol{\nabla} \cdot (\mathbf{M} \times \mathbf{E}) + \mathbf{M} \cdot (\boldsymbol{\nabla} \times \mathbf{E})$$

together with $\mathbf{B} \cdot \partial_t \mathbf{B}$ from Faraday's law, we can group all divergence terms into an effective Poynting vector and energy density :

$$u_{\text{eff}} = \frac{1}{2} \big(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H} \big), \qquad \mathbf{S}_{\text{eff}} = \mathbf{E} \times \mathbf{H}$$

leading to a very similar conservation equation :

$$\frac{\partial u_{\text{eff}}}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{S}_{\text{eff}} + \mathbf{E} \cdot \mathbf{J} = 0$$

7.5 Electromagnetic Waves in Isotropic Linear Media

We now consider electromagnetic wave propagation in an isotropic, linear medium characterized by ε and μ. From Maxwell's equations (in absence of free charges and currents),

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \boldsymbol{\nabla} \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}, \qquad \boldsymbol{\nabla} \cdot \mathbf{D} = 0, \qquad \boldsymbol{\nabla} \cdot \mathbf{B} = 0$$

Using the constitutive relations $\mathbf{D} = \varepsilon \mathbf{E}$ and $\mathbf{B} = \mu \mathbf{H}$, we can proceed the same as in chapter 2 to obtain the wave equation for the magnetic field :

$$\boldsymbol{\nabla}^2 \mathbf{H} - \mu \varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0$$

The phase velocity of electromagnetic waves in the medium is then :

$$v = \frac{1}{\sqrt{\mu \varepsilon}}$$

In vacuum, $c = 1/\sqrt{\mu_0 \varepsilon_0}$, hence the refractive index is :

$$n = \frac{c}{v} = \sqrt{\frac{\mu \varepsilon}{\mu_0 \varepsilon_0}} \geq 1$$

The plane wave solution to the wave equation is of the form :

$$\mathbf{H}(\mathbf{x}, t) = \mathbf{H}_0 \, e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} + \text{c.c.}, \qquad \omega = v |\mathbf{k}|.$$

By carrying out the same derivation using the scalar and vector potential as in Chapter 4, one can find that the magnetic and electric fields are still related by :

$$\mathbf{B} = \frac{1}{\omega} (\mathbf{k} \times \mathbf{E}) \Leftrightarrow \mathbf{H} = \frac{1}{\mu \omega} (\mathbf{k} \times \mathbf{E}).$$

Reflection and transmission at an interface

A good application of the matching conditions derived in the previous chapter are the laws of reflection and refraction. Consider an interface at $z = 0$ separating two media (ε_1, μ_1) and (ε_2, μ_2). An incident plane wave in medium 1 arrives on the interface with incidence angle θ_i, giving rise to a reflected and a transmitted wave :

$$\text{Incident wave : } \mathbf{E}_i = \mathbf{E}_{0i}\, e^{i(\mathbf{k}_i\cdot\mathbf{r}-\omega_i t)} + c.c.$$

$$\text{Reflected wave : } \mathbf{E}_r = \mathbf{E}_{0r}\, e^{i(\mathbf{k}_r\cdot\mathbf{r}-\omega_r t)} + c.c.$$

$$\text{Transmitted wave : } \mathbf{E}_t = \mathbf{E}_{0t}\, e^{i(\mathbf{k}_t\cdot\mathbf{r}-\omega_t t)} + c.c.$$

We will make use of the matching conditions to determine the transmitted and reflected amplitudes and wave vectors. The situation is shown on Fig.7.4.

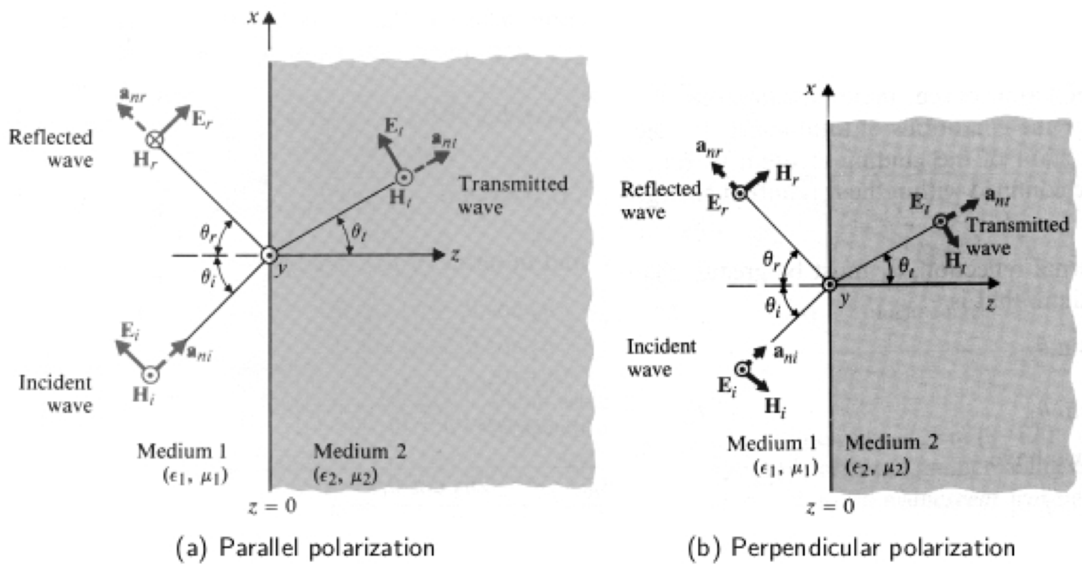


Figure 7.4: Study case for the two components $\mathbf{E}^{\parallel}$ and $\mathbf{E}^{\perp}$.

We need to match E^x at the boundary $z = 0$ by the matching conditions. We can first place ourselves at $\mathbf{x} = \mathbf{0}$. This gives :

$$E_i^x + E_r^x = E_t^x \iff E_{0i}^x\, e^{-i\omega_i t} + E_{0r}^x\, e^{-i\omega_r t} = E_{0t}^x\, e^{-i\omega_t t}$$

Since this relation has to hold for any time t, we can extract the following relations on the frequencies

$$\omega_i = \omega_r = \omega_t,$$

In terms of wavevectors, using $\omega_\alpha = v_\alpha|\mathbf{k}_\alpha| = c|\mathbf{k}_\alpha|/n_\alpha$, we have :

$$\boxed{|\mathbf{k}_i| = |\mathbf{k}_r| = \frac{n_1}{n_2}|\mathbf{k}_t|}$$

Now, let's place ourselves on a more general position on the plane $z = 0$. This gives :

$$E^x_{0i}\, e^{i(k^x_i x + k^y_i y)} + E^x_{0r}\, e^{i(k^x_r x + k^y_r y)} = E^x_{0t}\, e^{i(k^x_t x + k^y_t y)} \quad \forall x, y$$

We conclude that the tangential wavevectors must also match for all (x, y):

$$k^x_i = k^x_r = k^x_t, \qquad k^y_i = k^y_r = k^y_t \iff (\mathbf{k}_i)_\parallel = (\mathbf{k}_r)_\parallel = (\mathbf{k}_t)_\parallel$$

Writing this using the angles of incidence, reflection and transmission θ_i, θ_r and θ_t :

$$k^x_i = |\mathbf{k}_i| \sin\theta_i, \quad k^x_r = |\mathbf{k}_r| \sin\theta_r, \quad k^x_t = |\mathbf{k}_t| \sin\theta_t$$

We find :

$$\boxed{\theta_r = \theta_i} \qquad \boxed{n_1 \sin\theta_i = n_2 \sin\theta_t}$$

These are, respectively, the law of reflection and Snell's law.

Let's now focus on the 2 polarizations of the waves. The complex amplitudes can be decomposed in two polarizations $\mathbf{E_{0\alpha}} = \mathbf{E}^{\parallel}_{\mathbf{0}\alpha} + \mathbf{E}^{\perp}_{\mathbf{0}\alpha}$. The parallel component lies in the $x - z$ plane (perpendicular to the wave vector $\mathbf{k}_\alpha$) and the perpendicular one is orthogonal to the $x - z$ plane (see Fig.7.4).

For the parallel polarization, we can write :

$$\begin{aligned}
\mathbf{E}^{\parallel}_{0i} &= |\mathbf{E}^{\parallel}_{0i}|(\cos\theta_i \hat{\mathbf{x}} - \sin\theta_i \hat{\mathbf{z}}) \\
\mathbf{E}^{\parallel}_{0r} &= |\mathbf{E}^{\parallel}_{0r}|(\cos\theta_i \hat{\mathbf{x}} + \sin\theta_i \hat{\mathbf{z}}) \\
\mathbf{E}^{\parallel}_{0t} &= |\mathbf{E}^{\parallel}_{0t}|(\cos\theta_t \hat{\mathbf{x}} - \sin\theta_t \hat{\mathbf{z}})
\end{aligned}$$

Let's define the wave independence in each medium as :

$$Z_\alpha = \sqrt{\frac{\mu_\alpha}{\varepsilon_\alpha}}$$

The magnetic field is along the y direction. We can then write :

$$H^y_i = \frac{|\mathbf{E}^{\parallel}_{0i}|}{Z_1}, \qquad H^y_r = -\frac{|\mathbf{E}^{\parallel}_{0r}|}{Z_1}, \qquad H^y_t = \frac{|\mathbf{E}^{\parallel}_{0t}|}{Z_2}$$

Continuity of the tangential components E_x and H_y gives

$$\text{(i)} \quad E^x_i + E^x_r = E^x_t \iff |\mathbf{E}^{\parallel}_{0i}| \cos\theta_i + |\mathbf{E}^{\parallel}_{0r}| \cos\theta_i = |\mathbf{E}^{\parallel}_{0t}| \cos\theta_t \tag{7.5}$$

$$\text{(ii)} \quad H^y_i + H^y_r = H^y_t \iff \frac{|\mathbf{E}^{\parallel}_{0i}|}{Z_1} - \frac{|\mathbf{E}^{\parallel}_{0r}|}{Z_1} = \frac{|\mathbf{E}^{\parallel}_{0t}|}{Z_2}. \tag{7.6}$$

Using these two equations allows to find the Fresnel relations :

$$r_\parallel \equiv \frac{|\mathbf{E}^{\parallel}_{0r}|}{|\mathbf{E}^{\parallel}_{0i}|} = \frac{Z_2\cos\theta_t - Z_1\cos\theta_i}{Z_2\cos\theta_t + Z_1\cos\theta_i}, \qquad t_\parallel \equiv \frac{|\mathbf{E}^{\parallel}_{0t}|}{|\mathbf{E}^{\parallel}_{0i}|} = \frac{2\, Z_2\cos\theta_i}{Z_2\cos\theta_t + Z_1\cos\theta_i}$$

where $r_{\parallel}$ and $t_{\parallel}$ are respectively the reflection and transmission coefficients. A similar derivation can be carried out for the perpendicular polarization, which leads to :

$$r_{\perp} \equiv \frac{|\mathbf{E}_{0r}^{\perp}|}{|\mathbf{E}_{0i}^{\perp}|} = \frac{Z_2 \cos\theta_i - Z_1 \cos\theta_t}{Z_2 \cos\theta_i + Z_1 \cos\theta_t}, \qquad t_{\perp} \equiv \frac{|\mathbf{E}_{0t}^{\perp}|}{|\mathbf{E}_{0i}^{\perp}|} = \frac{2\, Z_2 \cos\theta_i}{Z_2 \cos\theta_i + Z_1 \cos\theta_t}$$

7.6 Study case : dielectric sphere

Consider a dielectric sphere of radius a and permittivity ε_1 placed in a constant electric field $\mathbf{E}_0$ in vacuum. Rotational invariance justify the following Ansatz for the scalar potential :

$$\Phi(\mathbf{r}) = f(r) \cos\theta\,,$$

The origin is taken at the center of the sphere and θ as the angle between $\mathbf{z}$ and $\mathbf{E}_0$. When the sphere is not there, the electric field is $\mathbf{E} = \mathbf{E}_0 = E_0 \mathbf{e}_z$. The potential is thus :

$$\Phi = -E_0 z = -E_0 r \cos\theta,$$

So $f(r) = -E_0 r$. To find the exact form of f, let us write the matching conditions at the interface of the sphere and the vacuum :

$$(\mathbf{D}_1 - \mathbf{D}_2)_{\perp} = \mathbf{0}, \tag{7.7}$$
$$(\mathbf{E}_1 - \mathbf{E}_2)_{//} = \mathbf{0}, \tag{7.8}$$

where the fields $\mathbf{E}_1, \mathbf{D}_1$ are ouside the dielectric sphere and the fields $\mathbf{E}_2, \mathbf{D}_2$ are inside. Note that the r.h.s. of the first condition is generally $\rho_s \mathbf{n}$, where $\mathbf{n}$ is a unit normal vector pointing outwards the sphere, and ρ_s is the surface density of free charges, which is 0 in our case. We relate the fields $\mathbf{E}$ and $\mathbf{D}$ with :

$$\mathbf{D}_1 = \varepsilon_0 \mathbf{E}_1, \qquad \mathbf{D}_2 = \varepsilon_1 \mathbf{E}_2. \tag{7.9}$$

To get a condition on the potential, we use:

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi = -f'(r) \cos\theta\, \mathbf{e}_r + \frac{f(r)}{r} \sin\theta\, \mathbf{e}_\theta.$$

This reduces (7.7) and (7.8) to

$$\varepsilon_1 f'(a^-) - \varepsilon_0 f'(a^+) = 0, \tag{7.10}$$
$$f(a^+) - f(a^-) = 0, \tag{7.11}$$

where $f(a^\pm)$ means $\lim_{r\to a^\pm} f(r)$.
We now solve for the potential everywhere. The boundary conditions at $r = a$ are given by (7.10),(7.11). At $r = 0$, the potential should actually not depend on θ so we impose

$$f(0) = 0. \tag{7.12}$$

When we are really far away of the sphere, its effect should vanish and the potential should converge to the result of question **a**)

$$\lim_{r\to\infty} \left(f(r) - (-E_0 r)\right) = 0. \tag{7.13}$$

We use the Maxwell equation:

$$\boldsymbol{\nabla}\cdot\mathbf{D} = \rho(x) = 0$$

which, using (7.9), indicates we simply have to solve the Laplace equation $\nabla^2\Phi = 0$, separately in regions $r < a$ and $r > a$.

$$\boldsymbol{\nabla}^2\Phi = \left(f''(r) + \frac{2}{r}f'r - \frac{2}{r^2}f(r)\right)\cos\theta = 0.$$

This must be verified for any θ. Making the Ansatz $f(r) \propto r^l$, we get that the parenthesis vanishes if

$$l^2 + l - 2 = 0 \quad\Rightarrow\quad l = 1 \text{ or } l = -2.$$

Since the equation is linear in Φ, any superposition of these solutions is valid, so we get:

$$f(r) = \begin{cases} K_1 r + K_2 r^{-2} & r < a, \\ C_1 r + c_2 r^{-2} & r > a. \end{cases}$$

Boundary conditions yield:

$$\begin{aligned}
&(7.12) \Rightarrow K_2 = 0,\\
&(7.13) \Rightarrow C_1 = -E_0,\\
&(7.10) \Rightarrow \varepsilon_1 K_1 - \varepsilon_0(C_1 - 2C_2 a^3) = 0,\\
&(7.11) \Rightarrow K_1 a - (C_1 a + C_2 a^{-2}) = 0.
\end{aligned}$$

The last two equations solve to:

$$K_1 = -3\frac{\varepsilon_0}{2\varepsilon_0 + \varepsilon_1}E_0, \quad C_2 = \frac{\varepsilon_1 - \varepsilon_0}{2\varepsilon_0 + \varepsilon_1}a^3 E_0,$$

which give as final answer:

$$f(r) = \begin{cases} -3\frac{\varepsilon_0}{2\varepsilon_0+\varepsilon_1}E_0 r & r < a, \\ -E_0 r + \frac{\varepsilon_1-\varepsilon_0}{2\varepsilon_0+\varepsilon_1}\frac{a^3}{r^2}E_0 & r > a. \end{cases}$$

The potential we found is thus :

$$\Phi = \begin{cases} -3\frac{\varepsilon_0}{2\varepsilon_0+\varepsilon_1}E_0 z & r < a, \\ -E_0 z + \frac{\varepsilon_1-\varepsilon_0}{2\varepsilon_0+\varepsilon_1}\frac{a^3}{r^3}E_0 z & r > a. \end{cases}$$

At $r > a$, we found the term from the external field $\mathbf{E}_0$, and the second term looks like a dipole potential, which means the polarized sphere produces a dipole field:

$$\frac{\varepsilon_1 - \varepsilon_0}{2\varepsilon_0 + \varepsilon_1}\frac{a^3}{r^3}E_0 z = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{d}\cdot\mathbf{x}}{r^3} \quad \text{for} \quad \mathbf{d} = \underbrace{4\pi\varepsilon_0\frac{\varepsilon_1 - \varepsilon_0}{2\varepsilon_0 + \varepsilon_1}a^3}_{\alpha}\,\mathbf{E}_0. \tag{7.14}$$

Now, imagine that the gas fills a sphere of radius R around the origin, with $r \gg R \gg n^{-1/3} \gg a$, with n the number of spheres per unit volume. This hierarchy of scales means the following. The typical distance $\lambda \sim n^{-1/3}$ between the small beads is much larger than their radius a, which means the beads don't interact (at the position of one bead, the dipole potential from an other bead scales like a^2/λ^2 which is negligible). But the sphere of radius R contains many beads, which means we can consider the gas homogeneous. And we look at that sphere from a very large distance r. Then we compare the effect of this sphere of gas, to the effect we computed for a sphere of continuous material, to extract its effective permittivity ε.

