Quantum mechanics II, Solutions 1 : Recap of the basics

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Problem 1 : Bloch sphere for pure states

1. Show that any quantum state $|\psi\rangle$ of a 2 level system with classical states $|0\rangle$ and $|1\rangle$ can be written

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle, \quad \theta \in [0,\pi], \quad \phi \in [0,2\pi), \tag{1}$$

and conclude that the unit sphere in 3D allows for the representation of a pure state of a 2 level system (qubit, spin 1/2, photon,...). We call this unit sphere the Bloch sphere

Any state in a 2 level system can be written as

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \tag{2}$$

where $\alpha, \beta \in \mathbb{C}$. We can write these complex numbers in polar form, that is $\alpha = pe^{i\gamma}$ and $\beta = qe^{i\delta}$, to see more explicitly that, prior to applying any constraint, $|\psi\rangle$ is really characterised by 4 real quantities : p, q, γ and δ, with $p, q \in [0,\infty)$ and $\gamma, \delta \in [0, 2\pi)$. Now if we apply the normalisation constraint $|\langle\psi|\psi\rangle| = 1$ we get

$$|\alpha|^2 + |\beta|^2 = p^2 + q^2 = 1. \tag{3}$$

Under this constraint, notice that now $p, q \in [0,1]$. It is always possible to find a $\theta \in [0,\pi)$ such that $p = \cos\theta/2$. From Eq. 3 we then deduce that $q = \sin\theta/2$. So now Eq. 2 reads

$$\begin{aligned} |\psi\rangle &= \cos\frac{\theta}{2}e^{i\gamma}|0\rangle + \sin\frac{\theta}{2}e^{i\delta}|1\rangle \qquad (4)\\ &= e^{i\gamma}\left(\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i(\delta-\gamma)}|1\rangle\right), \qquad (5) \end{aligned}$$

where, without loss of generality, we extracted the phase of $|0\rangle$ as a global phase. Recall that a global phase is not physical and cannot be measured experimentally in the case of a single qubit. Hence we can fix the global phase to be $\gamma = 0$. By defining $\phi \equiv \delta - \gamma$ we recover the result of interest.

2. In the case of a 1/2 spin, the convention for the north pole is $|0\rangle$ (resp. the south pole $|1\rangle$) which is the eigenstate of $S_z = \frac{\hbar}{2}\sigma_z$ of eigenvalue $+\hbar/2$ (resp. $-\hbar/2$). Show that state $|\psi\rangle$ of Eq. (1) is the eigenstate of $\boldsymbol{\sigma_n} = \boldsymbol{\sigma}.\boldsymbol{n}$, where $\boldsymbol{n}$ is a unit vector with direction parametrized by θ and ϕ in spherical coordinates and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. In a classical picture of the spin, $\boldsymbol{n}$ would then designate the direction in which that spin is pointing.

We are told that $\boldsymbol{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. We can then readily apply the operator

$$\boldsymbol{\sigma_n} = \boldsymbol{\sigma}\cdot\boldsymbol{n} = \begin{pmatrix} \cos\theta & \sin\theta\cos\phi - i\sin\theta\sin\phi \\ \sin\theta\cos\phi + i\sin\theta\sin\phi & -\cos\theta \end{pmatrix} \tag{6}$$

to the state $|\psi\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}$ in the following way

$$\begin{aligned}
\boldsymbol{\sigma_n}|\psi\rangle &= \begin{pmatrix} \cos\theta & \sin\theta\cos\phi - i\sin\theta\sin\phi \\ \sin\theta\cos\phi + i\sin\theta\sin\phi & -\cos\theta \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} && (7)\\
&= \begin{pmatrix} \cos\theta\cos\frac{\theta}{2} + e^{i\phi}\sin\frac{\theta}{2}(\sin\theta\cos\phi - i\sin\theta\sin\phi) \\ \cos\frac{\theta}{2}(\sin\theta\cos\phi + i\sin\theta\sin\phi) - e^{i\phi}\cos\theta\sin\frac{\theta}{2} \end{pmatrix} && (8)\\
&= \begin{pmatrix} \cos\theta\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\sin\theta \\ e^{i\phi}\cos\frac{\theta}{2}\sin\theta - e^{i\phi}\cos\theta\sin\frac{\theta}{2} \end{pmatrix} && (9)\\
&= \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} && (10)\\
&= |\psi\rangle, && (11)
\end{aligned}$$

where, in the first vector entry, we expanded $e^{i\phi} = \cos\phi + i\sin\phi$, used the identities $\cos^2\phi + \sin^2\phi = 1$ and $\sin(\theta/2)\sin\theta + \cos(\theta/2)\cos\theta = \cos(\theta/2)$, while for the second entry, we identified $\cos\phi + i\sin\phi = e^{i\phi}$ and then used the identity $\cos(\theta/2)\sin\theta - \cos\theta\sin(\theta/2) = \sin(\theta/2)$. So we conclude that indeed $|\psi\rangle$ is an eigenstate of $\boldsymbol{\sigma_n}$ with eigenvalue 1.

3. Sketch on the Bloch sphere the effect of applying the operators σ_x, σ_y and σ_z to an arbitrary state $|\psi\rangle$.

