Quantum mechanics II, Chapter 2 : Entanglement (Part 3)

TA : Slimane Thabet, Sofia Brizigotti, Alba Miren Taddei, Reyhaneh Aghaei Saem, Mehrad Sahebi, Ricard Puig, Sacha Lerch

Sometimes observation kills.

```
                 |\---/|               |\---/|
                 | o_o |               | x_x |
         .-----..-\_‘_/                 \_‘_/-..-----.
        / +"", ’   ‘ \___           ___/ ‘   ’  ,""+ \
:’.___.‘|     /__  ’...__)         (__...’   __\     |‘.___.’;
 +.....’\(__‘.,_)’...,_)              (_,...’(_,.‘__)/’.....+
```

Problem 1 : Bell inequality (quantum psychics version)

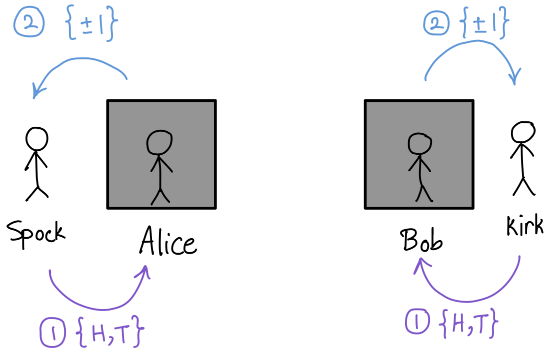

Figure 1 – **The Quantum Psychics Game.**

Alice and Bob claim to share a psychic connection. Some sceptics seek to test this by locking Alice and Bob into isolated rooms with no way to pass any messages between them. Outside Alice's room is a sceptic, let's call him Spock, who tosses a coin and tells Alice the outcome. Outside Bob's room is another sceptic, Kirk, who similarly tosses a coin and tells Bob the outcome. Alice and Bob must each then respond with a single bit of information ; a yes 'Y' or no 'N'. Spock and Kirk ask Alice and Bob to perform the following test :

If Alice and Bob ***both*** *are told 'H' they must give* ***opposite*** *answers, but* ***otherwise*** *(when one or both are told 'T') they must give the* ***same*** *answer.*

Secretly, Alice and Bob are not *psychics* - they are quantum *physicists* ! They share an entangled Bell state, $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, of which they can use the non-classical correlations stored within to pass the sceptics' test. Alice and Bob's strategy do so is as follows.

- If Alice gets told 'H' she measures in the Z basis and says 'Y' if she gets '$|0\rangle$' and 'N' if she gets '$|1\rangle$'.
- If Alice gets told 'T' she measures in the X basis and says 'Y' if she gets '$|+\rangle$' and 'N' if she gets '$|-\rangle$'.
- If Bob gets told 'H' he measures in the basis

$$\left\{|h\rangle = \sin(\pi/8)|0\rangle + \cos(\pi/8)|1\rangle\, ,\, |\overline{h}\rangle = \cos(\pi/8)|0\rangle - \sin(\pi/8)|1\rangle\right\} \qquad (1)$$

and says ‘Y’ if he gets ‘$|h\rangle$’ and ‘N’ if she gets ‘$|\overline{h}\rangle$’.

— If Bob gets told ‘T’ he measures in the basis

$$\big\{|t\rangle = \cos(\pi/8)|0\rangle + \sin(\pi/8)|1\rangle\,,\ |\overline{t}\rangle = \sin(\pi/8)|0\rangle - \cos(\pi/8)|1\rangle\big\} \tag{2}$$

and says ‘Y’ if he gets ‘$|t\rangle$’ and ‘N’ if she gets ‘$|\overline{t}\rangle$’.