From that distance, at first order every bead is located at the origin. Moreover, since they don't interact, they all get polarized like we computed before. So, we can simply add up their dipole contribution to the potential:

$$\begin{aligned}
\Phi =& - E_0 z + \sum_i \frac{1}{4\pi\varepsilon_0}\frac{\alpha\mathbf{E}_0\cdot(\mathbf{x}-\mathbf{x}_i)}{|\mathbf{x}-\mathbf{x}_i|^3} \\
\simeq& - E_0 z + \frac{1}{4\pi\varepsilon_0}\frac{\alpha\mathbf{E}_0\cdot\mathbf{x}}{r^3}\frac{4\pi R^3 n}{3} \\
=& - E_0 z + \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{d}'\cdot\mathbf{x}}{r^3}.
\end{aligned}$$

Here the index i runs over all the beads. Like previously, we find the effect of the big sphere to amount to a dipole $\mathbf{d}' = \alpha'\mathbf{E}_0$, with

$$\alpha' = 4\pi\varepsilon_0\left(\frac{4}{3}\pi n a^3\frac{\varepsilon_1-\varepsilon_0}{2\varepsilon_0+\varepsilon_1}\right)R^3 = 4\pi\varepsilon_0\left(\frac{\varepsilon-\varepsilon_0}{2\varepsilon_0+\varepsilon}\right)R^3.$$

The last equality defines ε, and is fixed by analogy with the definition of α in (7.14). Thus, ε is given by

$$\frac{\varepsilon-\varepsilon_0}{2\varepsilon_0+\varepsilon} = \frac{4}{3}\pi n a^3\frac{\varepsilon_1-\varepsilon_0}{2\varepsilon_0+\varepsilon_1},$$

which yields:

$$\varepsilon = \varepsilon_0\frac{1+2\xi}{1-\xi}, \qquad \xi = \frac{4}{3}\pi n a^3\frac{\varepsilon_1-\varepsilon_0}{2\varepsilon_0+\varepsilon_1}.$$

Chapter 8

Special relativity and Covariant formulation of classical electrodynamics

8.1 Introduction to Special Relativity

Galileo (1564–1642) postulated that the *laws of physics are invariant under a change of reference frame*. If two frames F and F' move with a constant relative velocity $\mathbf{v}$, the transformation of coordinates reads :

$$\mathbf{x}' = \mathbf{x} + \mathbf{V}t, \qquad t' = t.$$

This is the **Galilean boost**. Together with spatial rotations :

$$\mathbf{x}' = \hat{R} \cdot \mathbf{x} = R_{ij} x_j \hat{\mathbf{e}}_i$$

and translations (spatial and temporal) :

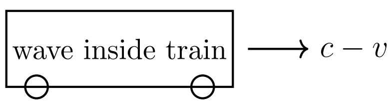
$$\mathbf{x}' = \mathbf{x} + \mathbf{a}, \qquad t' = t + T$$

these transformations form the *Galilean group*, the symmetry group of Newtonian mechanics (Newton 1643–1727).

By the end of the 19$^{\text{th}}$ century, light was understood as a wave. Since all known waves required a medium, physicists postulated the existence of the *ether*, characterized by:

$$c^2 = \frac{1}{\varepsilon_0 \mu_0}.$$

It was then natural to expect that the speed of light would depend on the observer's motion through the ether. Thus, in a rest frame, the light would propagate at velocity c but in a frame moving at velocity v, it would drop to $c - v$:



However, experiments such as the **Michelson–Morley** interferometer (1880s) showed that the measured value of c was constant in all inertial frames. This showed the failure of the wave equation to be invariant under Galilean transformations.

At the dawn of the 20th century, Lord Kelvin described two main "clouds" overshadowing classical physics:

- the **constancy of** c, i.e. the failure of classical mechanics to explain the invariance of light speed;
- the **ultraviolet catastrophe**, i.e. the failure of classical radiation theory to explain blackbody spectra.

The revolution of 1905 Einstein's *special theory of relativity* (1905) resolved the first cloud by postulating that:

1. the laws of physics are the same in all inertial frames;
2. the speed of light in vacuum, c, is invariant in all inertial frames.

These principles replaced Galilean invariance with **Lorentz invariance**, marking the birth of modern physics.

8.1.1 Symmetries of a Lagrangian

Consider a typical Lagrangian and its associated action :

$$\mathcal{L} = \sum_i \frac{1}{2} m_i \dot{\mathbf{x}}_i^2 - \sum_{i,j} V(|\mathbf{x}_i - \mathbf{x}_j|), \qquad S = \int dt\, \mathcal{L}.$$

The equations of motion follow from the variational principle :

$$\frac{\delta S}{\delta \mathbf{x}_i(t)} = 0.$$

If the Lagrangian (and hence the action) is invariant under a certain transformation, the equations of motion remain invariant as well. It is possible to show that the action is left unchanged under any Galilean transformation. We will show only the invariance under a Galilean boost, corresponding to changing to a reference frame moving with a constant velocity $\mathbf{v}$ with respect to the original one :

$$\mathbf{x}_i' = \mathbf{x}_i + \mathbf{v}t, \qquad t' = t.$$

Under this transformation, the velocities transform as $\dot{\mathbf{x}}_i' = \dot{\mathbf{x}}_i + \mathbf{v}$, and the Lagrangian becomes :

$$\mathcal{L}' = \sum_i \frac{1}{2} m_i (\dot{\mathbf{x}}_i + \mathbf{v})^2 - \sum_{i,j} V(|\mathbf{x}_i - \mathbf{x}_j|) = \mathcal{L} + \frac{d}{dt}\left(\sum_i m_i \mathbf{v} \cdot \mathbf{x}_i + \tfrac{1}{2} m_i \mathbf{v}^2 t \right).$$

Since the difference between $\mathcal{L}'$ and $\mathcal{L}$ is a total time derivative, the action S is invariant:

$$S' = S + (\text{boundary term}) \quad \Rightarrow \quad S' = S.$$

Therefore, the equations of motion are unchanged by a Galilean boost.

It is sometimes convenient to represent the Galilean transformation as a linear operation on an extended coordinate vector[1] :

$$(t, \mathbf{x}) \equiv x^\mu = \begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad x'^\mu = \hat{G}(\mathbf{v})\, x^\mu,$$

with :

$$\hat{G}(\mathbf{v}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ v_1 & 1 & 0 & 0 \\ v_2 & 0 & 1 & 0 \\ v_3 & 0 & 0 & 1 \end{pmatrix}.$$

This expresses the transformation :

$$(t, \mathbf{x}) \;\longrightarrow\; (t, \mathbf{x} + \mathbf{v}t)$$

in compact form.

The set of Galilean boosts forms a group under composition. Indeed, applying two boosts with velocities $\mathbf{v}$ and $\mathbf{v}'$ successively is equivalent to a single boost with velocity $\mathbf{v} + \mathbf{v}'$:

$$\hat{G}(\mathbf{v})\, \hat{G}(\mathbf{v}') = \hat{G}(\mathbf{v} + \mathbf{v}').$$

Hence, the Galilean boosts satisfy the usual group properties: closure, associativity, existence of an identity element ($\mathbf{v} = 0$), and an inverse element ($\mathbf{v} \mapsto -\mathbf{v}$).

Group of rotations Consider a spatial rotation acting on the coordinates :

$$x'_i = R_{ij} x_j, \qquad \mathbf{x}' = \hat{R}\, \mathbf{x}$$

We can generalize this definition to our extended vector x^μ for a more compact and uniform writing :

$$x'^\mu = \tilde{R}^\mu x^\mu, \qquad \tilde{R}^\mu = \begin{pmatrix} 1 & 0 \\ 0 & \hat{R} \end{pmatrix}.$$

Here $\hat{R}$ is a 3×3 matrix representing a spatial rotation.

We will now state properties of the rotation group.

[1] This is called a Lorentz 4-vector but we will come back to it later with a much rigorous definition.

Arbitrary rotation. Any spatial rotation can be written as a combination of three rotations about the coordinate axes :

$$\hat{R} = \hat{R}_{xy}\,\hat{R}_{yz}\,\hat{R}_{xz},$$

where, for example,

$$\hat{R}_{xy}(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

is a rotation around z of an angle α.

Orthogonality. Rotations preserve scalar products :

$$\mathbf{x}' \cdot \mathbf{y}' = x_i' y_i' = R_{ij} x_j R_{ik} y_k = x_j R^T_{ji} R_{ik} y_k = x_j (\hat{R}^T \hat{R})_{jk} \doteq x_j y_j = \mathbf{x} \cdot \mathbf{y}.$$

This condition requires rotation matrices to be orthogonal, meaning that :

$$\hat{R}^T \hat{R} = \hat{I}, \qquad R_{ij} R_{ik} = \delta_{jk} \equiv \hat{I}_{jk}, \qquad \det(\hat{R}^T \hat{R}) = 1 \;\Rightarrow\; \det(\hat{R}) = \pm 1.$$

Preservation of orientation. Proper rotations also preserve the orientation of space, which requires :

$$\det(\hat{R}) = +1.$$

In contrast, orthogonal matrices with $\det(\hat{P}) = -1$ correspond to improper transformations such as spatial reflections (e.g. parity $x \to -x$).

Rotation group. The set of all 3×3 orthogonal matrices with determinant $+1$ forms the group :

$$SO(3) = \{\, \hat{R} \in O(3) \mid \hat{R}^T \hat{R} = \hat{I},\ \det \hat{R} = 1 \,\}.$$

It is a 3-dimensional Lie group describing all possible spatial rotations. If we also include reflections ($\det = -1$), we obtain the full orthogonal group $O(3) = \hat{R} \oplus \hat{P}$.

8.1.2 Failure of invariance of the wave equation

Consider the standard wave equation :

$$\Box\psi(\mathbf{x}, t) = 0, \qquad \text{i.e.} \quad \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \psi = \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^i} \right) \psi = 0.$$

Where we used Einstein summation convention for repeated indices. This equation is invariant under spatial and temporal translations, and under spatial rotations. However, it is *not* invariant under Galilean transformations :

$$\mathbf{x}' = \mathbf{x} + \mathbf{v}t \implies x'^i = x^i + v^i t, \qquad t' = t.$$

The time and spatial derivatives transform as

$$\frac{\partial}{\partial t} = \frac{\partial t'}{\partial t} \frac{\partial}{\partial t'} + \frac{\partial x'^i}{\partial t} \frac{\partial}{\partial x'^i} = \frac{\partial}{\partial t'} + v^i \frac{\partial}{\partial x'^i}, \qquad \frac{\partial}{\partial x^i} = \frac{\partial}{\partial x'^i}.$$

Hence,

$$\frac{\partial^2}{\partial t^2} = \left(\frac{\partial}{\partial t'} + v^i \frac{\partial}{\partial x'^i}\right)^2 = \frac{\partial^2}{\partial t'^2} + 2v^i \frac{\partial^2}{\partial x'^i \partial t'} + (v^i v^j) \frac{\partial^2}{\partial x'^i \partial x'^j}.$$

Substituting into the wave operator shows that cross-terms appear, so the wave equation is *not invariant* under Galilean boosts. This is because we did not transform c. This is where Galilean relativity fails and the constancy of c enters. As a consequence, Maxwell's equations are also not invariant under Galilean transformations.

8.1.3 Invariance of the speed of light and wave equation under Lorentz transformations

We would like to suggest that the *principle of equivalence of inertial reference frames* remains valid in electromagnetism. However, the transformation laws for coordinates must be modified.

Galilean boosts are thus replaced by **Lorentz boosts**. We now ask:

Which transformation leaves the wave equation invariant?

$$\left[\frac{1}{c^2}\left(\frac{\partial}{\partial t}\right)^2 - \left(\frac{\partial}{\partial x_1}\right)^2 - \left(\frac{\partial}{\partial x_2}\right)^2 - \left(\frac{\partial}{\partial x_3}\right)^2\right] \psi = 0.$$

There are several ways to derive it, here is one of them using what we call a *Wick rotation.* The usual wave equation can be written as :

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2}\right] \psi = 0.$$

The spatial part is invariant under the rotation group $SO(3)$. To make the equation invariant under a larger group, we introduce an "Euclidean time" variable (Wick rotation) :

$$x_0^E = -ict, \qquad t = \frac{i}{c} x_0^E, \qquad \frac{\partial}{\partial t} = -ic \frac{\partial}{\partial x_0^E}.$$

The equation then becomes :

$$\left[\frac{\partial^2}{\partial (x_0^E)^2} + \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right] \psi = 0,$$

which is invariant under the $SO(4)$ group of 4D rotations. They are completely analogous to 3D rotations. A 4D vector $x_\mu^E = (x_0^E, x_1, x_2, x_3)^T$ transforms under a 4D rotation as :

$$x_\mu^{E\,\prime} = \hat{\Lambda}_{\mu\nu}^E \, x_\nu^E, \qquad \hat{\Lambda}_E^T \hat{\Lambda}_E = \hat{I}$$

The last expression can only be formulated component wise :

$$\left(\hat{\Lambda}_E^T \hat{\Lambda}_E\right)_{\mu\nu} = (\hat{\Lambda}^E)^T_{\mu\lambda} \hat{\Lambda}^E_{\lambda\nu} = \hat{\Lambda}^E_{\lambda\mu} \hat{\Lambda}^E_{\lambda\nu} = \delta_{\mu\nu}$$

This implies that scalar products are preserved:

$$x_\nu^{E\prime} y_\nu^{E\prime} = \left(\Lambda^E_{\nu\mu} x^E_\mu\right)\left(\Lambda^E_{\nu\lambda} y^E_\lambda\right) = x^E_\mu \left(\Lambda^E\right)^T_{\mu\nu} \Lambda^E_{\nu\lambda} y^E_\lambda = x^E_\mu \delta_{\mu\lambda} y^E_\lambda = x^E_\mu y^E_\mu = x^E_0 y^E_0 + x_1 y_1 + x_2 y_2 + x_3 y_3.$$

where we used that the terms commute since we are summing. Now, the sum implicitly goes from 0 to 3.
The $SO(4)$ group contains both the usual spatial rotations and additional "mixing" rotations between time and space coordinates. An arbitrary rotation is then given by :

$$\hat{\Lambda}^E = \hat{\Lambda}^E_{01}\, \hat{\Lambda}^E_{02}\, \hat{\Lambda}^E_{03}\, \underbrace{\hat{\Lambda}^E_{12}\, \hat{\Lambda}^E_{23}\, \hat{\Lambda}^E_{13}}_{SO(3) \text{ rotations}}$$

For instance, a rotation in the (x_0, x_1) plane reads

$$\hat{\Lambda}^E_{01} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \begin{cases} x_0' = x_0 \cos\alpha + x_1 \sin\alpha \\ x_1' = x_1 \cos\alpha - x_0 \sin\alpha \\ x_2' = x_2 \\ x_3' = x_3 \end{cases}$$

As in the 3D case, these rotations do not commute, and the same holds for their 4D analogues. Going back to Lorentzian time, we can see how the 01 boost acts on the relevant parts (2 and 3 are unchanged) :

$$\begin{cases} x_0' = x_0 \cos\alpha + x_1 \sin\alpha \\ x_1' = x_1 \cos\alpha - x_0 \sin\alpha \end{cases} \iff \begin{cases} -ict' = -ict\cos\alpha + x_1 \sin\alpha \\ x_1' = x_1 \cos\alpha + ict \sin\alpha \end{cases}$$

Now, let's define the *rapidity* $\chi = i\alpha$ and using that $\cos(\alpha) = \cosh(\chi)$ and $i\sin(\alpha) = \sinh(\chi)$, we get :

$$\begin{cases} t' = t\cosh\chi + \frac{x_1}{c}\sinh\chi \\ x_1' = x_1 \cosh\chi + ct\sinh\chi \end{cases}$$

We have derived the so-called *Lorentz boost* along the x-direction. Introducing the usual parameters[2] :

$$\beta = \frac{v}{c}, \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \qquad \text{and} \qquad \tanh\chi = \beta,$$

we can express the hyperbolic functions as :

$$\cosh\chi = \gamma, \qquad \sinh\chi = \gamma\beta.$$

Hence, the Lorentz boost becomes :

$$\begin{cases} t' = \gamma\left(t + \frac{v}{c^2}x_1\right) \\ x_1' = \gamma(x_1 + vt) \\ x_2' = x_2 \\ x_3' = x_3 \end{cases}$$

[2] We will show how we got the correspondence between the rapidity and β and γ in the following section.