Let's separately apply the Pauli operators to the state $|\psi\rangle$:

$$\sigma_x|\psi\rangle = \begin{pmatrix} e^{i\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix} = e^{i\phi}\begin{pmatrix} \sin\frac{\theta}{2} \\ e^{-i\phi}\cos\frac{\theta}{2} \end{pmatrix} = e^{i\phi}\begin{pmatrix} \cos\left(\frac{\pi-\theta}{2}\right) \\ e^{-i\phi}\sin\left(\frac{\pi-\theta}{2}\right) \end{pmatrix} \implies \begin{cases} \theta' = \pi - \theta \\ \phi' = -\phi \end{cases}, \quad (12)$$

$$\sigma_y|\psi\rangle = \begin{pmatrix} -ie^{i\phi}\sin\frac{\theta}{2} \\ i\cos\frac{\theta}{2} \end{pmatrix} = -ie^{i\phi}\begin{pmatrix} \sin\frac{\theta}{2} \\ -e^{-i\phi}\cos\frac{\theta}{2} \end{pmatrix} = -ie^{i\phi}\begin{pmatrix} \cos\left(\frac{\pi-\theta}{2}\right) \\ e^{i(\pi-\phi)}\sin\left(\frac{\pi-\theta}{2}\right) \end{pmatrix} \implies \begin{cases} \theta' = \pi - \theta \\ \phi' = \pi - \phi \end{cases}, \quad (13)$$

$$\sigma_z|\psi\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ -e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i(\pi+\phi)}\sin\frac{\theta}{2} \end{pmatrix} \implies \begin{cases} \theta' = \theta \\ \phi' = \pi + \phi \end{cases}. \quad (14)$$

We manipulated the expressions to be able to directly read off the new angles θ' and ϕ' parameterizing the state obtained after applying a Pauli operator. Again, the global phase is neglected. To visualise these transformations, in Figure 1 we show the effect of each Pauli operator when applied to the state $|\psi\rangle$ characterised by $(\theta, \phi) = (\pi/4, 0)$. From Figure 1, we see that the Pauli α operator, $\alpha \in \{x, y, z\}$, rotates the state by 180° around the α-axis.

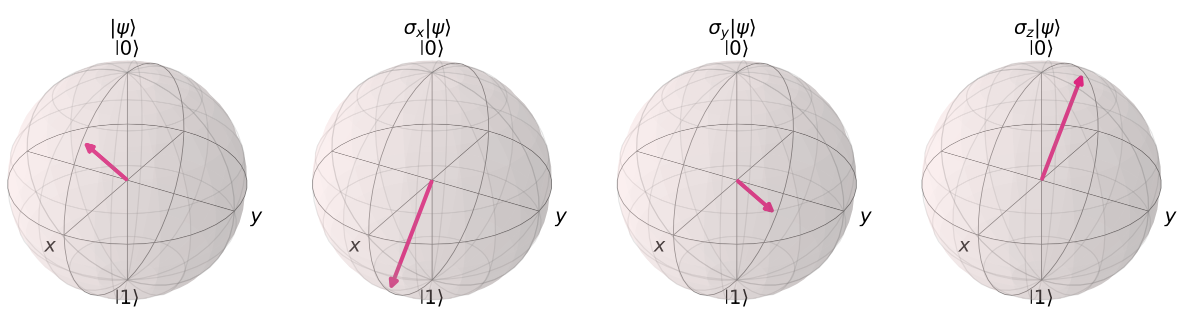

Figure 1 – Effect of the Pauli matrices.

4. Prove that the Pauli matrices together with the identity form an orthogonal basis of all (hermitian) operators acting on a 2 level system.

We first show that the Pauli matrices form a basis in the space of 2×2 Hermitian matrices.
Spanning the space : Any Hermitian matrix H satisfies $H = H^\dagger$ and so can be written as follows

$$H = \begin{pmatrix} a & \beta^* \\ \beta & d \end{pmatrix} = \begin{pmatrix} a & b - ic \\ b + ic & d \end{pmatrix}, \tag{15}$$

with $a, b, c, d \in \mathbb{R}$. Our goal is then to show that this general expression for H can be written as a linear combination of the Pauli matrices, that is

$$p\sigma_0 + q\sigma_1 + r\sigma_2 + s\sigma_3 = \begin{pmatrix} p+s & q-ir \\ q+ir & p-s \end{pmatrix}, \tag{16}$$

for some $p, q, r, s \in \mathbb{R}$. This is easily achieved with $p = (a+d)/2, s = (a-d)/2, q = b$ and $r = c$. We can thus conclude that the Pauli matrices span the space of 2×2 Hermitian matrices.
Linear independence : We need to show that it is not possible to express one Pauli matrix as a linear combination of the other Pauli matrices. We do this by contradiction ; we seek p, q, r, s which weight the Pauli matrices to sum to zero, implying linear dependence :

$$p\sigma_0 + q\sigma_1 + r\sigma_2 + s\sigma_3 = \begin{pmatrix} p+s & q-ir \\ q+ir & p-s \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{17}$$

The only solution to this set of 4 linear equations is $p = q = r = s = 0$, hence the Pauli matrices are linearly independent.

