QP II

Final exam
Autumn term 2025

Assignment date: Jan 29th, 2025, 15h15
Due date: Jan 29th, 2025, 18h15

PHYS-314 – Exam – room PO 01

- You must answer ALL questions in the short answer section.
- You must answer precisely 2 (out of 3) of the questions in the long answer section.

  Please mark clearly which two you have answered below and **start a new sheet for each of the long answer questions**.
- **Write your solutions in the indicated space.** Scrap paper will not be corrected.
- You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.
- A simple calculator (without internet access) is allowed.
- Please write your name on the top right corner of each sheet you use.
- Good luck! Enjoy!

NAME STICKER GOES HERE

Short answers:	/ 50
Problem A: YES or NO	/ 25
Problem B: YES or NO	/ 25
Problem C: YES or NO	/ 25
Total	/100

Short questions

1. **Basics of the quantum formalism.**

   Consider the state $|\Psi_{\mathrm{GHZ}}\rangle := \frac{1}{\sqrt{2}}(|+\rangle^{\otimes n} + |-\rangle^{\otimes n})$ where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$, and $|0\rangle$ and $|1\rangle$ are orthogonal basis states. Assume n is an arbitrary, positive integer. What are the expectation values of the following observables?

   (a) $|+\rangle\langle+| \otimes \hat{\mathbb{1}}^{\otimes(n-1)}$

   (b) $|0\rangle\langle 0| \otimes \hat{\mathbb{1}}^{\otimes(n-1)}$

   (c) $|+\rangle\langle+|^{\otimes 2} \otimes \hat{\mathbb{1}}^{\otimes(n-2)}$

   (d) $|0\rangle\langle 0|^{\otimes 2} \otimes \hat{\mathbb{1}}^{\otimes(n-2)}$

   (e) $|+\rangle\langle+|^{\otimes n}$

   (f) $|0\rangle\langle 0|^{\otimes n}$

   (9 marks)

2. **Decoherence.**

   A quantum system is initially in the superposition state:

   $$|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle),$$

   where $|0\rangle$ and $|1\rangle$ are orthogonal basis states. The system interacts with its environment, leading to decoherence. This interaction can be modeled by the density matrix of the system evolving as:

   $$\rho(t) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2}e^{-\gamma t} \\ \frac{1}{2}e^{-\gamma t} & \frac{1}{2} \end{bmatrix}$$

   where the parameter $\gamma > 0$ determines the system-environment interaction strength.

   (a) Define the decoherence time t_d as the time when the off-diagonal elements of $\rho(t)$ reduce to $1/e$ of their initial value. Solve for t_d in terms of γ.

   (2 marks)

   (b) The purity of a quantum state is given by $\mathrm{Tr}(\rho^2)$. Calculate the purity of the state $\rho(t)$ at time t and describe what it indicates about the system's transition from a pure state to a mixed state.

   (3 marks)

   (c) Describe how the process of decoherence, *as modeled above*, could explain the classical outcome observed in a quantum measurement.

   (4 marks)

3. **Indistinguishable Particles.**

Two indistinguishable bosons are confined in a one-dimensional harmonic potential, described by the Hamiltonian:

$$\hat{H} = \frac{\hat{p}_1^2}{2m} + \frac{m\omega^2 \hat{x}_1^2}{2} + \frac{\hat{p}_2^2}{2m} + \frac{m\omega^2 \hat{x}_2^2}{2},$$

where m is the particle mass, ω is the trap frequency, and $\hat{x}_1, \hat{x}_2$ are the position operators of the two particles.

(a) Calculate the total energy of the two-boson system in the ground state.
(2 marks)

(b) What is the first excited state of the two-boson system? Calculate its total energy.
(3 marks)

4. **Perturbation Theory.**

Consider a quantum harmonic oscillator with the unperturbed Hamiltonian

$$\hat{H}_0 = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2.$$

A perturbation $\hat{H}' = \alpha \hat{x}^4$ is added, where α is a small constant.

Using first-order perturbation theory, compute the correction to the ground state energy due to the perturbation.

(10 marks)

5. **Symmetry.**

Consider the 1-dimensional Hamiltonian:

$$\hat{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial \hat{x}^2} + V_0 \cos\left(\frac{2\pi \hat{x}}{a}\right),$$

where $a > 0$. Suppose that the system is confined to a region of width $L = l\,a$, where l is a positive integer, with periodic boundary conditions, i.e. the wavefunction must obey $\psi(L) = \psi(0)$.