This is the standard form of a Lorentz boost in the x-direction.
In the limit of small velocities $\beta = v/c \ll 1$, we have

$$\gamma \approx 1 + \tfrac{1}{2}\beta^2 \approx 1$$

and therefore,

$$t' \approx t, \qquad x_1' \approx x_1 + vt$$

This shows that the Lorentz transformation reduces smoothly to the Galilean transformation when $v \ll c$, as expected from the correspondence principle.
In a matrix form, the Lorentz boost along the x-axis can be expressed compactly as :

$$\hat{\Lambda}_{01} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

More generally, we may consider a boost along an arbitrary direction $\mathbf{v} = v\,\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is a unit vector. In this case, the Lorentz transformation acts separately on the components parallel and perpendicular to the boost.
We decompose any spatial vector as

$$\mathbf{x} = \mathbf{x}_{\parallel} + \mathbf{x}_{\perp}, \qquad \mathbf{x}_{\parallel} = (\mathbf{x}\cdot\hat{\mathbf{n}})\,\hat{\mathbf{n}}.$$

The Lorentz boost with velocity $\mathbf{v}$ then reads :

$$t' = \gamma\left(t + \frac{\mathbf{v}\cdot\mathbf{x}}{c^2}\right),$$
$$\mathbf{x}_{\parallel}' = \gamma\left(\mathbf{x}_{\parallel} + \mathbf{v}\,t\right),$$
$$\mathbf{x}_{\perp}' = \mathbf{x}_{\perp}.$$

Equivalently, in compact form :

$$\mathbf{x}' = \mathbf{x}_{\perp} + \gamma\left(\mathbf{x}_{\parallel} + \mathbf{v}\,t\right).$$

This is the Lorentz boost in an arbitrary direction. Notice that when $\hat{\mathbf{n}} = \hat{\mathbf{x}}_1$, we recover the standard x–direction boost introduced above.

8.2 Lorentz Four-vectors

8.2.1 Covariant and contravariant vectors

Let's introduce the so-called *Lorentz Four-vectors*. First, introduce the *Four-position* x^{μ}. This is a 4-component vector, defined with the upper index μ as :

$$x^{\mu} = \begin{pmatrix} ct \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

where $x^0 = ct$. This is called a *contravariant* vector. One can wonder what is the difference between an Euclidian 4-vector as defined above and the Lorentz 4-vector. Euclidian 4-vectors work exactly the same as our usual 3 vectors, with a scalar product satisfying :

$$x_E^\mu \cdot x_E^\mu = (x_E^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$$

However, since we want to work with real time, we find that Lorentz 4-vectors, defined with real time $x_E^0 = -ix^0$, should fulfill the following dot product relation :

$$x^\mu \cdot x^\mu = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 := x_\mu x^\mu$$

Where we define the lowered indexed *covariant* vector (or covector) as :

$$x_\mu = \begin{pmatrix} -ct \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

We see that the only difference between the lowered and upper indexed quantity is the 0-th component defined as $x_0 = -ct = -x^0$. This pushes us to define a way to raise and lower indices. This is done through the *Minkowski metric* in the $(-+++)$ convention[3]:

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad x^\mu = \eta^{\mu\nu} x_\nu, \quad x_\mu = \eta_{\mu\nu} x^\nu$$

This metric tensor can be seen as the Dirac delta in our space time vectorial space. Indeed :

$$x^\mu \cdot y^\mu := x_\mu y^\mu = x_\mu \eta^{\mu\nu} y_\nu = x^\mu \eta_{\mu\nu} y^\nu = x^\mu y_\mu$$

This relation shows important properties.

1. The choice $x_\mu y^\mu$ or $x^\mu y_\mu$ does not matter: both represent the same scalar quantity,

$$x_\mu y^\mu = x^\mu y_\mu = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2.$$

2. When summing over repeated indices, one must always have one index up and one index down. For example,

$$A_{\mu\nu\rho} = \eta_{\lambda\nu} \, A_\mu{}^\lambda{}_\rho,$$

means that we **contract** the index λ using the metric $\eta_{\lambda\nu}$. This can be seen as taking a "sum along the diagonal" between a contravariant and a covariant component.

3. Indices that are summed over (repeated once up and once down) are said to be *contracted.*

[3]Sometimes, the opposite convention exists, the $(+---)$ convention, where $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

4. Any expression in which all repeated indices are contracted is a **Lorentz scalar**, i.e. invariant under Lorentz transformations. This can be expressed by :

$$A'_\mu B'^\mu = A_\mu B^\mu$$

where the primed quantities refer to the same vectors expressed in the boosted frame.

To develop more on this last remark, let's express the Lorentz boost from a rest frame $\mathcal{R}$ to a moving frame $\mathcal{R}'$ along x^1. This can be expressed through the usual boost matrix :

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus, we can express how a contravariant 4-vector x^μ transforms under a change of frame encoded by $\Lambda^\mu{}_\nu$:

$$x^{\mu\prime} = \Lambda^\mu{}_\nu \, x^\nu$$

One can remark the similarity of the previous expression with the usual boost $\mathbf{x}' = \hat{\Lambda}\mathbf{x}$. The reciprocal boost can be expressed as :

$$\left(\Lambda^{-1}\right)^\mu{}_\nu = \Lambda_\nu{}^\mu = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Proof : To show this, we use that the Minkowski interval is invariant :

$$\eta_{\alpha\beta}\, x'^\alpha x'^\beta = \eta_{\rho\sigma}\, x^\rho x^\sigma$$

Since $x'^\alpha = \Lambda^\alpha{}_\mu \, x^\mu$, we have :

$$\eta_{\alpha\beta}\, x'^\alpha x'^\beta = \eta_{\alpha\beta}\, \Lambda^\alpha{}_\mu x^\mu \, \Lambda^\beta{}_\nu x^\nu = \left(\eta_{\alpha\beta}\, \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu\right) x^\mu x^\nu$$

Because this equality must hold for any four-vector x^μ, we obtain the following identity :

$$\eta_{\alpha\beta}\, \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu = \eta_{\mu\nu}$$

or, after renaming the dummy indices,

$$\Lambda^\mu{}_\rho \, \eta_{\mu\nu}\, \Lambda^\nu{}_\sigma = \left(\Lambda^T\right)_\rho{}^\mu \, \eta_{\mu\nu}\, \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$$

which is the component form of the Lorentz-invariance condition :

$$\hat{\Lambda}^T \, \hat{\eta} \, \hat{\Lambda} = \hat{\eta}$$

One can also rewrite the Lorentz invariance condition in the following way:

$$\eta_{\alpha\beta}\,\Lambda^{\alpha}{}_{\mu}\Lambda^{\beta}{}_{\nu} = \Lambda^{\alpha}{}_{\mu}\Lambda_{\alpha\nu} = \Lambda_{\beta\mu}\Lambda^{\beta}{}_{\nu} = \eta_{\mu\nu}$$

Let's now contract this expression with $\eta^{\mu\rho}$:

$$\Lambda_{\beta\mu}\,\eta^{\mu\rho}\,\Lambda^{\beta}{}_{\nu} = \Lambda_{\beta}{}^{\rho}\,\Lambda^{\beta}{}_{\nu} = \eta^{\mu\rho}\,\eta_{\mu\nu} = \delta^{\rho}{}_{\nu}$$

Hence, we conclude that the inverse Lorentz transformation can be expressed as :

$$(\Lambda^{-1})^{\mu}{}_{\nu} = \eta^{\mu\rho}\,\Lambda^{\sigma}{}_{\rho}\,\eta_{\sigma\nu} \quad\Longleftrightarrow\quad \Lambda^{-1} = \eta^{-1}\Lambda^{T}\eta$$

It follows directly that the inverse transformation reads :

$$(\Lambda^{-1})^{\mu}{}_{\nu} = \Lambda_{\nu}{}^{\mu}$$

□

Now, let's come back to the 4-th remark we did before. Since any contracted expression must be Lorentz invariant, we can express how a covariant vector V_μ transforms under the same change of frame :

$$V'_{\mu} = \Lambda_{\mu}{}^{\nu}V_{\nu} = \left(\Lambda^{-1}\right)^{\nu}{}_{\mu}V_{\nu}$$

Indeed, this can be shown through the invariance of a scalar contraction. Consider the invariant scalar $V_\mu B^\mu$, where B^μ is any contravariant vector. Under a Lorentz boost we have $B'^{\mu} = \Lambda^{\mu}{}_{\rho}B^{\rho}$, and by invariance :

$$V'_{\mu}B'^{\mu} \;=\; V_{\nu}B^{\nu}$$

Substituting the transformation of B^μ :

$$V'_{\mu}\Lambda^{\mu}{}_{\rho}B^{\rho} \;=\; V_{\nu}B^{\nu}$$

Since this must hold for *any* B^ρ, we conclude :

$$V'_{\mu}\Lambda^{\mu}{}_{\rho} \;=\; V_{\rho} \quad\Longrightarrow\quad V'_{\mu} \;=\; \left(\Lambda^{-1}\right)^{\nu}{}_{\mu}\,V_{\nu} = \Lambda_{\mu}{}^{\nu}\,V_{\nu},$$

The transformation laws for covariant and contravariant components ensure that scalar quantities, such as $V_\mu B^\mu$, remain invariant for all inertial observers. This invariance reflects one of the most fundamental symmetries of nature: the Lorentz symmetry. Lorentz transformations form a group that preserves the structure of spacetime, meaning that the physical laws (in particular those of electromagnetism and relativistic mechanics) take the same form in every inertial frame.

Geometrically, a Lorentz boost can be seen as a hyperbolic rotation in the (ct, x^1) plane of Minkowski space, where the rapidity χ is defined by $\tanh\chi = \beta$. Just as ordinary rotations preserve distances in Euclidean space, Lorentz transformations preserve the Minkowski interval :

$$s^2 = -c^2t^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = x_{\mu}x^{\mu}$$

which remains identical for all inertial observers. This expresses the deep symmetry underlying special relativity: time and space are not independent, but form a unified spacetime whose geometry is invariant under Lorentz transformations.

Four-derivative The **four-derivative** (or four-gradient) is defined as the covariant vector

$$\partial_\mu = \begin{pmatrix} \frac{1}{c}\frac{\partial}{\partial t} \\ \boldsymbol{\nabla} \end{pmatrix} = \frac{\partial}{\partial x^\mu}$$

and its contravariant counterpart as :

$$\partial^\mu = \eta^{\mu\nu}\,\partial_\nu = \begin{pmatrix} -\frac{1}{c}\frac{\partial}{\partial t} \\ \boldsymbol{\nabla} \end{pmatrix}$$

The corresponding Lorentz-invariant operator is

$$\partial_\mu\partial^\mu = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2 \;\equiv\; -\Box,$$

known as the *d'Alembert operator*. It directly appears in the relativistic wave equation, confirming that the latter is Lorentz invariant. This is written :

$$\Box\,\psi(\mathbf{x},t) \equiv -\partial_\nu\partial^\nu\psi(x^\mu)$$

Transformation of functions Let's now see how a scalar function $f(x^\mu)$ transforms under a Lorentz boost.
A scalar field is, by definition, invariant under a change of frame :

$$f'(x') = f(x),$$

where the primed coordinates are related to the unprimed ones by

$$x'^\nu = \Lambda^\nu{}_\rho\, x^\rho, \qquad \text{or equivalently} \qquad x^\mu = (\Lambda^{-1})^\mu{}_\nu\, x'^\nu.$$

Let us now look at how derivatives transform. Taking the derivative of $f'(x') = f(x(x'))$ with respect to x'^ν gives :

$$\partial'_\nu f' = \frac{\partial f'(x')}{\partial x'^\nu} = \frac{\partial f(x)}{\partial x^\mu}\,\frac{\partial x^\mu}{\partial x'^\nu} = \Lambda_\nu{}^\mu\,\frac{\partial f(x)}{\partial x^\mu} = \Lambda_\nu{}^\mu\,\partial_\mu f.$$

Therefore, the derivative operator itself transforms as :

$$\partial'_\nu = \Lambda_\nu{}^\mu\,\partial_\mu,$$

which shows that ∂_μ is indeed a covariant vector.

The same reasoning applies to higher-rank tensors. For instance, a rank-2 covariant tensor transforms as :

$$A'_{\mu\nu} = \Lambda_\mu{}^\rho\,\Lambda_\nu{}^\sigma\,A_{\rho\sigma} \qquad A'^{\mu\nu} = \Lambda^\mu{}_\rho\,\Lambda^\nu{}_\sigma\,A^{\rho\sigma}$$

Addition of velocities Starting from the Lorentz transformation for a boost along x:

$$\begin{cases} ct' = \gamma\,(ct - \beta x), \\ x' = \gamma\,(x - \beta ct), \\ y' = y, \qquad z' = z, \end{cases} \qquad \beta = \frac{V}{c}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

Let's differentiate :

$$\begin{cases} d(ct') = \gamma\big(d(ct) - \beta\, dx\big), \\ dx' = \gamma\big(dx - \beta\, d(ct)\big), \\ dy' = dy, \qquad dz' = dz. \end{cases}$$

Recall that, in $\mathcal{R}$, we have :

$$dx = v_x\, dt, \quad dy = v_y\, dt, \quad dz = v_z\, dt, \quad d(ct) = c\, dt.$$

Thus,

$$\begin{cases} c\, dt' = \gamma\big(c - \beta\, v_x\big)\, dt, \\ dx' = \gamma\big(v_x - \beta\, c\big)\, dt, \\ dy' = v_y\, dt, \qquad dz' = v_z\, dt \end{cases}$$

This lead us to the expression of the velocities in $\mathcal{R}'$:

$$v'_x \equiv \frac{dx'}{dt'} = \frac{\gamma\,(v_x - \beta c)\, dt}{\gamma\,(c - \beta v_x)\, dt/c} = \frac{v_x - \beta c}{1 - \dfrac{\beta v_x}{c}} = \frac{v_x - V}{1 - \dfrac{v_x V}{c^2}}$$

$$v'_y \equiv \frac{dy'}{dt'} = \frac{v_y\, dt}{\gamma\,(c - \beta v_x)\, dt/c} = \frac{v_y}{\gamma\left(1 - \dfrac{\beta v_x}{c}\right)} = \frac{v_y}{\gamma\left(1 - \dfrac{v_x V}{c^2}\right)}$$

$$v'_z = \frac{v_z}{\gamma\left(1 - \dfrac{v_x V}{c^2}\right)}$$

In summary, the velocity addition formulas are :

$$v'_x = \frac{v_x - V}{1 - \dfrac{v_x V}{c^2}}, \qquad v'_y = \frac{v_y}{\gamma\left(1 - \dfrac{v_x V}{c^2}\right)}, \quad v'_z = \frac{v_z}{\gamma\left(1 - \dfrac{v_x V}{c^2}\right)}.$$

Let's consider 2 limits :

- $V, v_{x,y,z} \ll c$: We go back to our usual Galilean formulas (since $\gamma \approx 1$) :

$$v'_x \approx v_x - V, \qquad v'_y \approx v_y, \quad v'_z \approx v_z.$$

- $v_x = c$: this corresponds to the case of a photon moving along the x-axis. Using the addition formulas, we obtain :

$$v'_x = \frac{c - V}{1 - \dfrac{cV}{c^2}} = c, \qquad v'_y = \frac{v_y}{\gamma\left(1 - \dfrac{V}{c}\right)} = 0, \quad v'_z = \frac{v_z}{\gamma\left(1 - \dfrac{V}{c}\right)} = 0.$$

Hence, the speed of light remains invariant ($v' = c$) as required by special relativity. However, if the photon is not exactly collinear with the boost direction ($v_y, v_z \neq 0$), we observe an aberration of light: the direction of propagation is tilted toward the x-axis while keeping $|\mathbf{v}'| = c$.
This is for this reason why Michelson and Morlay did not see a change in the speed of light during their experiment. This is also the reason why the wave equation is invariant under a Lorentz boost.