Hence we conclude that the Pauli matrices indeed form a basis. Now we finally show that this basis is orthogonal. As a reminder, two vectors $v, w \in \mathbb{R}^n$ are said to be orthogonal if $v \cdot w = 0$, or more explicitly, $\sum_{i=1}^n v_i w_i = 0$. For matrices, the analogue of this sum (where all indices are "contracted") is the trace operation. One can verify that $\mathrm{Tr}[\sigma_i \sigma_j] = 2\delta_{ij}$, meaning that while the Pauli matrices are orthogonal, they are not orthonormal because of this factor of 2. Note that the case $i = j$ can be treated rapidly because $\mathrm{Tr}[\sigma_i \sigma_i] = \mathrm{Tr}[\sigma_i^2] = \mathrm{Tr}[\mathbb{I}] = 2$, using the involutory property of the Pauli matrices (i.e. $\sigma_i^2 = \mathbb{I}$).

5. Show that

$$e^{-i\vartheta \mathbf{n}.\boldsymbol{\sigma}} = \cos(\vartheta)\mathbb{I} - i\sin(\vartheta)\mathbf{n}.\boldsymbol{\sigma}\,. \tag{18}$$

We can use the exponential series expansion to recover the desired result. First, to give us a bit of intuition on the general result, we consider the special case where $\boldsymbol{n} = (1, 0, 0)$, so that $\mathbf{n} \cdot \boldsymbol{\sigma} = \sigma_x$. In this case we have

$$\begin{aligned} e^{-i\theta\sigma_x} &= \mathbb{I} - i\theta\sigma_x - \frac{1}{2!}\theta^2\sigma_x^2 + i\frac{1}{3!}\theta^3\sigma_x^3 + \mathcal{O}(\theta^4) \\ &= \left(\mathbb{I} - \frac{1}{2!}\theta^2\mathbb{I} + \mathcal{O}(\theta^4)\right) - i\left(\theta - \frac{1}{3!}\theta^3 + \mathcal{O}(\theta^5)\right)\sigma_x \\ &= \cos\theta\mathbb{I} - i\sin\theta\sigma_x, \end{aligned} \tag{19}$$

where we used $\sigma_x^2 = \mathbb{I}$. Now to prove the general case, we can proceed similarly as above and split the even and odd terms

$$e^{-i\theta\mathbf{n}\cdot\boldsymbol{\sigma}} = \sum_{k=0}^{\infty} \frac{(-i\theta)^k}{k!}(\mathbf{n}\cdot\boldsymbol{\sigma})^k = \sum_{k=0}^{\infty} \frac{(-i\theta)^{2k}}{(2k)!}(\mathbf{n}\cdot\boldsymbol{\sigma})^{2k} + \sum_{k=0}^{\infty} \frac{(-i\theta)^{2k+1}}{(2k+1)!}(\mathbf{n}\cdot\boldsymbol{\sigma})^{2k+1}. \tag{20}$$

The expression involves terms of the form

$$(\mathbf{n}\cdot\boldsymbol{\sigma})^2 = \left(\sum_{i=1}^3 n_i\sigma_i\right)^2 = \sum_{i=1}^3\sum_{j=1}^3 n_i n_j \sigma_i\sigma_j \equiv n_i n_j \sigma_i \sigma_j, \tag{21}$$

where in the last equality we have adopted the Einstein summation convention where repeated indices are summed over. Now using the identity $\sigma_i\sigma_j = \delta_{ij}\mathbb{I} + i\epsilon_{ijk}\sigma_k$ (with the index k summed over) we get

$$\begin{aligned}(\mathbf{n}\cdot\boldsymbol{\sigma})^2 &= n_i n_j(\delta_{ij}\mathbb{I} + i\,\epsilon_{ijk}\sigma_k)\\ &= \mathbf{n}\cdot\mathbf{n}\mathbb{I} + \mathbb{I} + i\,n_i n_j\epsilon_{ijk}\sigma_k\\ &= \mathbb{I} + i\,\epsilon_{kij}n_i n_j\sigma_k\\ &= \mathbb{I} + i\,(\mathbf{n}\times\mathbf{n})_k\sigma_k \qquad (22)\\ &= \mathbb{I} \qquad (23)\end{aligned}$$

where for the first term we used that $\mathbf{n}$ is a vector of unit norm, and for the second term we swapped twice the indices of the Levi-Civita tensor (leading to 2 minus signs – yielding a plus sign), identified the definition of the cross product in terms of the Levi-Civita tensor, and noticed that the cross product of a vector with itself gives 0. Now more generally we have

$$(\mathbf{n}\cdot\boldsymbol{\sigma})^{2k} = \mathbb{I}, \tag{24}$$

$$(\mathbf{n}\cdot\boldsymbol{\sigma})^{2k+1} = \mathbf{n}\cdot\boldsymbol{\sigma}. \tag{25}$$

Putting everything together we obtain the desired result

$$e^{-i\theta\mathbb{n}\cdot\boldsymbol{\sigma}} = \sum_{k=0}^{\infty}(-1)^k\frac{\theta^{2k}}{(2k)!}\mathbb{I} - i\sum_{k=0}^{\infty}(-1)^k\frac{\theta^{2k+1}}{(2k+1)!}\mathbf{n}\cdot\boldsymbol{\sigma} = \cos\theta\mathbb{I} - i\sin\theta\mathbf{n}\cdot\boldsymbol{\sigma}. \tag{26}$$

6. Hence sketch the action of an arbitrary state on the Bloch sphere under $e^{-i\vartheta\mathbf{n}.\boldsymbol{\sigma}}$.

