Cherenkov Radiation

Last lecture we derived KK dispersion relations:

$$\frac{\varepsilon'(\omega)}{\varepsilon_0} = 1 + \frac{2}{\pi} P \int_0^\infty \frac{x \frac{\varepsilon''(x)}{\varepsilon_0}}{x^2 - \omega^2} dx$$

$$\frac{\varepsilon''(\omega)}{\varepsilon_0} = -\frac{2\omega}{\pi} P \int_0^\infty \frac{\varepsilon'(x)/\varepsilon_0 - 1}{x^2 - \omega^2} dx$$

Very general property of analytic functions

We then considered a charged particle moving through a media and derived equation for force acting on it:

$$F = \frac{i e^2}{4\pi^2} \int_{-\infty}^{\infty} d\omega \int_0^{Q_0} \frac{\left(\frac{1}{v^2} - \frac{\varepsilon(\omega)}{c^2}\right) \omega q \, dq \, d\omega}{\varepsilon \left[Q^2 + \omega^2 \left(\frac{1}{v^2} - \frac{\varepsilon(\omega)}{c^2} \right) \right]}$$

For details see L.L. vol. 8 113-115.

This force decellerates the particle and its energy gets transmitted into exciting the energy levels of molecules in the media (set $\mu_0 = 1$ here)

However, when $v > \frac{c}{\sqrt{\varepsilon \mu_0}} = \frac{c}{n(\omega)}$,

that is particle is moving faster than the speed of light in the media, there is a new interesting effect.

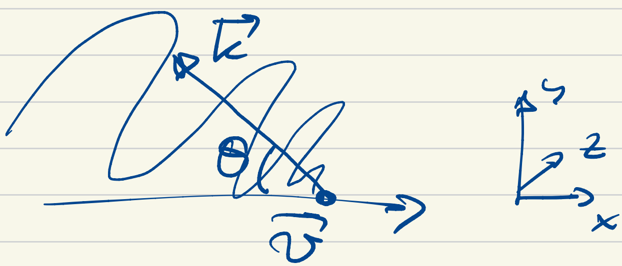
We saw last time that moving particle produces an electric field

with $\omega = v k_x$ or $k_x = \frac{\omega}{v}$

if $v > \frac{c}{n}$ we can find a plane wave solution with

$$\vec{k} = (k_x, \vec{k}_\perp)$$

$$\omega = v k_x = \frac{c}{n} |\vec{k}|$$



$$\cos\theta = \frac{k_x}{|\vec{k}|} = \frac{c}{nv} < 1$$

$$n = n(\omega) \Rightarrow \theta = \theta(\omega)$$

It means that there is a plane wave solution that is compatible with a motion of fast particle (at a constant velocity!)

take $M \to \infty$ $\quad a = \frac{F}{M} \to 0$

This type of radiation is called Cherenkov radiation, and it was discovered experimentally by Cherenkov in 1934, and explained theoretically by Frank and Tamm in 1937, for which they shared a Nobel prize in '58.

Let us derive the power of Cherenkov radiation.

Total energy loss in a frequency range $d\omega$ and per distance dx traveled is

$$\frac{d\mathcal{E}}{dx} = dF =$$

$$= d\omega \frac{i q^2}{4\pi^2} \sum_{\omega = \pm|\omega|} \omega \left(\frac{1}{c^2} - \frac{1}{v^2 \varepsilon} \right) \cdot \int_0^{Q_0} \frac{Q\, dQ}{Q^2 - \omega^2 \left(\frac{\varepsilon}{c^2} - \frac{1}{v^2} \right)}$$

condition $v > \frac{c}{\sqrt{\varepsilon}}$ ensures that there is a pole: this pole crossing our integration contour — this pole corresponds to emitted plane waves.

$$\left(Q = |k_\perp| \right)$$

Let's introduce variable

$$\xi = Q^2 - \omega^2\left(\frac{\varepsilon(\omega)}{c^2} - \frac{1}{v^2}\right)$$

$$dF = -d\omega \frac{i q^2}{4\pi^2} \sum_{\omega=\pm|\omega|} \omega\left(\frac{1}{c^2} - \frac{1}{v^2\varepsilon}\right) \int \frac{d\xi}{2\xi}$$

$\xi = 0$ corresponds to $Q^2 + k_x^2 = |\vec{k}|^2$

To have radiation we need $\operatorname{Im}\varepsilon = \varepsilon'' \ll \varepsilon_0$, because otherwise media is not transparent, however, there is always a small ε'': $\varepsilon'' > 0$, $\omega > 0$