Show that Alice and Bob can win the test and fool the sceptics with probability

$$P_{\text{Quantum}} = \cos(\pi/8)^2 = \frac{2+\sqrt{2}}{4} \approx 0.854\,. \tag{3}$$

Problem 2 : Tsirelson’s Bound

Suppose

$$Q = \mathbf{q}\cdot\sigma, \quad R = \mathbf{r}\cdot\sigma, \quad S = \mathbf{s}\cdot\sigma, \quad T = \mathbf{t}\cdot\sigma,$$

where $\mathbf{q}, \mathbf{r}, \mathbf{s}$ and $\mathbf{t}$ are real unit vectors in three dimensions. Let

$$A = Q\otimes S + R\otimes S + R\otimes T - Q\otimes T \tag{4}$$

Show that

$$A^2 = (Q\otimes S + R\otimes S + R\otimes T - Q\otimes T)^2 = 4I + [Q,R]\otimes[S,T].$$

Use this result to prove that

$$\langle A\rangle = \langle Q\otimes S\rangle + \langle R\otimes S\rangle + \langle R\otimes T\rangle - \langle Q\otimes T\rangle \le 2\sqrt{2}.$$

What is the maximum value of $\langle A\rangle$ in the classical case ? What do you infer from the difference between the classical and the quantum result ?

Hint : Assume any classical observables Q, R, S, T with possible measurement outcomes ± 1. In classical physics all observables commute with each other.

Problem 3 : Mermin-Peres and Telepathy (**non-examinable**)

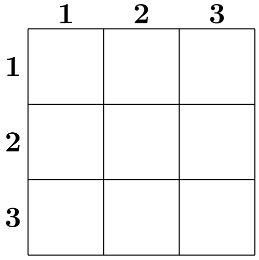

FIGURE 2 – Mermin-Peres Game Square

This exercise is another demonstration of quantum vs. classical correlations.
We consider Alice and Bob playing a cooperative game on a 3×3 square (see Figure 2). Alice will be assigned a secret row, and Bob assigned a secret column. The goal of the game is for Alice and Bob to fill their column/row with +1 or -1, without seeing each other’s values, while respecting the following constraints :

— Alice's row must have an even number of -1 (None or 2) i.e. the product of the entries is +1.

— Bob's column must contain an odd number of -1 (1 or 3) i.e. the product of the entries is -1.

— The square which is in both, Alice's row and Bob's column, must be filled with the same number by both (both +1 or both -1).

Alice and Bob are allowed to strategize before the game, however any communication is forbidden after they have been assigned their respective row or column.

1. Suppose they only have access to classical resources. Find a strategy that works out with a probability of success of 8/9 (It can be shown that this is the optimal success probability using classical resources). *Hint : Try to fill as many squares as possible of the 3×3 square with +1 and -1, while respecting the given constraints. How many squares can you fill until you reach a contradiction ?*
2. What changes if Bob and Alice can communicate *after* learning the row/column they've been assigned ?

	1	**2**	**3**
1	$+1/-1$	+1	+1
2	+1		
3	+1		

FIGURE 3 – Mermin-Peres Game Square after measuring $Z \otimes Z \otimes Z \otimes Z$ and obtaining $\{1, 1, 1, 1\}$ as measurement outcome.

3. Now suppose Alice and Bob share two copies of a maximally entangled Bell state i.e. the full quantum state is $|\psi\rangle = \frac{1}{2}(|0000\rangle + |0011\rangle + |1100\rangle + |1111\rangle)$. Each of them has one part of the entangled state, and they can choose to perform on each spin a measurement of $\{X, Y, Z, \mathbb{1}\}$ (by "measuring $\mathbb{1}$" on spin i, it is meant that no measurement is performed on the i-th spin). The goal is to find a measurement strategy that beats the classical one in terms of success chances.

   Before the game starts, they start to think about a strategy to win the game and come up with the following idea :

   They want to agree upon measurements they can perform on their qubits, for every combination of row and column they could possibly get. If they find 9 measurements (for every combination of row and column) that will successfully fill their row and columns (successful means they agree in the ij-th square, while keeping their constraints), they would win the game always. They start trying out different measurments they could perform. For example : If Alice got row 1 and Bob got column 1 they would have to agree in the top left corner of the 3×3 square. They decide to try both measure Z on both their qubits i.e. they measure $Z \otimes Z \otimes Z \otimes Z$ on $|\psi\rangle$. This results in 4 different possibilities to fill the squares : If Alice and Bob both measure $\{1, 1\}$, the state collapsed to $|0000\rangle$. Alice will put $\{1, 1\}$ in the second and third square of the first row, and Bob will put $\{1, 1\}$ in the second and third square of the first column. To respect the constraints Alice needs to fill another 1 in the top left square while Bob has to fill a -1 (see Figure 3). With this measurement they would therefore lose the game. Had they measured $\{1, 1, -1, -1\}$ (corresponding to the collapse to $|0011\rangle$), Alice would fill again $\{1, 1\}$ while Bob would fill $\{-1, -1\}$. Again respecting the constraints, Alice has to fill a 1, while Bob needs to fill a -1. They again lose.