In the lecture notes, we explicitly demonstrate that $e^{-i\vartheta\sigma_z}$ induces the following transformations on the angles of the state $|\psi\rangle$: $\theta \to \theta$ and $\phi \to 2\vartheta + \phi$. This corresponds to a rotation by an angle of 2ϑ with respect to the z-axis. Similarly, $e^{-i\vartheta\sigma_x}$ and $e^{-i\vartheta\sigma_y}$ correspond to 2ϑ rotations about the x and y axes respectively. Since we chose $|0\rangle$ and $|1\rangle$ to be the eigenstates of σ_z, if we write these rotations operators in this so-called computational basis of $|0\rangle$ and $|1\rangle$, we get

$$e^{-i\vartheta\sigma_x} \doteq \begin{pmatrix}\cos\vartheta & -i\sin\vartheta\\ -i\sin\vartheta & \cos\vartheta\end{pmatrix}, \tag{27}$$

$$e^{-i\vartheta\sigma_y} \doteq \begin{pmatrix}\cos\vartheta & -\sin\vartheta\\ \sin\vartheta & \cos\vartheta\end{pmatrix}, \tag{28}$$

$$e^{-i\vartheta\sigma_z} \doteq \begin{pmatrix}e^{-i\vartheta} & 0\\ 0 & e^{i\vartheta}\end{pmatrix}, \tag{29}$$

where the dot on top of the equal sign is there to remind us that a choice of basis has been made (and a different choice would result in different expressions). In particular, we see that under this convention, σ_z is diagonal. As a side note, in terms of the z-basis states ($|0\rangle$ and $|1\rangle$), the orthogonal x-basis states read

$$|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle). \tag{30}$$

Hence in the $|\pm\rangle$ basis (the eigenstates of σ_x), the operator $e^{-i\vartheta\sigma_x}$ would have the same matrix representation as in Eq. 29. The same holds true in the y direction, where the orthogonal y-basis states are

$$|R/L\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm i|1\rangle). \tag{31}$$

This is just a statement that there is no preferred direction.

As an example, we show in Figure 2 how the operator $e^{-i\vartheta\sigma_z}$, with $\vartheta = \pi/4$, acts on the state $|\psi\rangle$ parameterised by $(\theta, \phi) = (\pi/4, 0)$. As we can see, $e^{-i\vartheta\sigma_z}$ corresponds to a rotation about the z-axis by an angle 2ϑ. In general, $e^{-i\vartheta\mathbf{n}\cdot\boldsymbol{\sigma}}$ corresponds to a rotation by an angle 2ϑ about the axis of rotation specified by the unit vector $\mathbf{n}$. Hence the Pauli matrices are often called the *generators of rotation* in this context.

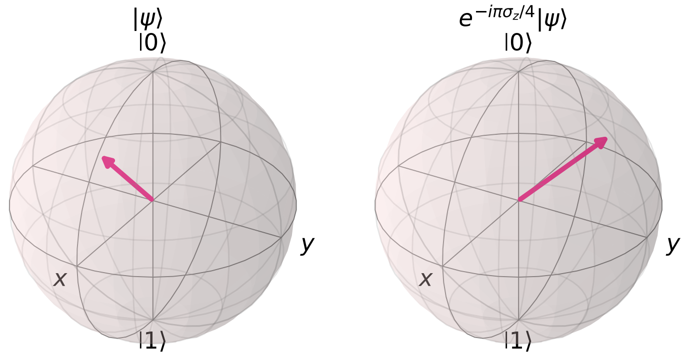

FIGURE 2 – Action of $e^{-i\vartheta \mathbf{n}\cdot\boldsymbol{\sigma}}$ on the state $|\psi\rangle$.

7. Compute the expectation value of the Pauli operators σ_x, σ_y and σ_z for the arbitrary state $|\psi\rangle$. Explain your answer in terms of the Bloch sphere.

After a bit of algebra, one obtains

$$\langle\psi|\sigma_x|\psi\rangle = \sin\theta\cos\phi, \tag{32}$$

$$\langle\psi|\sigma_y|\psi\rangle = \sin\theta\sin\phi, \tag{33}$$

$$\langle\psi|\sigma_z|\psi\rangle = \cos\theta. \tag{34}$$

$$\tag{35}$$

which corresponds respectively to the x, y and z spherical coordinate components, respectively, of the unit vector parameterized by (θ, ϕ) on the Bloch sphere.

Problem 2 : Rethinking the thought experiment

Consider the thought experiment described in Section 1.4 of the course notes. I'll copy it over here for simplicity.

Thought experiment 1 : We start with a system in state $|0\rangle$. We wait half an hour (in our units, from $t = 0$ to $t = 1/2$) before measuring it. We then find that 50% of the time it is in state $|1\rangle$ and that 50% of the time it is in state $|0\rangle$.

Thought experiment 2 : We start with a system in state $|1\rangle$. We wait half an hour before measuring it. We then find that 50% of the time it is in state $|1\rangle$ and that 50% of the time it is in state $|0\rangle$.

Let $U = e^{-iHt}$ represent the evolution of the system as we "wait for t hours". The state of the system pre-measurement in the two experiments can therefore be written as $|\psi\rangle = U|0\rangle$ and $|\phi\rangle = U|1\rangle$ respectively.

1. Write down the most general form for the states $|\psi\rangle$ and $|\phi\rangle$ that are consistent with the observed outcomes in experiments 1 and 2.