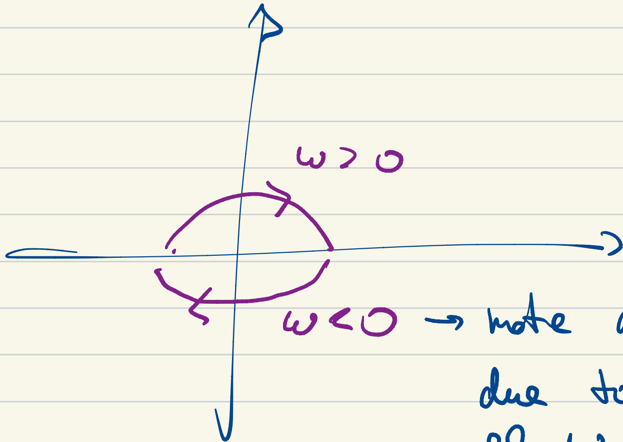
$\varepsilon'' < 0$, $\omega < 0$

$\omega > 0$

$$\xi = \left(Q - \omega\sqrt{\frac{\varepsilon(\omega)}{c^2} - \frac{1}{v^2}}\right)\left(Q + \omega\sqrt{\frac{\varepsilon(\omega)}{c^2} - \frac{1}{v^2}}\right)$$

$\omega > 0$ $\quad \xi \sim -i\varepsilon''(\omega)\omega^2$ $\hookrightarrow$

$\omega < 0 \qquad \xi \sim -i\,\varepsilon^4(\omega)\,\omega^2 = A$



$\rightarrow$ note direction change due to the factor of $\omega = -|\omega|$

$$|\omega|\left(\frac{1}{c^2} - \frac{1}{v^2\varepsilon}\right)\oint \frac{d\xi}{2\xi} = \pi i\,|\omega|\left(\frac{1}{c^2} - \frac{1}{v^2\varepsilon}\right)$$

We conclude that

$$\frac{d\mathcal{E}}{dx} = dF = \frac{q^2}{4\pi c^2}\left(1 - \frac{c^2}{v^2 n^2}\right)\omega\, d\omega$$

$$\frac{d\mathcal{E}}{dt} = \frac{d\mathcal{E}}{dx}\cdot\frac{dx}{dt} = \frac{d\mathcal{E}}{dx}\cdot v$$

$$d\theta = \frac{c}{v n^2 \sin\theta} \frac{dn}{d\omega} d\omega \rightarrow$$

$\rightarrow$ angle in which radiation is pointing.

Compare with Bremsstrahlung:

$$\frac{d\mathcal{E}}{dt} = \frac{q^2}{6\pi\varepsilon_0 c} \gamma^6 \left[|\dot{\vec{\beta}}|^2 - |\vec{\beta} \times \dot{\vec{\beta}}|^2 \right]$$

Review:

Since last review two main topics

• Media $\longrightarrow$

$\nearrow$ $\varepsilon, \mu = \text{const}$

$\searrow$ $\varepsilon(\omega)$ $[\mu \approx \text{const}]$

const:

ε_1, μ_1 | $\varepsilon_2\ \mu_2$

macroscopic

$$\vec{E} = \varepsilon \vec{D}$$

$$\vec{H} = \frac{\vec{B}}{\mu}$$

$$\vec{\nabla} \cdot \vec{D} = 0 \qquad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$\vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = 0 \qquad \vec{\nabla} \cdot \vec{B} = 0$$

•

dispersion

$$\varepsilon(\omega) \Longleftrightarrow \varepsilon = \varepsilon'(\omega) + i\varepsilon''(\omega)$$

• Relativity

Lorentz as transformation of coordinates : $x^{\mu'} = \Lambda^\mu{}_\nu x^\nu$

But more generally, other vectors $a^\mu = \Lambda^\mu{}_\nu a^\nu$ and tensors

it is a direct generalization of rotation : $\vec{x} \sim x^i \to x^{i'} = R^i{}_j x^j$

lower-index:

$$\eta_{\mu\nu} x^\nu \Rightarrow x_\mu \sim \frac{\partial}{\partial x^\mu}$$

$$x_\mu x^\mu = \text{invariant}$$

$$\varphi, \vec{A} \to A_\mu \to F_{\mu\nu} = \overbrace{\partial_\mu A_\nu - \partial_\nu A_\mu}^{\text{gauge invariant}} \to$$

$$\to S_{QED} \sim F_{\mu\nu} F^{\mu\nu} \to \partial_\mu F^{\mu\nu} = 0$$

Newtonian mechanics: $d\tau = \frac{dt}{\gamma}$

$$d\tau = \sqrt{dt^2 - dx^2} = dt\sqrt{1-\beta^2}$$