Interval The spacetime interval between two events is defined as :

$$s^2 = -c^2(\Delta t)^2 + (\Delta\vec{x})^2 = \Delta x^\mu \Delta x_\mu, \qquad \Delta x^\mu = (ct_2 - ct_1,\ x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1).$$

The interval corresponds to the "internal" or *proper* time of the object when $s^2 < 0$, with :

$$\tau = \sqrt{-s^2}/c.$$

It represents the quantity invariant under Lorentz transformations. Three kind of space-time events can be defined :

$$\begin{cases} s^2 < 0 : & \text{time-like interval (a signal can reach the point)} \\ s^2 > 0 : & \text{space-like interval (a signal cannot reach)} \\ s^2 = 0 : & \text{null interval, light cone.} \end{cases}$$

Time-like intervals correspond to events that can be causally connected by a subluminal signal, space-like intervals correspond to events that cannot be connected by any physical signal ($v > c$), and null intervals correspond to light rays that define the boundaries of the light cone.

8.3 Covariant form of Maxwell's equations

8.3.1 Four-current and Four-potential

Going back to classical electrodynamics, recall how the charge and current densities transforms under a Galilei transform :

$$\rho(t, \mathbf{x}) \longrightarrow \rho'(t', \mathbf{x}') = \rho\big(t(t'),\, \mathbf{x}(t', \mathbf{x}')\big)$$

$$\mathbf{J} \longrightarrow \mathbf{J}'(t'(t),\, \mathbf{x}'(\mathbf{x}, t)) = \mathbf{J}\big(t(t'),\, \mathbf{x}(t', \mathbf{x}')\big) + \mathbf{v}\,\rho\big(t(t'),\, \mathbf{x}(t', \mathbf{x}')\big)$$

This reminds us of how $x^\mu = (ct, \mathbf{x})$ transforms in the non relativistic limit :

$$\begin{cases} t \to t' = t \\ \mathbf{x} \to \mathbf{x}' = \mathbf{x} + \mathbf{v}\,t \end{cases}$$

This motivates us into defining a new four-vector in a similar way of the 4-position, the **Four-current**, defined as :

$$J^\mu(x^\mu) = \begin{pmatrix} c\rho\,(\mathbf{x},t) \\ \mathbf{J}\,(\mathbf{x},t) \end{pmatrix}$$

The charge conservation then reads :

$$\partial_t \rho + \boldsymbol{\nabla}\cdot\mathbf{J} = 0 \Longleftrightarrow \partial_\mu J^\mu = 0$$

Under a Lorentz boost, the 4-current transforms as a contravariant vector :

$$J^{\mu\prime}(x') = \Lambda^\mu{}_\nu J^\nu(x(x')) = \Lambda^\mu{}_\nu J^\nu(\Lambda^{-1}x')$$

We can also define the **Four-potential** as :

$$A^\mu = \begin{pmatrix} \Phi \\ c\mathbf{A} \end{pmatrix}$$

We can see that this definition is well acquainted with the gauge symmetry condition :

$$\begin{cases} \Phi \;\to\; \Phi - \partial_t\alpha \\ A_i \;\to\; A_i + \partial_i\alpha \end{cases}$$

Which reads in covariant formalism :

$$A_\mu = \begin{pmatrix} -\Phi \\ c\mathbf{A} \end{pmatrix} \;\to\; A_\mu + c\,\partial_\mu\alpha = \begin{pmatrix} -\Phi + c\,\dfrac{\partial_t\alpha}{c} \\ cA_i + c\,\partial_i\alpha \end{pmatrix}$$

Now, recall the Maxwell's equations in Lorenz gauge, using the box operator defined as : $\Box \equiv \dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2} - \nabla^2 = -\partial_\mu\partial^\mu$, we have :

$$\Box\phi = \frac{\rho}{\varepsilon_0}, \quad \Box\mathbf{A} = \mu_0\mathbf{J} = \frac{1}{c^2\varepsilon_0}\mathbf{J}$$

with the Lorenz gauge condition :

$$\partial_\mu A^\mu = \frac{1}{c}\frac{\partial\phi}{\partial t} + c\nabla\cdot\mathbf{A} = 0$$

In terms of four-vectors, we can write the Maxwell's equations as :

$$\Box\begin{pmatrix} \Phi \\ c\mathbf{A} \end{pmatrix} = \frac{1}{c\varepsilon_0}\begin{pmatrix} c\rho \\ \mathbf{J} \end{pmatrix} \Longleftrightarrow \Box A^\mu = \frac{1}{c\varepsilon_0}J^\mu$$

8.3.2 Field strength

The next logical step is to derive the covariant form of the electric and magnetic fields **E** and **B**. To do so, let's define the **Field strength** :

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

This is a 4×4 covariant tensor, fully antisymmetric $F_{\mu\nu} = -F_{\nu\mu}$ and Gauge invariant :

$$F_{\mu\nu} \to \partial_\mu A_\nu + c\partial_\mu \partial_\nu \alpha - \partial_\nu A_\mu - c\partial_\nu \partial_\mu \alpha = F_{\mu\nu}$$

$F_{\mu\nu}$ transforms under a Lorentz boost as :

$$F_{\mu\nu}(x^\lambda) \to F'_{\mu\nu}(x'^\lambda) = \Lambda_\mu{}^\rho \Lambda_\nu{}^\gamma F_{\rho\gamma}(x^\lambda(x'))$$

Let's check what are the components of the field strength tensor. Recall first the fields :

$$E_i = -\frac{\partial \Phi}{\partial x_i} - \frac{\partial \mathbf{A}_i}{\partial t}, \qquad B_i = (\boldsymbol{\nabla} \times \mathbf{A})_i = \epsilon_{ijk} \partial_{x_j} A_k$$

The first line of the field strength tensor is given by :

$$F_{0\nu} = \partial_0 A_\nu - \partial_\nu A_0 = \frac{1}{c}\partial_t A_\nu + \partial_\nu \Phi = \begin{pmatrix} 0 & \dot{A}_1 + \partial_1 \Phi & \dot{A}_2 + \partial_2 \Phi & \dot{A}_3 + \partial_3 \Phi \end{pmatrix}$$

Thus, we have :

$$F_{0i} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \end{pmatrix}, \quad i = 1, 2, 3 \equiv x, y, z$$

Recall that the Greek indices μ, ν, ρ can take values from 0 to 3 whereas the latin indices i, j, k from 1 to 3. They label spatial coordinates only. For the other components, we can check for example :

$$F_{21} = \partial_2 A_1 - \partial_1 A_2 = c\partial_y \mathbf{A}_x - c\partial_x \mathbf{A}_y = -c\left(\boldsymbol{\nabla} \times \mathbf{A}\right)_z = -cB_z$$

This lead to the final form of the field strength :

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & cB_z & -cB_y \\ E_y & -cB_z & 0 & cB_x \\ E_z & cB_y & -cB_x & 0 \end{pmatrix}$$

And it's contravariant form :

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & cB_z & -cB_y \\ -E_y & -cB_z & 0 & cB_x \\ -E_z & cB_y & -cB_x & 0 \end{pmatrix}$$

8.3.3 Covariant Maxwell equations

Before writing Maxwell's equations in covariant form, let us briefly recall the role of **Lorentz transformations** and **gauge transformations** in electrodynamics.

Lorentz invariance. The electromagnetic theory must be invariant under Lorentz transformations:

$$x^\mu \;\to\; x'^\mu = \Lambda^\mu{}_\nu\, x^\nu, \qquad A^\mu \;\to\; A'^\mu = \Lambda^\mu{}_\nu A^\nu,$$

with $\Lambda^\mu{}_\nu$ satisfying $\eta_{\alpha\beta}\Lambda^\alpha{}_\mu\Lambda^\beta{}_\nu = \eta_{\mu\nu}$. The fields $\mathbf{E}$ and $\mathbf{B}$ then mix into each other in a consistent way. Lorentz invariance reflects a *physical symmetry* of spacetime.

Gauge invariance. In contrast, gauge transformations

$$A_\mu \;\to\; A_\mu + c\,\partial_\mu\alpha(x),$$

do not correspond to a physical symmetry. They express the fact that many different potentials correspond to the same physical fields $(\mathbf{E},\mathbf{B})$ or equivalently to the same field tensor $F_{\mu\nu}$. Gauge symmetry is therefore a *redundancy of description*.

Goal. We want equations that are:

- Lorentz covariant (so that the form of the equations is the same in all inertial frames),
- Gauge invariant (so that physics does not depend on the choice of potential).

This motivates the introduction of the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, since it is:

(i) antisymmetric, (ii) Lorentz covariant, (iii) gauge invariant.

Now that we have the field strength tensor $F_{\mu\nu}$, we want to rewrite Maxwell's equations in a compact and Lorentz-covariant form.
Recall the inhomogeneous Maxwell equations:

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0}, \qquad \nabla\times\mathbf{B} - \mu_0\varepsilon_0\,\partial_t\mathbf{E} = \mu_0\mathbf{J}.$$

In terms of the four-current $J^\mu = (c\rho,\,\mathbf{J})$, these two equations combine into a single covariant expression:

$$\partial_\mu F^{\mu\nu} = -\,\frac{1}{c\varepsilon_0}\,J^\nu.$$

This equation is Lorentz covariant (good) and gauge invariant (also good), since $F_{\mu\nu}$ does not change under $A_\mu \to A_\mu + c\,\partial_\mu\alpha$.

The homogeneous Maxwell equations,

$$\nabla\cdot\mathbf{B} = 0, \qquad \nabla\times\mathbf{E} + \partial_t\mathbf{B} = 0,$$

must also be written in covariant form.

To do so, introduce the Levi-Civita tensor $\varepsilon^{\mu\nu\rho\sigma}$, defined by

$$\varepsilon^{0123} = +1,$$

and fully antisymmetric under any exchange of indices. It is a Lorentz-invariant tensor density:

$$\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\Lambda^{\rho}{}_{\gamma}\Lambda^{\sigma}{}_{\delta}\,\varepsilon^{\alpha\beta\gamma\delta} = \varepsilon^{\mu\nu\rho\sigma}.$$

Using this object, the homogeneous Maxwell equations take the compact form:

$$\varepsilon^{\mu\nu\rho\sigma}\,\partial_\nu F_{\rho\sigma} = 0.$$

This expression encodes both $\nabla\cdot\mathbf{B} = 0$ and $\nabla\times\mathbf{E} + \partial_t\mathbf{B} = 0$.

Even more importantly, this equation is **identically satisfied** when $F_{\mu\nu}$ is expressed in terms of a potential A_μ:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Indeed,

$$\varepsilon^{\mu\nu\rho\sigma}\,\partial_\nu(\partial_\rho A_\sigma) = 0,$$

because $\partial_\nu\partial_\rho$ is symmetric under $\nu \leftrightarrow \rho$ whereas $\varepsilon^{\mu\nu\rho\sigma}$ is antisymmetric. Thus the homogeneous equations impose no additional constraint once we define $F_{\mu\nu}$ via a potential.

8.4 Covariant form of Newtonian mechanics

To solve the motion of charged objects, we need forces (Lorentz force). We also need Newton's law $\mathbf{F} = m\mathbf{a}$ in a Lorentz-covariant form.

We define the **proper time** as :

$$d\tau = \frac{1}{c}\sqrt{-dx^\mu dx_\mu} = dt\sqrt{1 - \frac{d\mathbf{x}^2}{c^2dt^2}} = dt\sqrt{1-\beta^2} = \frac{dt}{\gamma}.$$

Thus :

$$\Delta\tau = \int_{t_1}^{t_2} dt\sqrt{1-\beta^2}.$$

We define the **Four-velocity** as :

$$u^\mu = \frac{dx^\mu}{d\tau} = \gamma\begin{pmatrix} c \\ \mathbf{v}\end{pmatrix}$$

with a fixed norm $u^\mu u_\mu = \gamma^2(-c^2 + v^2) = -c^2$

Finally, the **Four-acceleration** can be defined in a similar way

$$a^\mu = \frac{du^\mu}{d\tau} = \gamma\frac{du^\mu}{dt} = \gamma\begin{pmatrix} c\dfrac{d\gamma}{dt} \\ \dfrac{d(\gamma\mathbf{v})}{dt}\end{pmatrix} \to_{\beta\to 1} \begin{pmatrix} 0 \\ \mathbf{a}\end{pmatrix}$$

with $\mathbf{a} = d_t\mathbf{v}$ the usual acceleration.

Recall the non-relativistic Lorentz force :

$$m\mathbf{a} = \mathbf{F}, \qquad \frac{d\mathbf{p}}{dt} = \mathbf{F}, \qquad \mathbf{F} = q\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right).$$

A natural question is: *what is the relativistic generalisation of Newton's law* $m\mathbf{a} = \mathbf{F}$ *?* Since u^μ is the 4-velocity, the only Lorentz-covariant way to define a force is through a tensor that already mixes electric and magnetic fields. The only available rank-2 tensor is the field strength $F^{\mu\nu}$.

Thus the only linear, Lorentz-covariant equation of motion is :

$$ma^\mu \;=\; \frac{q}{c}\,F^{\mu\nu}u_\nu.$$

This equation satisfies:

- linearity in the charge q (superposition principle),
- correct non-relativistic limit,
- gauge invariance, because $F^{\mu\nu}$ is gauge invariant.

In components, its spatial part reproduces the usual Lorentz force:

$$\frac{d\mathbf{p}}{dt} = q\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right).$$

Whereas its time component gives the work–energy relation. Indeed, taking $\mu = 0$ gives:

$$ma^0 = \frac{q}{c}F^{0\nu}u_\nu.$$

Using $F^{0i} = E_i$, $u_i = \gamma v_i$ and $a^0 = c\,d_t\gamma$, we have :

$$\frac{d}{dt}(\gamma mc^2) = q\,\mathbf{E}\cdot\mathbf{v}.$$

Thus the relativistic energy $\mathcal{E}$ satisfies

$$\frac{d\mathcal{E}}{dt} = q\,\mathbf{E}\cdot\mathbf{v}, \qquad \mathcal{E} = \gamma mc^2.$$

This is the relativistic work–energy theorem.

It is natural to define the **4-momentum** :

$$p^\mu = m\,u^\mu = \left(\frac{\mathcal{E}}{c}, \mathbf{p}\right), \qquad \mathbf{p} = \gamma m\mathbf{v}, \qquad a^\mu = \gamma\frac{dp^\mu}{dt}$$

The 4-momentum has an invariant norm:

$$p^\mu p_\mu = -m^2c^2,$$

which is the relativistic analogue of "mass is constant".

In the non-relativistic limit, expanding γ at small velocities gives :

$$\mathcal{E} = \gamma mc^2 = mc^2 + \frac{1}{2}mv^2 + \mathcal{O}(v^4/c^2).$$

Subtracting the rest energy mc^2, we recover the classical kinetic energy:

$$K = \frac{1}{2}mv^2.$$

Thus the covariant equation $ma^\mu = \frac{q}{c}F^{\mu\nu}u_\nu$ contains both Newton's law and the Lorentz force as limiting cases.

From the invariant norm of the 4-momentum, we can relate the energy to the momentum **p** :

$$p^\mu p_\mu = -m^2c^2,$$

Writing

$$p^\mu = \left(\frac{\mathcal{E}}{c}, \mathbf{p}\right), \qquad p_\mu = \left(-\frac{\mathcal{E}}{c}, \mathbf{p}\right),$$

the invariant norm gives

$$-\left(\frac{\mathcal{E}}{c}\right)^2 + \mathbf{p}^2 = -m^2c^2.$$

Multiplying by c^2 yields the fundamental **Energy–momentum relation** :

$$\mathcal{E}^2 = \mathbf{p}^2c^2 + m^2c^4.$$

This relation holds for any relativistic particle.