Both states have the same probability distribution over σ_z-basis states $|0\rangle$ and $|1\rangle$. Let us first consider $|\psi\rangle$. We know that

$$|\langle 0|\psi\rangle|^2 = |\langle 1|\psi\rangle|^2 = \frac{1}{2}. \tag{36}$$

Which states fulfil that ? We saw earlier that the general state of a qubit can be expressed as

$$|\varphi\rangle = \cos\frac{\alpha}{2}|0\rangle + \sin\frac{\alpha}{2}e^{-i\beta}|1\rangle, \tag{37}$$

with general probability

$$|\langle 0|\psi\rangle|^2 = \left|\cos\frac{\alpha}{2}\right|^2, \tag{38}$$

which imposed to be $\frac{1}{2}$, admits

$$\cos^2\frac{\alpha}{2} = \frac{1}{2} \quad \Rightarrow \quad \alpha = \frac{\pi}{2}. \tag{39}$$

This constraint does not impose anything in β, thus the most general state is

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{-i\beta}|1\rangle). \tag{40}$$

The negative sign of β is completely arbitrary. Because $|\psi\rangle$ and $|\phi\rangle$ both satisfy the constraints from which we produced this form, they are both described by it, with different β, e.g. β and β' respectively.

However, for U to be a valid unitary, $|\psi\rangle$ and $|\phi\rangle$ cannot be completely independent ; satisfying $\hat{U}\hat{U}^\dagger$ will constrain the relative values of β and β'. A quick way to derive this constraint is to realise that because the input states are orthogonal :

$$\langle 1|0\rangle = 0 \tag{41}$$

so too must the output states :

$$\langle\phi|\psi\rangle = \langle 1|U^\dagger U|0\rangle = \langle 1|0\rangle = 0 \tag{42}$$

Ergo if

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{-i\beta}|1\rangle) \tag{43}$$

$$|\phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{-i\beta'}|1\rangle) \tag{44}$$

then unitarity requires

$$\langle\phi|\psi\rangle = \frac{1}{2}(1 + e^{-i(\beta-\beta')}) = 0 \tag{45}$$

This implies that $e^{-i(\beta-\beta')} = -1$, or in other words $\beta' = \beta + \pi$.

Thought experiment 3 : We start with a system in state $|0\rangle$. We wait half an hour before measuring it. We then find that 50% of the time it is in state $|1\rangle$ and that 50% of the time it is in state $|0\rangle$. Then we wait another half an hour before measuring again. We then find that 50% of the time it is in state $|1\rangle$ and that 50% of the time it is in state $|0\rangle$.

2. Use your answer to (a) to explain the result of the third thought experiment.

We previously showed that both $|\psi\rangle$ and $|\phi\rangle$, which result from evolving states $|0\rangle$ or $|1\rangle$ for half an hour, have the form :

$$|\varphi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{-i\beta}|1\rangle). \tag{46}$$

When we measure in the σ_z-basis, we collapse the state into classical states $|0\rangle$ and $|1\rangle$. We have reproduced our initial states. Ergo subsequent evolution and measurement is just a repetition of the initial process, and ergo yields the same statistics.

Thought experiment 4 : We start with a system in state $|0\rangle$. We *wait a full hour* before measuring it. We find that the system is always in state $|0\rangle$.

3. For concreteness, let's now assume that $|\psi\rangle = U|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Write down two non-equivalent [1] unitaries that U could be and the corresponding Hamiltonian H (hint - think about Problem 1) which would generate them.

Geometric solution

We can approach this geometrically, recognising that $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ (i.e. the $|+\rangle$ state) can be reached from $|0\rangle$ by a $\pi/2$ rotation about the σ_y axis.

$$e^{-\frac{\pi}{4}i\,\sigma_y}|0\rangle = |+\rangle = |\psi\rangle\,. \tag{47}$$

The same rotation would transform the $|1\rangle$ state to :

$$e^{-\frac{\pi}{4}i\,\sigma_y}|1\rangle = -\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = |-\rangle\,. \tag{48}$$

So a $U = e^{-\frac{\pi}{4}i\,\sigma_y}$ achieves $U|0\rangle = |\psi\rangle$, but there may still exist *other* unitaries which produce the same state. We can find them by considering subsequent operations which do not change the state $|\psi\rangle = |+\rangle$, but which are free to change $|-\rangle$. We recognise that $|+\rangle$ and $|-\rangle$ are eigenstates of the σ_x operator, i.e.

$$\begin{cases} \sigma_x|+\rangle = |+\rangle \\ \sigma_x|-\rangle = -|-\rangle\,. \end{cases} \tag{49}$$

Ergo realise that $\sigma_x\, e^{-\frac{\pi}{4}i\,\sigma_y}|0\rangle = \sigma_x|+\rangle = |\psi\rangle$. Two candidate unitaries (which we here needlessly express as σ_z-basis matrices) are :

$$U_1 = e^{-\frac{\pi}{4}i\,\sigma_y} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{50}$$

$$U_2 = \sigma_x\, U_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}. \tag{51}$$

1. i.e. not the same up to a global phase

You might wonder; are these the *only* two unitaries that satisfy $U|0\rangle = |\psi\rangle$? Absolutely not! We can think up more subsequent operations which induce only global phases upon $|\psi\rangle$ but which are themselves not merely a global phase operator (i.e. they are not of the form $e^{\theta\, i}\mathbb{1}$). We realise that any *rotation* about the σ_x-axis will not modify its eigenkets $|+\rangle$ and $|-\rangle$, but *will* apply distinct phases to them. The spectral theorem lets us write this explicitly as

$$\begin{cases} e^{\theta i\,\sigma_x}|+\rangle = e^{\theta i}|+\rangle \\ e^{\theta i\,\sigma_x}|-\rangle = e^{-\theta i}|-\rangle \end{cases} \tag{52}$$