	1	**2**	**3**	
1	$\mathbb{1} \otimes \sigma_z$	$\sigma_z \otimes \mathbb{1}$	$\sigma_z \otimes \sigma_z$	$+\mathbb{1}$
2	$\sigma_x \otimes \mathbb{1}$	$\mathbb{1} \otimes \sigma_x$	$\sigma_x \otimes \sigma_x$	$+\mathbb{1}$
3	$-\sigma_x \otimes \sigma_z$	$-\sigma_z \otimes \sigma_x$	$\sigma_y \otimes \sigma_y$	$+\mathbb{1}$
	$-\mathbb{1}$	$-\mathbb{1}$	$-\mathbb{1}$	

FIGURE 4 – Mermin-Peres Game Square

Here is a 4-qubit measurement that will result in a win every time, without failure. Convince yourself about this statement i.e. try to understand why this is true.

Problem 4 : Bloch sphere for pure or mixed states of a two-level system

In the lecture you have started to talk about density matrices. This exercise serves as a first introduction to the topic, connecting to the already known concept of the Bloch sphere.

1. *Derivation of Bloch vector from generic pure state.* Any pure one-qubit quantum state can be written as ket

$$|\psi\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\phi}|1\rangle \qquad \theta \in [0,\pi),\ \phi \in [0,2\pi)$$

   or as the density matrix,

$$|\psi\rangle\langle\psi| = \frac{1}{2}\left(\mathbb{1} + \hat{\boldsymbol{\sigma}} \cdot \boldsymbol{r}\right)$$

   Find an expression for $\mathbf{r}$ in terms of θ and ϕ. What does the vector $\mathbf{r}$ denote?

2. *Derivation of Bloch vector from properties of density operators.* Define the set of density matrices with the following 3 conditions :
   - The density matrix is Hermitian : $\hat{\rho}^\dagger = \hat{\rho}$
   - It has trace 1 : $\mathrm{Tr}\hat{\rho} = 1$
   - It is positive or null : $\langle\Psi|\hat{\rho}|\Psi\rangle \geq 0, \quad \forall\Psi$

   Show that any density matrix $\hat{\rho}$ of the 2 level system can be written

$$\hat{\rho} = \frac{1}{2}(\mathbb{1} + \hat{\boldsymbol{\sigma}} \cdot \boldsymbol{r}), \tag{5}$$

   where $\hat{\boldsymbol{\sigma}} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$. Argue that $\boldsymbol{r}$ is a real vector of 3D space and $|\boldsymbol{r}| \leq 1$.

3. Show that the state is pure iff $||\boldsymbol{r}|| = 1$. Explain why $\mathrm{Tr}[\rho^2]$ is a measure of the 'purity' of a quantum state.

4. Sketch on the Bloch sphere the states :
   a) $|0\rangle$
   b) $|+\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

c) $|-\rangle := \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

d) $\frac{1}{\sqrt{3}}(|0\rangle - i\sqrt{2}\,|1\rangle)$.

e) $\frac{1}{2}\mathbb{1}$

f) $\frac{1}{3}|+\rangle\langle+| + \frac{2}{3}|-\rangle\langle-|$

5. Give a geometric argument to show that $\frac{1}{2}(|+\rangle\langle+| + |-\rangle\langle-|) = \frac{1}{2}(|0\rangle\langle0| + |1\rangle\langle1|)$. (Is this surprising ?)

   Disclaimer : think about the meaning of this state. How would you represent $|0\rangle\langle0|$ on the Bloch sphere ? And $|1\rangle\langle1|$? Now if the space of qubits is a convex space, what is the point in the Bloch sphere that represents the combination of the previous one ?

   Now make the same reasoning for $|+\rangle\langle+|$, $|-\rangle\langle-|$.