8.5 Relativistic motion in a constant electric field

We consider a particle of charge q and mass m in a constant, uniform electric field

$$\mathbf{E} = (E, 0, 0), \qquad \mathbf{B} = 0.$$

The Lorentz force gives

$$\frac{d\mathbf{p}}{dt} = q\mathbf{E} \quad \Longrightarrow \quad \begin{cases} \dfrac{dp_x}{dt} = qE, \\ \dfrac{dp_y}{dt} = 0, \end{cases}$$

so

$$p_x(t) = qEt + p_x^0, \qquad p_y(t) = p_y^0 = \text{const}.$$

The relativistic energy is

$$\mathcal{E} = \sqrt{m^2c^4 + p_x^2c^2 + p_y^2c^2}.$$

The velocity is related to the momentum by

$$\mathbf{v} = \frac{c^2\mathbf{p}}{\mathcal{E}}, \qquad \Rightarrow \qquad \frac{dx}{dt} = v_x = \frac{c^2p_x}{\mathcal{E}}, \quad \frac{dy}{dt} = v_y = \frac{c^2p_y}{\mathcal{E}}.$$

To simplify, choose the origin of time such that

$$t = 0 \;\Rightarrow\; p_x(0) = 0.$$

Then

$$p_x(t) = qEt, \qquad p_y(t) = p_y^0.$$

Define the constant

$$\kappa^2 \equiv m^2c^4 + (p_y^0)^2c^2 \quad \Rightarrow \quad \mathcal{E}(t) = \sqrt{\kappa^2 + (qEt)^2}.$$

We first study the **longitudinal motion** $x(t)$. We have :

$$\frac{dx}{dt} = \frac{c^2p_x}{\mathcal{E}} = \frac{c^2qEt}{\sqrt{\kappa^2 + (cqEt)^2}}$$

Then

$$x = \int_{x_0}^{x} dx' = c\int_{t_0}^{t} dt' \frac{cqEt'}{\sqrt{\kappa^2 + (cqEt')^2}}$$

Introduce

$$z = (cqEt')^2 \quad \Rightarrow \quad dz = 2(cqE)^2t'\,dt',$$

Then

$$x(t) = \frac{c}{2cqE}\int_{(cqEt_0)^2}^{(cqEt)^2} dz \frac{1}{\sqrt{\kappa^2 + z}} = x_0 + \frac{1}{qE}\sqrt{\kappa^2 + (cqEt)^2}$$

For large times, $cqEt \gg \kappa$, we have

$$\sqrt{\kappa^2 + (cqEt)^2} \simeq |cqEt|, \qquad \Rightarrow \quad x(t) \sim x_0 + ct,$$

so the particle asymptotically approaches the speed of light.
Inverting the last results allows to find $t(x)$:

$$cqEt = \sqrt{q^2E^2(x - x_0)^2 - \kappa^2}$$

The **Transverse motion** $y(t)$, can be treated similarly.

We are mostly interested in the **Trajectory** $x(y)$. First, write :

$$\frac{dx}{dy} = \frac{\frac{dx}{dt}}{\frac{dy}{dt}}$$

where we had :

$$\frac{dx}{dt} = \frac{c^2 qEt}{\sqrt{\kappa^2 + (cqEt)^2}}, \qquad \frac{dy}{dt} = \frac{c^2 p_y^0}{\sqrt{\kappa^2 + (cqEt)^2}}$$

Which yields :

$$\frac{dx}{dy} = \frac{cqEt}{cp_y^0} = \frac{1}{cp_y^0}\sqrt{q^2E^2(x-x_0)^2 - \kappa^2}$$

Introduce $w \equiv qE(x-x_0)$, $dw = qE\,dx$:

$$\frac{dx}{dy} = \frac{1}{cp_y^0}\sqrt{q^2E^2(x-x_0)^2 - \kappa^2} = \frac{1}{cp_y^0}\sqrt{w^2 - \kappa^2}.$$

Thus

$$dy = \frac{cp_y^0}{\sqrt{w^2-\kappa^2}}\,dx = \frac{cp_y^0}{qE}\frac{dw}{\sqrt{w^2-\kappa^2}}.$$

To integrate this, set

$$w = \kappa\cosh\alpha, \qquad dw = \kappa\sinh\alpha\,d\alpha, \qquad \sqrt{w^2-\kappa^2} = \kappa\sinh\alpha.$$

Then

$$dy = \frac{cp_y^0}{qE}\,\frac{\kappa\sinh\alpha\,d\alpha}{\kappa\sinh\alpha} = \frac{cp_y^0}{qE}\,d\alpha.$$

Integrating:

$$y = \frac{cp_y^0}{qE}\,\alpha + y_0.$$

Hence

$$\alpha = \frac{qE}{cp_y^0}(y-y_0).$$

Since

$$w = \kappa\cosh\alpha = qE(x-x_0),$$

we obtain the trajectory

$$x(y) = x_0 + \frac{\kappa}{qE}\cosh\left(\frac{qE}{cp_y^0}(y-y_0)\right).$$

Explicitly,

$$\boxed{x(y) = x_0 + \frac{1}{qE}\sqrt{m^2c^4 + (p_y^0)^2c^2}\,\cosh\left(\frac{qE}{cp_y^0}(y-y_0)\right)}.$$

For large y:

$$\cosh(C\,y) \simeq \tfrac{1}{2}e^{Cy}, \qquad x \sim C_1\,e^{(qE/cp_y^0)\,y}.$$

Thus

$$y \sim \log x, \qquad \dot{y} \sim \frac{1}{t}.$$

8.6 Relativistic motion in a constant magnetic field

We consider a relativistic particle of charge q and mass m in a uniform magnetic field, with no electric field :

$$\mathbf{E} = 0, \qquad \mathbf{B} = B_z\,\hat{\mathbf{z}}.$$

The Lorentz force gives :

$$\frac{d\mathbf{p}}{dt} = q\,\mathbf{v}\times\mathbf{B},$$

and since $\mathcal{E}^2 = m^2c^4 + p^2c^2$, we find that :

$$\frac{d\mathcal{E}}{dt} = \frac{c^2}{\mathcal{E}}\,\mathbf{p}\cdot\frac{d\mathbf{p}}{dt} = \frac{c^2}{\mathcal{E}}\,q\,\mathbf{p}\cdot(\mathbf{v}\times\mathbf{B}) = 0,$$

Which implies that $\mathcal{E}$ and therefore $|\mathbf{v}|$ are constant.
Using $\mathbf{p} = \gamma m\mathbf{v}$ with $\gamma = \mathcal{E}/(mc^2)$, we find that :

$$\frac{d\mathbf{v}}{dt} = \frac{q}{\gamma m}\,\mathbf{v}\times\mathbf{B} \equiv \mathbf{v}\times\boldsymbol{\omega}_B, \qquad \boldsymbol{\omega}_B = \frac{q\mathbf{B}}{\gamma m} = \frac{q\mathbf{B}c^2}{\mathcal{E}},$$

Where we have defined the relativistic cyclotron frequency ω_B. The equations of motion are the same as in Galilean mechanics :

$$\frac{dv_z}{dt} = 0, \qquad \frac{dv_x}{dt} = \omega_B v_y, \qquad \frac{dv_y}{dt} = -\omega_B v_x,$$

They describe a uniform rotation in the (x, y) plane.
For completely general initial conditions :

$$\begin{cases} x(0) = x_0 \\ y(0) = y_0 \\ z(0) = z_0 \end{cases} \qquad \begin{cases} v_x(0) = v_{x0} \\ v_y(0) = v_{y0} \\ v_z(0) = v_{z0} \end{cases}$$

The solution reads :

$$\begin{cases} v_x(t) = v_{x0}\cos(\omega_B t) + v_{y0}\sin(\omega_B t) \\ v_y(t) = v_{y0}\cos(\omega_B t) - v_{x0}\sin(\omega_B t) \\ v_z(t) = v_{z0} \end{cases}$$

and, after integration :

$$\begin{cases} x(t) = x_0 + \dfrac{v_{x0}}{\omega_B}\sin(\omega_B t) + \dfrac{v_{y0}}{\omega_B}\big(1 - \cos(\omega_B t)\big) \\ y(t) = y_0 + \dfrac{v_{y0}}{\omega_B}\sin(\omega_B t) - \dfrac{v_{x0}}{\omega_B}\big(1 - \cos(\omega_B t)\big) \\ z(t) = z_0 + v_{z0}t \end{cases}$$

The motion is therefore a circular orbit in the (x, y) plane with constant angular frequency.

$$\omega_B = \frac{qB_z}{\gamma m}, \qquad \gamma = \frac{1}{\sqrt{1 - \dfrac{v_{x0}^2 + v_{y0}^2 + v_{z0}^2}{c^2}}},$$

determined entirely by the initial velocity. The Larmor radius now reads :

$$\rho_L = \frac{v_\perp}{\omega_B} = \frac{\sqrt{v_{x0}^2 + v_{y0}^2}}{\omega_B} = \frac{\gamma m v_\perp}{qB_z}.$$

8.7 Radiation of a Moving Charge

We start from the non–relativistic Larmor formula (see Chapter 5), which gives the instantaneous radiated power of an accelerated charge :

$$d\mathcal{E} = \frac{q^2}{6\pi\varepsilon_0 c^5}\,\dot{\mathbf{u}}^2\,d\tau = c dp^0$$

where $\mathbf{u} = d\mathbf{x}/d\tau$ is the spatial part of the four–velocity.

Relativistic generalization To obtain a covariant expression valid for any velocity, it is natural to promote the Larmor formula to a four–vector relation. The only Lorentz–invariant scalar one can form from the four–acceleration $a^\mu = du^\mu/d\tau$ is:

$$a^\mu a_\mu = \frac{du^\mu}{d\tau}\frac{du_\mu}{d\tau}.$$

The radiated four–momentum must therefore be proportional to this invariant times the four–displacement dx^μ. This leads to the compact relativistic Larmor formula:

$$dp^\mu = \frac{q^2}{6\pi\varepsilon_0 c^5}\left(\frac{du^\nu}{d\tau}\frac{du_\nu}{d\tau}\right) dx^\mu.$$

This expression reduces to the usual Larmor formula when $v \ll c$, and it immediately implies that the radiated four–momentum is parallel to the particle's four–velocity.

To make contact with the three–dimensional formulation, we now compute explicitly the invariant $a^\nu a_\nu$.
Recall :

$$a^\mu = \frac{du^\mu}{d\tau} = \gamma\frac{du^\mu}{dt}$$

We first compute :

$$\frac{d\gamma}{dt} = \frac{d}{dt}(1-\beta^2)^{-1/2} = \gamma^3\,\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}$$

Thus

$$\frac{du^\mu}{dt} = c\left(\dot\gamma,\ \dot\gamma\,\boldsymbol{\beta} + \gamma\dot{\boldsymbol{\beta}}\right)$$

and multiplying by the factor γ :

$$a^\mu = c\gamma\left(\dot\gamma,\ \dot\gamma\,\boldsymbol{\beta} + \gamma\dot{\boldsymbol{\beta}}\right)$$

We now evaluate the invariant :

$$a^\mu a_\mu = c^2\gamma^2\left[-\dot\gamma^2 + \left|\dot\gamma\,\boldsymbol{\beta} + \gamma\dot{\boldsymbol{\beta}}\right|^2\right]$$

Using$\dot\gamma = \gamma^3\,(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})$, one arrives, after simplification, at the identity:

$$\boxed{a^\mu a_\mu = \gamma^6\left[\dot{\boldsymbol{\beta}}^2 - (\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}})^2\right]c^2.}$$

This is the key relativistic generalization of the non–relativistic acceleration squared. Plugging this result into the covariant radiation formula gives the total radiated power:

$$P = \frac{d\mathcal{E}}{dt} = \frac{q^2}{6\pi\varepsilon_0 c^3}\,\gamma^6\left[\dot{\boldsymbol{\beta}}^2 - (\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}})^2\right].$$

This is known as the *Liénard formula*. It reduces to the familiar Larmor expression in the non–relativistic limit $\beta \ll 1$, for which $\gamma \to 1$ and $\dot{\boldsymbol{\beta}} \simeq \dot{\mathbf{v}}/c$.
A full derivation using the Liénard–Wiechert potentials yields the radiated power per solid angle:

$$\frac{d\mathcal{E}}{d\Omega\, dt} = \frac{q^2}{16\pi^2\varepsilon_0 c}\frac{\left|\mathbf{n}\times\left[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]\right|^2}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^6}.$$

Here $\mathbf{n}$ is the direction to the observer. Integrating this expression over Ω reproduces the total radiated power P above.
Using the Lorentz force $\frac{q}{c}F^{\mu\nu}u_\nu$, one can make use of Newton's second law to rewrite the Larmor formula as :

$$\frac{d\mathcal{E}}{dt} = \frac{q^2}{6\pi\epsilon_0 c^3}a^\mu a_\mu = \frac{q^4}{6\pi\varepsilon_0 m^2 c^5}\,F^{\mu\nu}u_\nu F_{\mu\rho}u^\rho$$

8.8 Electromagnetic stress-energy tensor

We define the **Electromagnetic stress-tensor** as the following quantity :

$$T^{\mu\nu} = \varepsilon_0\left(F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right)$$

This symmetric tensor encodes four conservation laws (one for energy and three for momentum). To see this, we first look at its components in terms of $\mathbf{E}$ and $\mathbf{B}$.
A straightforward calculation gives :

$$T^{00} = \frac{\varepsilon_0}{2}\left(E^2 + c^2B^2\right) \equiv u_{\text{em}}$$

$$T^{0i} = \frac{1}{c}\,S_i = \frac{1}{\mu_0 c}\,(\mathbf{E}\times\mathbf{B})_i$$

$$T^{ij} = -\varepsilon_0\Big(E_iE_j + c^2B_iB_j - \tfrac{1}{2}\delta_{ij}(E^2 + c^2B^2)\Big) \equiv -\sigma_{ij}$$

where $\mathbf{S} = \frac{1}{\mu_0}\,\mathbf{E}\times\mathbf{B}$ is the Poynting vector (energy flux density) and u_{em} is the electromagnetic energy density. Thus :

- T^{00} is the energy density
- T^{0i} (or T^{i0}) is the energy flux density (Poynting vector) along spatial direction i

- T^{ij} is the *Maxwell stress tensor*, describing momentum flow and mechanical stresses carried by the field.

We can reduce the stress-energy tensor to :

$$T^{\mu\nu} = \begin{pmatrix} u_{\text{em}} & \mathbf{S}/c \\ \mathbf{S}/c & -\sigma_{ij} \end{pmatrix}$$

Using Maxwell's equations in covariant form :

$$\partial_\mu F^{\mu\nu} = -\frac{J^\nu}{c\varepsilon_0}$$

as well as the *Bianchi identity* :

$$\partial^\nu F^{\alpha\beta} + \partial^\alpha F^{\beta\nu} + \partial^\beta F^{\nu\alpha} = 0$$

One can find through some lines of algebra that the conservation laws reads in a compact form :

$$\partial_\mu T^{\mu\nu} = \frac{1}{c} J_\alpha F^{\alpha\nu} \equiv -f^\nu$$

Where the right–hand side is minus the *Lorentz force density* $f^\nu = \frac{1}{c} J_\alpha F^{\nu\alpha}$, whose components are :

$$\text{Work done on charges : } f^0 = \frac{1}{c}\mathbf{J}\cdot\mathbf{E}$$
$$\text{Lorentz force density : } f^i = \rho E_i + (\mathbf{J}\times\mathbf{B})_i$$

Let's see explicitly the conservation laws by computing each components.

- For $\nu = 0$:

$$\frac{\partial u_{\text{em}}}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{S} + \mathbf{J}\cdot\mathbf{E} = 0$$

which is the **Poynting theorem/energy conservation law** : the variation of electromagnetic energy density is due to the flux of energy density (Poynting vector) and the dissipation of energy via Joule effect (work done on charges).