If we wish to keep $|+\rangle$ unchanged, we merely apply a global phase factor $e^{-\theta i}$ which undoes the phase $e^{\theta i}$ of $|+\rangle$ induced by the rotation. Ergo the continuously-parameterised unitary

$$U(\theta) = e^{-\theta i} \cdot e^{\theta i\,\sigma_x} \cdot e^{-\frac{\pi}{4} i\,\sigma_y} \tag{53}$$

$$= e^{\theta i\,(\sigma_x - \mathbb{I})} \cdot e^{-\frac{\pi}{4} i\,\sigma_y} \tag{54}$$

effects $U|0\rangle = |\psi\rangle$ for any $\theta \in [0, 2\pi)$.

Analytic solution

We can also approach this problem non-geometrically. We recognise that $U|0\rangle = |\psi\rangle$ immediately constraints the leftmost column of U :

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & a \\ \frac{1}{\sqrt{2}} & b \end{pmatrix} \tag{55}$$

where $a, b \in \mathbb{C}$. To be a valid unitary however, it must satisfy $U^\dagger U = \mathbb{I}$, and ergo (expanding the matrices) that (for any complex phases $\alpha, \beta \in \mathbb{R}$) :

$$\begin{cases} |a|^2 = \dfrac{1}{2}, & \implies a = \dfrac{1}{\sqrt{2}} e^{i\alpha} \\ |b|^2 = \dfrac{1}{2}, & \implies b = \dfrac{1}{\sqrt{2}} e^{i\beta} \\ a\, b^* = -\dfrac{1}{2}, & \implies b = -\dfrac{1}{2a^*} = -\dfrac{e^{i\alpha}}{\sqrt{2}} \end{cases} \tag{56}$$

Therefore, all unitaries of the below form, for any real value of α, produce $U|0\rangle = |\psi\rangle$.

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & e^{i\alpha} \\ 1 & -e^{i\alpha} \end{pmatrix}. \tag{57}$$

Corresponding Hamiltonian

We next seek the Hamiltonian which describes this evolution, and which generates our U operator. That is, we seek H which satisfies $H = H^\dagger$ and $\mathrm{e}^{-iHt} = U = e^{-\theta i} e^{\theta i\,\sigma_x} e^{-\frac{\pi}{4} i\,\sigma_y}$ at $t = \frac{1}{2}$.

There are several ways of doing so, one of which begins by expanding the exponential as performed in Problem 1.5 :

$$e^{-\theta i} e^{i\theta\sigma_x} e^{-i\pi\sigma_y/4} = e^{-\theta i}(\mathbb{1}\cos\theta - i\sin\theta\sigma_x)\frac{1}{\sqrt{2}}(\mathbb{1} - i\sigma_y) \tag{58}$$

$$= e^{-\theta i}\left(\mathbb{1}\cos\theta\cos\frac{\pi}{4} - i(\sin\theta\cos\frac{\pi}{4}\sigma_x + \sin\frac{\pi}{4}\cos\theta\sigma_y + \sin\theta\sin\frac{\pi}{4}\sigma_z)\right) \tag{59}$$

$$= e^{-\theta i}\,(\mathbb{1}\cos x + i\sin x(\sigma\cdot\mathbf{n})) \tag{60}$$

$$= e^{-\theta i} e^{-ix(\sigma\cdot\mathbf{n})} \tag{61}$$

where in the last equality we have defined $\cos x = \cos\theta\cos\frac{\pi}{4}$ and $\mathbf{n} = \frac{1}{\sin x}(\sin\theta\cos\frac{\pi}{4}, \cos\theta\sin\frac{\pi}{4}, \sin\theta\sin\frac{\pi}{4})$.

We can either discard the unphysical global phase $e^{-\theta i}$, or let it merely add an $e^{-\theta i}\mathbb{1}$ term to our Hamiltonian which shifts the energy spectrum (which does nothing). Discarding the global phase, our unitary has been expressed as $U = e^{-ix(\sigma\cdot\mathbf{n})}$, so our Hamiltonian H ergo satisfies

$$\mathrm{e}^{-iHt}|_{t=\frac{1}{2}} = e^{-ix(\sigma\cdot\mathbf{n})}, \tag{62}$$

$$\implies -iH\frac{1}{2} = -ix(\sigma\cdot\mathbf{n}), \tag{63}$$

$$\implies H = 2x\,(\sigma\cdot\mathbf{n}) \tag{64}$$

4. Compute the corresponding state $|\phi\rangle$ for the two cases.

We can apply the two unitary matrices U_1, U_2 in Eq. (51) to $|1\rangle$ to see that

$$U_1\,|1\rangle = e^{-\frac{\pi}{4}i\,\sigma_y}\,|1\rangle = |-\rangle \tag{65}$$

$$U_2\,|1\rangle = \sigma_x\,U_1\,|1\rangle \quad = -\,|-\rangle\,. \tag{66}$$

It is worth noting the two unitaries produce states $\pm\,|-\rangle$; this is the same physical state, modulo the fictitious global phase. They remain *distinct* unitaries however (i.e. they themselves differ by more than just a prescribing global phase). We can appreciate this by applying them to arbitrary superpositions, in which case the unitaries prescribe different relative phases, which *are* physical. Explicitly (foregoing normalisation for clarity) :

$$U_1(|0\rangle + |1\rangle) = |+\rangle + |-\rangle\,, \tag{67}$$

$$U_2(|0\rangle + |1\rangle) = |+\rangle - |-\rangle\,. \tag{68}$$

5. Given that in experiment 4 the system was always found in $|0\rangle$, what can we say about the Hamiltonian of the system ?

Period

This tells us the evolution is periodic (as is all unitary evolution) with a maximum period of $t = 1$ hour. This is because unitary evolution is linear ; if we ever evolve back to our initial state, we *must* have evolved for an integer multiple of the period.

We know that $t = 1$ cannot be an *even* multiple of the period, because the state is not $|0\rangle$ at $t = 1/2$, where we know it instead to be $|+\rangle$. It remains possible that $t = 1$ is an *odd* multiple of the period. The period can ergo be any element of the set

$$\{\frac{1}{n} : n \text{ is an odd, positive integer}\} \tag{69}$$

$$\equiv \{\frac{1}{2n+1} : n \in \mathbb{N}\} \tag{70}$$

Hamiltonian

What does that imply about the Hamiltonian ? There are multiple ways to think about this. One method is to consider how an eigenstate $|\lambda_i\rangle$ of the Hamiltonian is time-evolved :

$$e^{-iHt}\,|\lambda_i\rangle = e^{-i\lambda_i t}\,|\lambda_i\rangle\,. \tag{71}$$

This is a result of the spectral theorem, and tells us that eigenstates merely experience a phase oscillation at frequency $\omega_i = \frac{\lambda_i}{2\pi}$. An arbitrary state can always be expressed in the complete basis of Hamiltonian eigenstates, so all unitary-time evolution can be understood as merely eigenstates phase-oscillating at different rates :

$$e^{-iHt}\sum_i \alpha_i\,|\lambda_i\rangle = \sum_i \alpha_i\,e^{-i\lambda_i t}\,|\lambda_i\rangle\,. \tag{72}$$

The periodicity of unitary-time evolution tells us information about the Hamiltonian spectrum, i.e. the eigenvalues $\{\lambda_i : i\}$. Can you figure out what ? **:)**

The link is much simpler for the single qubit case, where we have eigenvalues λ_1 and λ_2. Because the absolute value of the energy eigenvalues is *unphysical* (that is, translating them all up or down does not change the described system), we can arbitrarily shift the lowest eigenvalues (the “ground state") to be zero. Now our two eigenvalues are 0 and $(\lambda_2 - \lambda_1)$. The time evolution is now :

$$e^{-iHt} \left(\alpha_1 |\lambda_1\rangle + \alpha_2 |\lambda_2\rangle\right) = \alpha_1 e^0 |\lambda_1\rangle + e^{-i(\lambda_2-\lambda_1)t}\alpha_2 |\lambda_2\rangle \tag{73}$$

All the time dependence is in the complex phase of the second eigenstate. The coefficient $e^{-i(\lambda_2-\lambda_1)t}$ oscillates with frequency $\frac{(\lambda_2-\lambda_1)}{2\pi}$, or equivalently with period $P = \frac{2\pi}{\lambda_2-\lambda_1}$. Therefore, knowing the period $P = 1$ tells us...

$$P = 1 = \frac{2\pi}{\lambda_2 - \lambda_1} \qquad \Longrightarrow \lambda_2 - \lambda_1 = 2\pi \tag{74}$$

that the difference between the energy eigenvalues is 2π.

Knowing therefore that the period is $P = \frac{1}{2n+1}$ for any positive integer n, tells us that the possible difference between the energy eigenvalues is :

$$\frac{1}{2n+1} = \frac{2\pi}{\Delta\lambda} \qquad \Longrightarrow \Delta\lambda = \frac{2\pi}{2n+1} \tag{75}$$

One qubit dynamics

Another way to interpret how the known period relates to the Hamiltonian is to realise that all two-level (i.e. one-qubit) dynamics is sinusoidal. We can use one-qubit Pauli matrices as a basis to represent the Hamiltonian H. Indeed, we previously demonstrated that $H = \omega\sigma \cdot \mathbf{n}$. We can apply the solution of problem 2.5 to see that

$$U(t=1)\,|0\rangle = \cos\omega\,|0\rangle - i\sin\omega(\sigma\cdot\mathbf{n})\,|0\rangle = e^{i\theta}\,|0\rangle \tag{76}$$

which means that $|\cos\omega| = 1$, or in other words $\omega = \pi n$ where $n \in \mathbb{N}$.

Problem 3 : Collection of tensor product exercises

Consider a state $|\psi\rangle$ of *five* quantum particles, each with two levels denoted $|0\rangle$ and $|1\rangle$, which is initially

$$\begin{aligned} |\psi\rangle &= |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \\ &= |00000\rangle \,. \end{aligned}$$

We will identify these five particles with an index $0 \leq i < 5$ beginning from zero, which indicates the rightmost *ket* above.

1. Write down the state produced by applying the σ_x operator to the rightmost particle (of index $i = 0$). Write the *bra* version of it.