- For $\nu = i$:

$$\frac{1}{c^2}\frac{\partial \mathbf{S}_i}{\partial t} - \partial_k \sigma_{ki} + \rho \mathbf{E}_i + (\mathbf{J}\times\mathbf{B})_i = 0$$

Which can be written :

$$\frac{\partial \mathbf{p}_{\text{em}}}{\partial t} - \boldsymbol{\nabla}\cdot\sigma + \rho\mathbf{E} + \mathbf{J}\times\mathbf{B} = 0$$

with $\mathbf{P}_{\text{em}} = \mathbf{S}/c^2$ is the *electromagnetic momentum density.* This law corresponds to a **momentum conservation law** : the time variation of the electromagnetic momentum density is exactly balanced by the divergence of the Maxwell stress tensor and by the Lorentz force density acting on charges and currents.

8.9 Transformations of EM fields under Lorentz boosts

To conclude this chapter, we now want to understand how the electric and magnetic fields transform from one inertial frame to another. Recall the antisymmetric field tensor :

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & cB_z & -cB_y \\ E_y & -cB_z & 0 & cB_x \\ E_z & cB_y & -cB_x & 0 \end{pmatrix}.$$

Under a Lorentz transformation $\Lambda^\mu{}_\alpha$ the tensor transforms as :

$$F^{\mu\nu} \longrightarrow F'^{\mu\nu} = \Lambda^\mu{}_\alpha\, \Lambda^\nu{}_\beta\, F^{\alpha\beta}.$$

Let us consider a Lorentz boost in the x–direction. For a boost with velocity $v = \beta c$, the matrix reads :

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

As an example, let us compute F'^{01} explicitly:

$$F'^{01} = \Lambda^0{}_\alpha \Lambda^1{}_\beta F^{\alpha\beta} = \gamma^2(1-\beta^2)F^{01} = F^{01}.$$

Thus :

$$E'_x = E_x.$$

Repeating the calculation for the other components of $F_{\mu\nu}$ gives the well–known transformation laws:

$$\begin{aligned} \mathbf{E}'_\parallel &= \mathbf{E}_\parallel, \\ \mathbf{E}'_\perp &= \gamma\left(\mathbf{E}_\perp + c\,\boldsymbol{\beta}\times\mathbf{B}\right), \\ \mathbf{B}'_\parallel &= \mathbf{B}_\parallel, \\ \mathbf{B}'_\perp &= \gamma\left(\mathbf{B}_\perp - \frac{1}{c}\boldsymbol{\beta}\times\mathbf{E}\right). \end{aligned}$$

A few comments :

- Components parallel to the boost remain unchanged.
- Perpendicular components mix electric and magnetic fields.

- A purely electric field in one frame generally appears as a mixture of **E** and **B** in another.

This mixing is a direct consequence of the fact that $F_{\mu\nu}$ is the "unified" relativistic object containing both fields. A Lorentz transformation acts on $F_{\mu\nu}$ linearly, and the usual 3–vector formulas follow automatically. Notice that the structure of these transformations is essentially the same as for space–time coordinates under a Lorentz boost.

Chapter 9

Action principle in classical electrodynamics

We now want to formulate our classical electrodynamics laws from a single object : the **action**. The idea is simple : write down the most general Lorentz-invariant and gauge-invariant quantity, our action, and derive both Maxwell equations and the Lorentz force from a variational principle. Recall the definition of the action :

$$S = \int dt \mathcal{L}, \quad \mathcal{L} = \int d^3x L$$

Where L is the *Lagrangian density* and $\mathcal{L}$ the *Lagrangian*. In this chapter, the action will be derived mainly from symmetry considerations. First, we split the total action as :

$$S = S_f + S_m + S_{mf}$$

where:

- S_f : free electromagnetic field
- S_m : matter part (we take a single relativistic point charge)
- S_{mf} : interaction between charge and field

9.1 Action of a point charge S_m

Consider a point particle of mass m moving along a worldline :

$$x^\mu(\lambda), \qquad \mu = 0, 1, 2, 3$$

Here λ is just a parameter labeling the points along the trajectory. It has no real physical meaning. We could choose $\lambda = t$, the coordinate time, or the proper time τ, or any weird parametrization like $\lambda = \tan(t)$, the physics cannot depend on this choice. This is called *reparameterization invariance*.

Recall the (invariant) length of an infinitesimal displacement in spacetime given by :

$$ds^2 = -c^2 d\tau^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu,$$

where τ is the particle's proper time.
For a *timelike* trajectory,

$$d\tau = \frac{1}{c}\sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu}.$$

Thus, the total proper time experienced by the particle is

$$\int d\tau = \frac{1}{c}\int \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu}.$$

The action for a free relativistic particle must satisfy two constraints:

1. **Lorentz invariance** : It must be built from invariant scalars such as $ds = \sqrt{-dx^\mu dx_\mu}$.
2. **No preferred parametrization** : The action cannot depend on how we choose λ to run along the worldline.

The only scalar quantity with the right physical dimension is the invariant proper length of the worldline, so the natural choice is :

$$S_m = -mc \int d\tau.$$

The minus sign is standard (it gives the correct Newtonian limit $L = T - V$ when expanded for small velocities).
If the trajectory is written as $x^\mu(\lambda)$, then :

$$dx^\mu = \dot{x}^\mu\, d\lambda, \qquad \dot{x}^\mu = \frac{dx^\mu}{d\lambda}$$

Using $d\tau = \frac{1}{c}\sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu} = \frac{1}{c}\sqrt{-\dot{x}^\mu \dot{x}_\mu}\, d\lambda$, the action can be written equivalently as:

$$S_m = -mc \int d\tau = -mc \int d\lambda\, \sqrt{-\dot{x}^\mu \dot{x}_\mu}.$$

This rewriting makes it clear that the action does not depend on the specific choice of λ.
If we change the parametrization :

$$\lambda \longrightarrow \lambda' = f(\lambda),$$

then :

$$\dot{x}^\mu = \frac{dx^\mu}{d\lambda} \quad \longrightarrow \quad x^{\mu\prime} = \frac{dx^\mu}{d\lambda'} = \frac{dx^\mu}{d\lambda}\frac{d\lambda}{d\lambda'} = \dot{x}^\mu \left(\frac{d\lambda}{d\lambda'}\right).$$

Thus :

$$\sqrt{-\dot{x}^\mu \dot{x}_\mu}\, d\lambda \quad \longrightarrow \quad \sqrt{-x^{\mu\prime} x'_\mu}\, d\lambda' = \sqrt{-\dot{x}^\mu \dot{x}_\mu}\, d\lambda.$$

So the integrand transforms in such a way that the product $\sqrt{-\dot{x}^\mu \dot{x}_\mu}\, d\lambda$ stays invariant. Therefore the action does not change:

$$S'_m = S_m.$$

This proves the reparameterization invariance. So the action we wrote down is indeed the right one for the assumptions we started from.

9.2 Field Action S_f

For the electromagnetic field, we want an action that is Lorentz invariant, gauge invariant, and involving the minimal number of derivatives. Since A_μ is not gauge invariant, the action must be built from the field strength tensor $F_{\mu\nu}$, which is both Lorentz covariant and gauge invariant.

At leading order, the only Lorentz scalar we can form is $F_{\mu\nu}F^{\mu\nu}$ (the other possible scalar $\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$ is a total derivative and does not affect the equations of motion).

Therefore the field action is uniquely fixed (up to an overall constant) as:

$$S_g = -\frac{\varepsilon_0}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu},$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

The normalization is chosen so that the usual energy density $\frac{\varepsilon_0}{2}(E^2 + c^2B^2)$ is recovered. Indeed, expanding the invariant :

$$F_{\mu\nu}F^{\mu\nu} = 2\left(c^2B^2 - E^2\right),$$

the action becomes

$$S_g = \frac{\varepsilon_0}{2}\int d^4x\left(E^2 - c^2B^2\right).$$

With this convention,

$$L_g = \frac{\varepsilon_0}{2}(E^2 - c^2B^2),$$

which yields the correct energy density :

$$u_{\text{em}} = \frac{\varepsilon_0}{2}(E^2 + c^2B^2),$$

and reproduces Maxwell's equations with the standard definitions of **E**, **B**, and the Poynting vector.

This fixes the coefficient $-\varepsilon_0/4$ in front of $F_{\mu\nu}F^{\mu\nu}$ and ensures full consistency with the dynamical and energetic quantities introduced earlier.

9.3 Interaction Action S_{mf}

To couple the particle to the electromagnetic field, the action must again respect Lorentz invariance, gauge invariance, and involve the lowest number of derivatives. Since the particle follows a worldline $x^\mu(\lambda)$, the only available 4-vector at that point is the gauge potential A_μ evaluated along the trajectory. The natural Lorentz scalar we can form with one derivative along the worldline is therefore $A_\mu \dot{x}^\mu$.
This leads to the interaction action

$$S_{mf} = -\frac{q}{c} \int A_\mu(x(\lambda))\ \dot{x}^\mu\, d\lambda.$$

The factor q/c is the correct normalization ensuring that, after variation, the equation of motion contains the Lorentz force $qF_{\mu\nu}\dot{x}^\nu/c$ and matches standard electromagnetism. This is the unique lowest-order coupling consistent with dimensions and symmetries.

Under a gauge transformation $A_\mu \to A_\mu + \partial_\mu \alpha$, the action transforms as :

$$S_{mf} \to S_{mf} - \frac{q}{c} \int (\partial_\mu \alpha)\, \dot{x}^\mu\, d\lambda.$$

But along the worldline,

$$(\partial_\mu \alpha)\dot{x}^\mu = \frac{\partial \alpha}{\partial x^\mu} \frac{\partial x^\mu}{\partial \lambda} = \frac{d\alpha}{d\lambda},$$

so the variation becomes :

$$-\frac{q}{c} \int \frac{d\alpha}{d\lambda}\, d\lambda = -\frac{q}{c} \big[\alpha(\lambda_f) - \alpha(\lambda_i)\big].$$

Since the endpoints of the variation are fixed, this boundary term does not contribute. Therefore,

$$S'_{mf} = S_{mf}.$$

This shows that the interaction action is manifestly gauge invariant, as expected for a consistent coupling between matter and the electromagnetic field.

9.4 Maxwell's equations and Lorentz force from least action principle

9.4.1 Variation with respect to the trajectory $x^\mu(\lambda)$

We now vary the combined action $S_m + S_{mf}$ with respect to the worldline $x^\mu(\lambda)$, keeping the gauge field $A_\mu(x)$ fixed. This will give the relativistic equation of motion, i.e. the Lorentz force law.

Variation of the matter action

Starting from :

$$S_m = -mc \int d\lambda \sqrt{-\dot{x}^2},$$

We vary with respect to $x^\mu(\lambda)$. Since the integrand depends on x^μ only through $\dot{x}^\mu$, the variation gives

$$\delta S_m = -mc \int d\lambda \frac{\dot{x}_\mu}{\sqrt{-\dot{x}^2}} \delta\dot{x}^\mu.$$

Integrating by parts (and using $\delta x^\mu(\lambda_i) = \delta x^\mu(\lambda_f) = 0$), we obtain

$$\delta S_m = mc \int d\lambda \frac{d}{d\lambda}\left(\frac{\dot{x}_\mu}{\sqrt{-\dot{x}^2}}\right) \delta x^\mu.$$

The quantity

$$u^\mu = \frac{dx^\mu}{d\tau} \qquad \Longleftrightarrow \qquad \frac{\dot{x}^\mu}{\sqrt{-\dot{x}^2}}$$

is the 4-velocity, so the term above will later become $m\, du_\mu/d\tau$.

Variation of the interaction action

The interaction part is :

$$S_{mf} = -\frac{q}{c} \int A_\mu(x)\, \dot{x}^\mu\, d\lambda.$$

Varying this with respect to x^μ gives two contributions :

$$\delta S_{mf} = -\frac{q}{c} \int [(\partial_\nu A_\mu)\dot{x}^\mu\, \delta x^\nu + A_\mu\, \delta\dot{x}^\mu]\, d\lambda.$$

Integrating the second term by parts yields :

$$\delta S_{mf} = -\frac{q}{c} \int \left[\partial_\nu A_\mu\, \dot{x}^\mu - \frac{dA_\nu}{d\lambda}\right] \delta x^\nu\, d\lambda.$$

Using :

$$\partial_\nu A_\mu\, \dot{x}^\mu - \frac{dA_\nu}{d\lambda} = F_{\nu\mu}\dot{x}^\mu,$$

we obtain the compact form :

$$\delta S_{mf} = -\frac{q}{c} \int F_{\nu\mu}\dot{x}^\mu\, \delta x^\nu\, d\lambda.$$

Equation of motion

Adding both variations gives :

$$\delta(S_m + S_{mf}) = \int d\lambda \left[mc \frac{d}{d\lambda}\left(\frac{\dot{x}_\nu}{\sqrt{-\dot{x}^2}}\right) - \frac{q}{c}F_{\nu\mu}\dot{x}^\mu\right] \delta x^\nu.$$

Since δx^ν is arbitrary, the term in brackets must vanish:

$$mc\,\frac{d}{d\lambda}\left(\frac{\dot{x}_\nu}{\sqrt{-\dot{x}^2}}\right) = \frac{q}{c}F_{\nu\mu}\dot{x}^\mu.$$

Switching from λ to the proper time τ gives the standard, manifestly covariant Lorentz force :

$$m\,\frac{du_\nu}{d\tau} = \frac{q}{c}F_{\nu\mu}u^\mu.$$

9.4.2 Variation with respect to the gauge field $A_\mu(x)$

We now vary the action with respect to the gauge potential A_μ. This will give the inhomogeneous Maxwell equations.

Variation of the field part

Starting from :

$$S_f = -\frac{\varepsilon_0}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu},$$

and using :

$$\delta F_{\mu\nu} = \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu,$$

integration by parts gives :

$$\delta S_f = \varepsilon_0 \int d^4x\,(\partial_\mu F^{\mu\nu})\,\delta A_\nu.$$

Variation of the interaction part

From :

$$S_{mf} = -\frac{q}{c}\int d\lambda\, A_\mu(x(\lambda))\dot{x}^\mu,$$

we obtain :

$$\delta S_{mf} = -\frac{q}{c}\int d\lambda\,\dot{x}^\mu \delta A_\mu(x(\lambda)) = -\frac{1}{c}\delta A_\mu(x)J^\mu = -\frac{1}{c}\int d^4x\, J^\nu(x)\,\delta A_\nu,$$

with the 4-current :

$$J^\nu(x) = q\int d\lambda\,\dot{x}^\nu\,\delta^{(4)}\big(x - x(\lambda)\big).$$

Maxwell equations

Setting $\delta(S_f + S_{mf}) = 0$ for arbitrary δA_ν gives:

$$\partial_\mu F^{\mu\nu} = \frac{1}{c\varepsilon_0}\,J^\nu.$$

These are the *inhomogeneous Maxwell equations*. The homogeneous equations

$$\varepsilon^{\alpha\nu\beta\gamma}\partial_\alpha F_{\beta\gamma} = 0$$

follow automatically from the definition of $F_{\mu\nu}$ (Bianchi identity).

Chapter 10

Dispersion and Kramers-Kronig relations

10.1 Linear response in a dispersive medium

In this chapter we come back to EM fields in matter, but now at a more advanced level. We want to understand:

- absorption of EM waves (imaginary part of the response),
- non-locality in time and dispersion in frequency,
- analytic properties of the dielectric function $\varepsilon(\omega)$,
- Kramers–Kronig relations,
- an example: resonant absorption on a single transition.

We start from the microscopic picture of a dielectric medium : charges are bound in atoms or molecules and develop electric dipole moments in response to an applied field. Averaging over many microscopic dipoles leads to the *polarization* :

$$\mathbf{P}(\mathbf{x}) = \sum_n \mathbf{d}_n \, f(\mathbf{x} - \mathbf{x}_n),$$

where $\mathbf{d}_n$ is the dipole moment of the n-th particle and f is a smearing function that averages over microscopic structure.
The averaged (macroscopic) charge density is

$$\langle\eta\rangle(\mathbf{x}) = \rho_f(\mathbf{x}) - \boldsymbol{\nabla} \cdot \mathbf{P}(\mathbf{x}),$$

where ρ_f is the free charge density. From this we get the macroscopic/averaged Maxwell equations :

$$\boldsymbol{\nabla} \cdot \mathbf{D} = \rho_f, \qquad \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}.$$

Similarly, for magnetism we define the magnetization $\mathbf{M}$ and write

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}.$$

In terms of $\mathbf{D}$ and $\mathbf{H}$, Maxwell's equations become

$$\begin{aligned}
\nabla \cdot \mathbf{D} &= \rho_f, \\
\nabla \cdot \mathbf{B} &= 0, \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\
\nabla \times \mathbf{H} &= \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t},
\end{aligned}$$

where $\mathbf{J}_f$ is the free current density.
To solve concrete problems we need *constitutive relations* connecting $\mathbf{D}, \mathbf{H}$ to $\mathbf{E}, \mathbf{B}$.