Because these are two-level particles, we call them "qubits" for brevity. We apply the one-qubit Pauli operator upon the qubit's corresponding *ket* in the tensor product. Let X_i denote a σ_x operator acting upon the i-th qubit, i.e.

$$\begin{aligned} X_0 |\psi\rangle &= |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes (\sigma_x |0\rangle) \\ &= |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |1\rangle \\ &= |00001\rangle \,. \end{aligned}$$

The corresponding *bra* state is simply

$$\langle\psi| X_0^\dagger = \langle 0| \otimes \langle 0| \otimes \langle 0| \otimes \langle 0| \otimes (\langle 0| \sigma_x) = \langle 00001| \,. \tag{77}$$

2. Write down the state produced by applying the σ_y operator to particle $i = 3$. Write the *bra* version of it.

As above, we apply σ_y upon particle $i = 3$, which is the *fourth* qubit from the right.

$$\begin{aligned} Y_3 |\psi\rangle &= |0\rangle \otimes (\sigma_y |0\rangle) \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \\ &= |0\rangle \otimes (i |1\rangle) \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \\ &= i |01000\rangle \end{aligned}$$

where we have commuted the scalar factor to the front of the tensor product. The corresponding *bra* state is

$$\langle\psi| Y_3^\dagger = (Y_3 |\psi\rangle)^\dagger = (i |01000\rangle)^\dagger = (i)^* |01000\rangle^\dagger = -i \langle 01000| \,. \tag{78}$$

3. For two arbitrary matrices A, B compute the commutator of $[A \otimes 1, 1 \otimes B]$ and the anti-commutator $\{A \otimes 1, 1 \otimes B\}$. Recall that $[A, B] = AB - BA$, $\{A, B\} = AB + BA$.

We simply substitute the definitions of the commutator and anti-commutator. Let us start with the commutator

$$\begin{aligned} [A \otimes 1, 1 \otimes B] &= A \otimes 1 \cdot 1 \otimes B - 1 \otimes B \cdot A \otimes 1 && (79) \\ &= A \otimes B - A \otimes B && (80) \\ &= 0 && (81) \end{aligned}$$

For the anti-commutator we can similarly do

$$\{A \otimes 1, 1 \otimes B\} = A \otimes 1 \cdot 1 \otimes B + 1 \otimes B \cdot A \otimes 1 = A \otimes B + A \otimes B = 2\, A \otimes B \tag{82}$$

4. The exponential of a sum of matrices satisfies $e^{A+B} = e^A e^B$ if $[A, B] = 0$. Use this and the previous exercise to show that $e^{A\otimes 1+1\otimes B} = e^A \otimes e^B$

In the previous exercise we showed that $[A \otimes 1, 1 \otimes B] = 0$. Thus we can use the identity in the question

$$e^{A\otimes 1+1\otimes B} = e^{A\otimes 1} e^{1\otimes B} \tag{83}$$

We can now see that if we Taylor expand $e^{A\otimes 1}$ and $e^{1\otimes B}$:

$$e^{A\otimes 1} = \sum_k \frac{(A\otimes 1)^k}{k!} = \sum_k \frac{A^k}{k!} \otimes 1 = e^A \otimes 1 \tag{84}$$

where we have recognised $(A \otimes 1)^k = (A \otimes 1)(A \otimes 1) \dots (A \otimes 1) = A^k \otimes 1^k = A^k \otimes 1$. Similarly

$$e^{1\otimes B} = \sum_k \frac{(1\otimes B)^k}{k!} = 1 \otimes \sum_k \frac{B^k}{k!} = 1 \otimes e^B \tag{85}$$

If we go back to Eq. (83) and apply this results we see that

$$e^{A\otimes 1+1\otimes B} = e^{A\otimes 1} e^{1\otimes B} = e^A \otimes 1 \cdot 1 \otimes e^B = e^A \otimes e^B \tag{86}$$

5. Write the matrix version of $a\,|0\rangle\langle 1| + b\,|1\rangle\langle 1|$.

Hint : recall that $|0\rangle = (1, 0)^T$, $|1\rangle = (0, 1)^T$ *and that* $|0\rangle\langle 1| = |0\rangle \otimes \langle 1|$.

$$|0\rangle\langle 1| = |0\rangle \otimes \langle 1| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes (0, 1) = \begin{pmatrix} 1 \cdot (0, 1) \\ 0 \cdot (0, 1) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{87}$$

Therefore

$$a\,|0\rangle\langle 1| = a \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}. \tag{88}$$

Similarly

$$b\,|1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \tag{89}$$

and we conclude

$$a\,|0\rangle\langle 1| + b\,|1\rangle\langle 1| = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \tag{90}$$

6. Let A, B be two 2×2 matrices. Compute $A \otimes B$.

$$A = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{pmatrix} \tag{91}$$

Then

$$A \otimes B = \begin{pmatrix} a_{11} \cdot B & a_{21} \cdot B \\ a_{12} \cdot B & a_{22} \cdot B \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{21} & a_{21}b_{11} & a_{21}b_{21} \\ a_{11}b_{12} & a_{11}b_{22} & a_{21}b_{12} & a_{21}b_{22} \\ a_{12}b_{11} & a_{12}b_{21} & a_{22}b_{11} & a_{22}b_{21} \\ a_{12}b_{12} & a_{12}b_{22} & a_{22}b_{12} & a_{22}b_{22} \end{pmatrix} \tag{92}$$