Linear, isotropic, static response In the simplest case of a linear, isotropic, homogeneous medium (and for static fields) we assume

$$\mathbf{D} = \varepsilon\, \mathbf{E}, \qquad \mathbf{B} = \mu\, \mathbf{H},$$

with constant ε, μ. This is the standard story used for electrostatics and magnetostatics. However, for time-dependent fields (EM waves in matter) this simple local, instantaneous relation is not enough. Two new effects appear :

- **absorption** : ε and/or μ acquire an imaginary part, the wave loses energy in the medium.
- **dispersion** : the response is non-local in time, the field at time t depends on the field at earlier times $t' < t$.

These two effects turn out to be closely related.

General linear response in time domain We now assume a linear, homogeneous medium and focus on the electric response (we keep μ real and constant for simplicity). The most general linear, causal relation between $\mathbf{D}$ and $\mathbf{E}$ is

$$\mathbf{D}(t) = \varepsilon_0\, \mathbf{E}(t) + \varepsilon_0 \int_0^{\infty} d\tau\, \chi(\tau)\, \mathbf{E}(t-\tau),$$

where $\chi(\tau)$ is the *electric susceptibility* in the time domain.
Key points:

- The integral starts at $\tau = 0$ (no dependence on future times): this is **causality**.
- We still assume locality in space (no spatial convolution), but not in time.

We now go to frequency space. We define the Fourier transforms :

$$\mathbf{E}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\mathbf{E}}(\omega)\, e^{-i\omega t}\, d\omega, \qquad \mathbf{D}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\mathbf{D}}(\omega)\, e^{-i\omega t}\, d\omega,$$

and similarly for other fields.

Using the convolution structure in time, the Fourier transform of $\mathbf{D}(t)$ gives :

$$\begin{aligned}\tilde{\mathbf{D}}(w) &= \varepsilon_0 \int_{-\infty}^{\infty} \frac{dt}{\sqrt{2\pi}} \mathbf{E}(t)\, e^{i\omega t} + \varepsilon_0 \int_0^{\infty} d\tau \int_{-\infty}^{\infty} \frac{dt}{\sqrt{2\pi}} \chi(\tau)\mathbf{E}(t-\tau)\, e^{i\omega t} \\ &= \varepsilon_0 \tilde{\mathbf{E}}(\omega) + \varepsilon_0 \tilde{\mathbf{E}}(\omega) \int_0^{\infty} d\tau\, \chi(\tau)\, e^{i\omega\tau}\end{aligned}$$

We can write in a more compact way :

$$\mathbf{D}(\omega) = \varepsilon(\omega)\, \mathbf{E}(\omega),$$

where

$$\varepsilon(\omega) = \varepsilon_0 \left[1 + \int_0^{\infty} d\tau\ \chi(\tau)\, e^{i\omega\tau} \right].$$

Thus the medium is characterized by a *frequency-dependent* dielectric function $\varepsilon(\omega)$.
We can write :

$$\varepsilon(\omega) = \varepsilon'(\omega) + i\, \varepsilon''(\omega),$$

with real and imaginary parts ε' and ε''.
Because $\chi(\tau)$ is real, one finds

$$\varepsilon(-\omega) = \varepsilon^*(\omega),$$

so that

$$\varepsilon'(-\omega) = \varepsilon'(\omega), \qquad \varepsilon''(-\omega) = -\varepsilon''(\omega).$$

In dielectric media, the static limit $\omega \to 0$ corresponds to the usual real static susceptibility:

$$\varepsilon(0) = \varepsilon_0\, \varepsilon_{\text{stat}}, \quad \text{with} \quad \varepsilon_{\text{stat}} \in \mathbb{R}.$$

Analyticity in the upper half-plane We now treat $\varepsilon(\omega)$ as a complex function of ω. Because the integral defining $\varepsilon(\omega)$ starts at $\tau = 0$ and is damped for large τ (finite memory), the integral :

$$\int_0^{\infty} \chi(\tau)\, e^{i\omega\tau}\, d\tau$$

converges for all complex ω with $\operatorname{Im}\omega > 0$:

$$e^{i\omega\tau} = e^{i\text{Re}\,\omega\,\tau} e^{-\text{Im}\,\omega\,\tau}.$$

Hence $\varepsilon(\omega)$ is analytic in the upper half-plane. For very large real frequencies, the external electric field oscillates so fast that the bound charges in the medium cannot follow its motion. Physically, the electrons behave almost like free particles : the restoring forces inside atoms do not have time to act during one period of the oscillation.
To see this more explicitly, consider the simple equation of motion for a charged particle of charge e and mass m driven by a high-frequency field $E(t) = E_0 e^{-i\omega t}$:

$$m\, \ddot{x} = eE_0 e^{-i\omega t}.$$

For large ω we can neglect all slower dynamics inside the atom (binding forces, damping, etc.), and solve (integrate over time twice) :

$$x(t) \simeq -\frac{eE_0}{m\omega^2}\, e^{-i\omega t}.$$

The induced dipole moment is $p = -ex \propto E_0/\omega^2$, and the macroscopic polarization scales the same way:

$$\mathbf{P}(\omega) \propto \frac{1}{\omega^2}\,\mathbf{E}(\omega).$$

Using $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$, this implies that

$$\varepsilon(\omega) = \varepsilon_0 \left(1 + \frac{\text{const}}{\omega^2}\right), \qquad |\omega| \to \infty,$$

or equivalently

$$\frac{\varepsilon(\omega)}{\varepsilon_0} \simeq 1 + \frac{\text{const}}{\omega^2}.$$

Thus the medium becomes "transparent" at very high frequencies, and the dielectric function approaches its vacuum value with a universal $1/\omega^2$ tail.

10.2 Cauchy's principle value

Before deriving the Kramers-Kronig relations, we make a little mathematical interlude that will be useful later. We define Cauchy's principle value as the following quantity :

$$\mathcal{P}\int dx\,\frac{1}{x}f(x) = \lim_{\delta\to 0}\left[\int_{-\infty}^{-\delta} dx\,\frac{1}{x}f(x) + \int_{\delta}^{\infty} dx\,\frac{1}{x}f(x)\right]$$

Without going deeper in the details, this limit exists for any smooth functions f.
We now introduce the distributions :

$$\frac{1}{x+i\delta} \qquad \text{and} \qquad \frac{1}{x-i\delta},$$

with $\delta > 0$, and study their behaviour as $\delta \to 0^+$. These objects appear naturally when evaluating integrals slightly above or below the real axis in complex analysis.
Consider first the following combination :

$$\int_{-\infty}^{\infty} dx\,\frac{f(x)}{x \pm i\delta}.$$

We can rewrite the denominator as :

$$\frac{1}{x\pm i\delta} = \frac{x \mp i\delta}{x^2+\delta^2} = \frac{x}{x^2+\delta^2} \mp i\,\frac{\delta}{x^2+\delta^2}.$$

Both terms have a well-defined distributional limit. The first one approaches the principal value :

$$\lim_{\delta\to 0}\frac{x}{x^2+\delta^2} = \mathcal{P}\frac{1}{x}.$$

The second term tends to a representation of the Dirac delta :

$$\lim_{\delta\to 0} \frac{1}{\pi} \frac{\delta}{x^2+\delta^2} = \delta(x).$$

Up to the constant π, the imaginary part thus becomes a delta function.
Putting everything together, we find the fundamental identity:

$$\frac{1}{x \pm i0} = \mathcal{P}\frac{1}{x} \mp i\pi\,\delta(x)$$

This identity is essential when taking limits from the upper or lower half-plane in contour integrals. It tells us that:

- the *real part* of $1/(x \pm i0)$ gives the principal value contribution,
- the *imaginary part* produces a delta peak at $x = 0$,
- choosing $+i0$ or $-i0$ determines the sign of this delta function.

10.3 Kramers–Kronig relations

Because $\varepsilon(\omega)$ is analytic in the upper half-plane (causality) and decays as $1/\omega^2$ for large $|\omega|$, we can apply Cauchy's theorem to relate its values on the real axis. Introduce the function :

$$F(z) = \frac{\varepsilon(z)}{\varepsilon_0} - 1,$$

and consider the Cauchy integral :

$$I(z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{F(x)}{x - z}\, dx, \qquad \operatorname{Im} z > 0.$$

Since $F(z)$ is analytic for $\operatorname{Im} z > 0$ and falls at least as $1/z^2$, the contribution of the large semicircle in the upper half-plane vanishes. Closing the contour above the real axis therefore gives, by Cauchy's theorem,:

$$I(z) = F(z) \qquad (\operatorname{Im} z > 0).$$

To obtain the limit on the real axis, take $z \to \omega + i0$ with $\omega \in \mathbb{R}$. Using the identity found above :

$$\frac{1}{x - \omega + i0} = \mathcal{P}\frac{1}{x - \omega} - i\pi\,\delta(x - \omega),$$

we get :

$$F(\omega + i0) = \frac{1}{2\pi i} \left[\mathcal{P} \int_{-\infty}^{\infty} \frac{F(x)}{x - \omega}\, dx - i\pi F(\omega) \right].$$

Separating real and imaginary parts and using

$$F(\omega) = \frac{\varepsilon(\omega)}{\varepsilon_0} - 1 = F'(\omega) + iF''(\omega),$$

we obtain the Kramers–Kronig relations:

$$\frac{\varepsilon'(\omega)}{\varepsilon_0} - 1 = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\varepsilon''(x)/\varepsilon_0}{x-\omega}\,dx,$$
$$\frac{\varepsilon''(\omega)}{\varepsilon_0} = -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\varepsilon'(x)/\varepsilon_0 - 1}{x-\omega}\,dx.$$

Using the symmetry $\varepsilon(-\omega) = \varepsilon^*(\omega)$, one can also rewrite them as integrals over positive frequencies only:

$$\frac{\varepsilon'(\omega)}{\varepsilon_0} = 1 + \frac{2}{\pi}\mathcal{P}\int_{0}^{\infty} \frac{x\,\varepsilon''(x)/\varepsilon_0}{x^2-\omega^2}\,dx,$$
$$\frac{\varepsilon''(\omega)}{\varepsilon_0} = -\frac{2\omega}{\pi}\mathcal{P}\int_{0}^{\infty} \frac{\varepsilon'(x)/\varepsilon_0 - 1}{x^2-\omega^2}\,dx.$$

Physical meaning. These relations tell us that:

- If we know the absorption spectrum $\varepsilon''(\omega)$ at all frequencies, we can reconstruct the full dispersion $\varepsilon'(\omega)$.
- Conversely, a measured dispersion law fixes the absorption.

This is a direct consequence of causality.

10.4 Example : resonant absorption on a single line

A standard model for a resonant transition is :

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \frac{\lambda}{\omega_0^2 - \omega^2 - i\gamma\omega},$$

with resonance frequency ω_0, damping rate γ and oscillator strength λ.
Separating real and imaginary parts :

$$\frac{\varepsilon'(\omega)}{\varepsilon_0} = 1 + \lambda\,\frac{\omega_0^2-\omega^2}{D(\omega)}, \qquad \frac{\varepsilon''(\omega)}{\varepsilon_0} = \lambda\,\frac{\gamma\omega}{D(\omega)},$$

with $D(\omega) = (\omega_0^2-\omega^2)^2 + (\gamma\omega)^2$. Let's assume that we are close to the resonance frequency of the transition, i.e. for $\omega = \omega_0 + \Delta$ with $|\Delta| \ll \omega_0$ and $\gamma \ll \omega_0$:

$$\omega_0^2 - \omega^2 \simeq -2\omega_0\Delta, \qquad D(\omega) \simeq \omega_0^2\big(4\Delta^2 + \gamma^2\big),$$

so that the absorption curve has a Lorentzian form :

$$\frac{\varepsilon''(\omega)}{\varepsilon_0} \simeq \frac{\lambda}{\omega_0}\,\frac{\gamma/4}{\Delta^2 + (\gamma/2)^2},$$

and the dispersive curve our usual shape :

$$\frac{\varepsilon'(\omega)}{\varepsilon_0} - 1 \simeq -\frac{\lambda}{2\omega_0}\,\frac{\Delta}{\Delta^2 + (\gamma/2)^2},$$

Check of Kramers–Kronig We already computed $\varepsilon'(\omega)$ and $\varepsilon''(\omega)$ algebraically. We now check that the Kramers–Kronig relation indeed reproduces the dispersive shape of $\varepsilon'(\omega)$ from the absorptive Lorentzian peak.

Close to the resonance, the imaginary part is sharply peaked at $x = \omega_0$. Thus the KK integral is dominated by values of x near ω_0, so we approximate

$$\frac{\varepsilon''(x)}{\varepsilon_0} \simeq \frac{\lambda}{\omega_0} \frac{\gamma/4}{(x-\omega_0)^2 + (\gamma/2)^2}.$$

Insert this into

$$\frac{\varepsilon'(\omega)}{\varepsilon_0} - 1 = \frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\varepsilon''(x)/\varepsilon_0}{x-\omega}\, dx.$$

Since only values close to ω_0 matter, we set

$$y = x - \omega_0, \qquad \Delta = \omega - \omega_0,$$

and approximate all slowly varying factors by their value at the resonance. The integral becomes

$$\frac{\varepsilon'(\omega)}{\varepsilon_0} - 1 \simeq \frac{\lambda}{\omega_0} \frac{\gamma}{4\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{1}{y^2 + (\gamma/2)^2} \frac{1}{y-\Delta}\, dy.$$

The remaining integral is the principal value of a Lorentzian divided by $y - \Delta$. It is well known (and easily checked by contour integration) that

$$\mathcal{P} \int_{-\infty}^{\infty} \frac{1}{y^2 + a^2} \frac{dy}{y-\Delta} = -\frac{\pi}{a} \frac{\Delta}{\Delta^2 + a^2}, \qquad a > 0.$$

With $a = \gamma/2$, this gives

$$\frac{\varepsilon'(\omega)}{\varepsilon_0} - 1 \simeq -\frac{\lambda}{2\omega_0} \frac{\Delta}{\Delta^2 + (\gamma/2)^2},$$

which is exactly the dispersive curve obtained earlier.

Thus the Lorentzian absorption peak $\varepsilon''(\omega)$ produces the correct dispersion of $\varepsilon'(\omega)$ through the Kramers–Kronig relation, as required by analyticity.

Thus the Lorentz oscillator model satisfies the Kramers–Kronig relations: the absorptive part ε'' produces the characteristic dispersive behavior of ε' around the resonance. Both curves are plotted in Fig.10.1

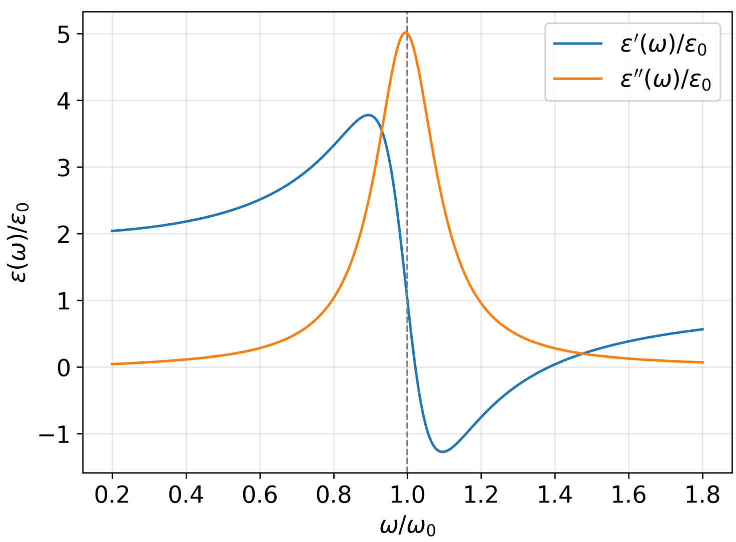


Figure 10.1: Real and imaginary parts of the dielectric constant ε as a function of the frequency ω.

10.5 Plane waves, refractive index and group velocity

Consider a monochromatic plane wave in a medium :

$$\mathbf{E}(\mathbf{x}, t) = \hat{\mathbf{E}}\, e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}.$$

For an isotropic medium with $\mathbf{D}(\omega) = \varepsilon(\omega)\mathbf{E}$ and $\mathbf{B}(\omega) = \mu(\omega)\mathbf{H}$, Maxwell's equations in Fourier space give the dispersion relation :

$$k^2 = \varepsilon(\omega)\mu(\omega)\,\frac{\omega^2}{c^2}.$$

We can define the (complex) refractive index $n(\omega)$ via $k = n(\omega)\,\omega/c$, with $n^2(\omega) = \varepsilon(\omega)\mu(\omega)$. If $\varepsilon(\omega)$ and $\mu(\omega)$ are real (in general, $\mu(\omega) \approx \text{const} \in \mathbb{R}$), then $n(\omega)$ is real and the wave propagates without absorption. The *group velocity* is :

$$v_g = \frac{d\omega}{dk} = \left(\frac{d}{d\omega}\frac{n(\omega)\omega}{c}\right)^{-1} = \frac{c}{n(\omega) + \omega\,\frac{dn}{d\omega}}.$$

If $\varepsilon(\omega)$ has an imaginary part, then $n(\omega)$ is complex :

$$n(\omega) = n'(\omega) + i\,n''(\omega),$$

and the wavevector :

$$k = (n'(\omega) + in''(\omega))\,\frac{\omega}{c}$$

acquires an imaginary part. The field then decays exponentially in space :

$$|\mathbf{E}(\mathbf{x})| \sim e^{-n''(\omega)\omega x/c},$$

where $n''(\omega)$ is the *extinction coefficient*.
Kramers–Kronig-like relations can also be written directly for $n'(\omega)$ and $n''(\omega)$, expressing again that dispersion and absorption are two sides of the same (causal) response function.

Chapter 11

Radiation in a medium

When a charged particle propagates through a material medium, it can emit electromagnetic radiation due to its interaction with the microscopic charges of the medium. Several radiation mechanisms exist, depending on the physical origin of the interaction and the kinematic regime of the particle. The main radiation mechanisms in matter are :

- Bremsstrahlung (braking radiation)
- Synchrotron radiation
- Cyclotron radiation
- Cherenkov radiation
- Transition radiation

Bremsstrahlung Bremsstrahlung is radiation emitted when a charged particle is accelerated or decelerated due to electromagnetic interactions, typically in the electric field of atomic nuclei. Even in the absence of external electric fields, a charged particle moving through a medium polarizes its surroundings, which leads to a deceleration and hence radiation emission. This can be directly seen through our usual Larmor formula :

$$\frac{d\mathcal{E}}{dt} = \frac{q^2}{6\pi\varepsilon_0 c^3}\,\gamma^6\left(|\dot{\boldsymbol{\beta}}|^2 - |\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}|^2\right)$$

Synchrotron and Cyclotron Radiation These types of radiation occur when charged particles move in magnetic fields. Cyclotron radiation corresponds to the non-relativistic regime, while synchrotron radiation refers to the relativistic regime. This is represented by the Larmor formula but with $\boldsymbol{\beta}\perp\dot{\boldsymbol{\beta}}$:

$$\frac{d\mathcal{E}}{dt} = \frac{q^2}{6\pi\varepsilon_0 c^3}\,\gamma^4|\dot{\boldsymbol{\beta}}|^2$$

Cherenkov Radiation Cherenkov radiation occurs when a charged particle propagates in a medium with a velocity exceeding the phase velocity of light in that medium.

Transition Radiation Transition radiation is emitted when a charged particle crosses an interface between two media with different dielectric properties.

11.1 Bremsstrahlung radiation and braking force

We now focus on the energy loss mechanism associated with Bremsstrahlung radiation in the non-relativistic regime. The goal is not to derive the full spectral radiation power, but rather to understand how energy loss arises from the electromagnetic self-interaction mediated by the medium.

One of Maxwell's equations for the scalar potential ϕ in a medium reads :

$$\nabla^2 \phi(\mathbf{x}, t) = -\frac{1}{\varepsilon} \rho_{\text{ext}}(\mathbf{x}, t),$$

where the external charge density for a particle of charge q moving with velocity $\mathbf{v}$ is :

$$\rho_{\text{ext}}(\mathbf{x}, t) = q\, \delta(\mathbf{x} - \mathbf{v}t).$$

Because the medium response is non-local in time, the potential is related to the source by a convolution in time.
We introduce the spatial Fourier transform :

$$\phi(\mathbf{x}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\, \phi_{\mathbf{k}}(t)\, e^{i\mathbf{k}\cdot\mathbf{x}},$$

and similarly for the charge density:

$$\rho_{\text{ext}}(\mathbf{x}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\, \rho_{\mathbf{k}}(t)\, e^{i\mathbf{k}\cdot\mathbf{x}}.$$

Using the definition of the delta function, we obtain

$$\begin{aligned} \rho_{\mathbf{k}}(t) &= \int \frac{d^3\mathbf{x}}{(2\pi)^{3/2}}\, \rho_{\text{ext}}(\mathbf{x}, t)\, e^{-i\mathbf{k}\cdot\mathbf{x}} \\ &= q \int \frac{d^3\mathbf{x}}{(2\pi)^{3/2}}\, \delta(\mathbf{x} - \mathbf{v}t)\, e^{-i\mathbf{k}\cdot\mathbf{x}} \\ &= \frac{q}{(2\pi)^{3/2}}\, e^{-i\mathbf{k}\cdot\mathbf{v}t} \end{aligned}$$

Applying the Laplacian to the Fourier expansion yields :

$$\nabla^2 \phi(\mathbf{x}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\, (-k^2)\, \phi_{\mathbf{k}}(t)\, e^{i\mathbf{k}\cdot\mathbf{x}},$$

where $k = |\mathbf{k}|$.

Equating both sides of Poisson's equation mode by mode, we obtain :

$$-k^2\,\phi_{\mathbf{k}}(t) = -\frac{1}{\varepsilon}\,\rho_{\mathbf{k}}(t),$$

which gives

$$\phi_{\mathbf{k}}(t) = \frac{q}{\varepsilon\,k^2}\,e^{-i\mathbf{k}\cdot\mathbf{v}t}\frac{1}{(2\pi)^{3/2}}$$

This shows that each Fourier mode oscillates at a *single frequency*

$$\omega = \mathbf{k}\cdot\mathbf{v}.$$

Recall that in this picture, the permittivity is frequency-dependent $\varepsilon \equiv \varepsilon(\omega = \mathbf{k}\cdot\mathbf{v})$.

The electric field is related to the scalar potential by :

$$\mathbf{E}(\mathbf{x},t) = -\boldsymbol{\nabla}\phi(\mathbf{x},t).$$

In Fourier space, this becomes :

$$\mathbf{E}_{\mathbf{k}}(t) = -i\mathbf{k}\,\phi_{\mathbf{k}}(t),$$

so that

$$\mathbf{E}_{\mathbf{k}}(t)\,e^{i\mathbf{k}\cdot\mathbf{x}} = -i\mathbf{k}\,\phi_{\mathbf{k}}(t)\,e^{i\mathbf{k}\cdot\mathbf{x}}.$$

Finally, the real-space electric field reads :

$$\mathbf{E}(\mathbf{x},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\,\mathbf{E}_{\mathbf{k}}(t)\,e^{i\mathbf{k}\cdot\mathbf{x}}.$$

Braking force The braking force $\mathbf{F}$ acting on the particle is given by the Lorentz force. In the present non-relativistic treatment, only the electric field contributes :

$$\mathbf{F} = q\,\mathbf{E}$$

Using the Fourier representation of the electric field and evaluating it at the particle position $\mathbf{x}(t) = \mathbf{v}t$, we obtain :

$$\mathbf{F}(t) = q\int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\,\mathbf{E}_{\mathbf{k}}(t)\,e^{i\mathbf{k}\cdot\mathbf{v}t}.$$

Which gives, using the previous results,

$$\mathbf{F} = -iq^2\int \frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{\mathbf{k}}{k^2\,\varepsilon(\mathbf{k}\cdot\mathbf{v})},$$

which is time-independent, as expected for a particle moving at constant velocity. Take $\mathbf{v} = v\,\mathbf{e_x}$, we can parametrize the argument of the integral in cylindrical coordinates as :

$$k_x v = \omega, \quad Q = \sqrt{k_y^2 + k_z^2}, \quad k^2 = \frac{\omega^2}{v^2} + Q^2, \quad d^3\mathbf{k} = 2\pi\,dQ\frac{d\omega}{v}$$

Which gives, using that $\mathbf{F} = F\mathbf{e_x}$:

$$F = i\,\frac{q^2}{4\pi^2}\int_{-\infty}^{\infty} d\omega \int_0^{Q_0} \frac{Q\,\omega}{\varepsilon(\omega)\left(Q^2v^2+\omega^2\right)}\,dQ$$

The integral is divergent. The cutoff Q_0 is a short-distance scale, reflecting the fact that the polarization fields from which this force derives are screened at short distances. As a result, one finds $F \sim \ln Q_0$.
Let us now study the divergence in the ω integral. Indeed, for large $\omega = \mathbf{k}\cdot\mathbf{v}$, the dielectric function approaches its high-frequency limit,

$$\varepsilon(\omega) \to \varepsilon_0,$$

and its imaginary part vanishes,

$$\operatorname{Im}\varepsilon(\omega) \to 0.$$

This contribution is therefore not important, since it corresponds to the high-frequency regime where the medium cannot absorb energy. It can be subtracted using a principal value prescription. Only the imaginary part of $1/\varepsilon$ (odd in ω) contributes.

Relativistic case We now consider the relativistic case. Here we need both the scalar potential ϕ and the vector potential $\mathbf{A}$. In the same gauge as before, they satisfy the wave equations

$$\nabla^2\mathbf{A} - \frac{\varepsilon}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{q\,\mathbf{v}}{c}\,\delta(\mathbf{r}-\mathbf{v}t),$$

and

$$\nabla^2\phi - \frac{\varepsilon}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{q}{\varepsilon}\,\delta(\mathbf{r}-\mathbf{v}t).$$

Again, we perform a spatial Fourier transform. Taking the Fourier components, we obtain :

$$\mathbf{A}_{\mathbf{k}} = \frac{q}{c}\,\frac{\mathbf{v}}{k^2 - \dfrac{\omega^2}{c^2}\varepsilon(\omega)}\,e^{-i\omega t}, \quad \phi_{\mathbf{k}} = \frac{q}{\varepsilon(\omega)}\,\frac{1}{k^2 - \dfrac{\omega^2}{c^2}\varepsilon(\omega)}\,e^{-i\omega t},$$

where again $\omega = \mathbf{k}\cdot\mathbf{v}$.
The Fourier component of the electric field is :

$$\mathbf{E}_{\mathbf{k}} = \frac{i\omega}{c}\,\mathbf{A}_{\mathbf{k}} - i\mathbf{k}\,\phi_{\mathbf{k}}.$$

The force acting on the particle is given by

$$\mathbf{F} = q\,\mathbf{E}.$$

Proceeding as in the non-relativistic case and taking $\mathbf{v} = v\,\mathbf{e}_x$, we obtain, using the same parametrization :

$$\mathbf{F} = i\,\frac{q^2}{4\pi^2}\int_{-\infty}^{\infty} d\omega \int_0^{Q_0} \frac{\left(\dfrac{1}{v^2} - \dfrac{\varepsilon(\omega)}{c^2}\right)\omega\,Q\,dQ}{\varepsilon(\omega)\left[Q^2 + \omega^2\left(\dfrac{1}{v^2} - \dfrac{\varepsilon(\omega)}{c^2}\right)\right]}.$$

In the limit $c \to \infty$, this expression reduces to the non-relativistic result.

11.2 Cherenkov radiation

This force decelerates the particle and its energy gets transmitted into exciting the energy levels of molecules in the medium (set $\mu_0 = 1$ here). However, when

$$v > \frac{c}{\sqrt{\varepsilon(\omega)}} = \frac{c}{n(\omega)},$$

that is, the particle is moving faster than the speed of light in the medium, there is a new interesting effect.

We know that a moving particle produces an electric field with

$$\omega = vk_x \qquad \text{or} \qquad k_x = \frac{\omega}{v}.$$

If $v > c/n$, we can find a plane wave solution with

$$\mathbf{k} = (k_x, \mathbf{k}_\perp), \qquad \omega = vk_x = \frac{c}{n}|\mathbf{k}|.$$

Therefore

$$\cos\theta = \frac{k_x}{|\mathbf{k}|} = \frac{c}{n(\omega)\, v} < 1,$$

and since $n = n(\omega)$ we have $\theta = \theta(\omega)$.
This means that there is a plane wave solution that is compatible with a motion of a fast particle at a constant velocity.
This type of radiation is called *Cherenkov radiation*. It was discovered experimentally by Cherenkov in 1934 and explained theoretically by Frank and Tamm in 1937, for which they shared a Nobel prize in 1958.

Power of Cherenkov radiation Let us derive the power of Cherenkov radiation. The total energy loss in a frequency range $d\omega$ and per distance dx traveled is

$$\frac{d\mathcal{E}}{dx} = \mathrm{d}F = -\mathrm{d}\omega \frac{iq^2}{4\pi^2} \sum_{\omega=\pm|\omega|} \omega \left(\frac{1}{c^2} - \frac{1}{\varepsilon v^2}\right) \int_0^{Q_0} \frac{Q\ \mathrm{d}Q}{Q^2 - \omega^2 \left(\frac{\varepsilon}{c^2} - \frac{1}{v^2}\right)}.$$

The condition $v > c/\sqrt{\varepsilon}$ ensures that there is a pole. This pole crosses our integration contour and corresponds to emitted plane waves. (Here $Q = |\mathbf{k}_\perp|$)
Let us introduce the variable

$$\zeta = Q^2 - \omega^2 \left(\frac{\varepsilon(\omega)}{c^2} - \frac{1}{v^2}\right).$$

We can then write :

$$dF = -\mathrm{d}\omega \frac{iq^2}{4\pi^2} \sum_{\omega=\pm|\omega|} \omega \left(\frac{1}{c^2} - \frac{1}{\varepsilon v^2}\right) \int \frac{d\zeta}{2\zeta}$$

And the singular point $\zeta = 0$ corresponds to $Q^2 + k_x^2 = |\mathbf{k}|^2$.
Although we take $\varepsilon(\omega)$ real (transparent medium), we introduce an infinitesimal imaginary part $\operatorname{Im}\varepsilon \equiv \varepsilon''$ to fix the prescription by bypassing the pole:

$$\varepsilon''(\omega) > 0 \text{ for } \omega > 0, \qquad \varepsilon''(\omega) < 0 \text{ for } \omega < 0.$$

Accordingly, the contour lies slightly below or above the real ζ-axis, when it is brought back onto the real axis, it must therefore bypass the pole at $\zeta = 0$ from below (for $\omega > 0$) or from above (for $\omega < 0$). Indenting the contour with infinitesimal semicircles, the principal value parts cancel in the sum and we get

$$\sum_{\omega} \int \frac{d\zeta}{\zeta} = \omega \left\{ \int \frac{d\zeta}{\zeta} - \int \frac{d\zeta}{\zeta} \right\} = 2i\pi\,\omega.$$

Therefore

$$dF = \frac{q^2}{4\pi c^2} \left(1 - \frac{c^2}{v^2 n^2(\omega)} \right) \omega\, d\omega, \qquad (n^2(\omega) = \varepsilon(\omega)).$$

Finally,

$$\frac{d\mathcal{E}}{dt} = v\,\frac{d\mathcal{E}}{dx} = v\,dF.$$

Recall that the radiation is emitted at an angle $\theta(\omega)$ with respect to the direction of motion, such that $\cos\theta(\omega) = c/n(\omega)v$, which gives :

$$d\theta = \frac{c}{v\,n^2 \sin\theta}\,\frac{dn}{d\omega}\,d\omega.$$

We thus obtain the radiated power per unit frequency, together with the corresponding emission angle $\theta(\omega)$, as $n(\omega)$ varies with ω, the radiation is distributed over angles according to $\cos\theta(\omega) = \frac{c}{vn(\omega)}$, while the total power follows from integrating $\frac{d\mathcal{E}}{dt} = v\,dF$ over the relevant frequency range for which the medium is transparent.