(a) Explain why the symmetry group of $\hat{H}$ is $\mathbb{Z}_l$, the cyclic group of order l.
(3 marks)

(b) What dimension are the irreducible representations of $\mathbb{Z}_l$? Explain your reasoning.
(2 marks)

(c) How many irreducible representations are there of $\mathbb{Z}_l$? Explain your reasoning.
(2 marks)

(d) Write down the irreducible representations of $\mathbb{Z}_l$. Show that they are indeed a representation of $\mathbb{Z}_l$.

(2 marks)

(e) Consider the following guess for the eigenfunction of $\hat{H}$:

$$\psi_k(x) = e^{i\alpha_k x} u_k(x) \tag{1}$$

What constraints do we need on α_k and $u_k(x)$ to satisfy the symmetry properties and boundary conditions of $\hat{H}$.

(4 marks)

(f) Suppose we now turn on an additional perturbation of the form $V_1 \sin(\frac{2\pi\hat{x}}{b})$ where $L = l'\ b$, where l' is a positive integer. Suppose $a = 2$ and $b = 3$. What is the symmetry group of $\hat{H}$ in this case?

(4 marks)

Longer questions

Please **pick 2 questions** to attempt - mark your choices clearly on the cover sheet.

Start a new sheet for each question.

Question A - Variational Principle

Let us consider one particle with mass m, in one dimension, subject to a potential $V(x) = r|x|$ where $r > 0$. Assume the following Gaussian trial wavefunction:

$$\psi(x) \propto e^{-\frac{x^2}{\sigma^2}}, \quad \sigma > 0\,.$$

1. Using the variational principle with the above trial wavefunction compute, as a function of r, the following:

   i) an estimate of the ground state energy E_0.

   ii) an estimate for the ground state wavefunction.

   (14 marks)

Now suppose the potential is modified to

$$V(x) = r\left(\frac{|x|}{L}\right)^p \tag{2}$$

where $r > 0$, $L > 0$ and $p > 0$.

2. Sketch $V(x)$ for:

   i) $p = 0$

   ii) $p \to \infty$

   (2 marks)

3. Explicitly state (or, if you prefer, carefully sketch) the exact ground state wavefunction in each of the following limits:

   i) $p = 0$

   ii) $p = 2$

   iii) $p \to \infty$

   (4 marks)

4. What do you expect is the optimal value of σ in the Gaussian trial function to make the variational energy as low as possible in the following regimes? (*You do not need to perform explicit integrals*)

   i) $p \to 0$

   ii) $p = 2$

   iii) $p \to \infty$

   (3 marks)

5. Comment on features of the Gaussian trial wavefunction that make it suitable and/or unsuitable for the regimes

   i) $0 < p \ll 1$

   ii) $p = 2$

   iii) large p

   (2 marks)

You may find the following Gaussian integrals helpful:

$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}, \quad \int_{-\infty}^{+\infty} x^2 e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

Question B - Leggett–Garg Inequality

Consider a system characterized by an observable $Q(t)$ that can take values $+1$ or -1 at any time t. To test the principles of *macroscopic realism* and *non-invasive measurability*, measurements of Q are performed at three distinct times: t_1, t_2, and t_3, with $t_1 < t_2 < t_3$.

The principle of *macroscopic realism* implies that the observable $Q(t)$, which takes values $+1$ or -1, must always have a definite value at any time t, even when not being observed.

The principle of *non-invasive measurability* implies that it is possible to measure a system's properties without affecting its subsequent evolution. For example, measuring the value of $Q(t_1)$ should not influence the outcomes at later times (t_2, t_3).

Define $P(Q_i, Q_j)$ as the joint probability that $Q(t_i) = Q_i$ and $Q(t_j) = Q_j$.

1. Explain why macroscopic realism and non-invasive measurability imply there is a well defined probability distribution $P(Q_1, Q_2, Q_3)$ for the outcomes at times t_1, t_2, and t_3.

   (1 mark)

The "two-time correlation function" $C_{ij} = \langle Q(t_i)Q(t_j)\rangle$ is the product of the outcomes $Q(t_i)$ and $Q(t_j)$ averaged over the possible outcomes Q_i, Q_j weighted by their joint probabilities:

$$C_{ij} = \sum_{Q_i,Q_j} Q_i Q_j P(Q_i, Q_j) \tag{3}$$

for $i = 1, 2, 3$ and $j = 1, 2, 3$.

Let us define the correlation coefficient

$$K_3 := C_{12} + C_{23} - C_{13}\,. \tag{4}$$

2. Given that macroscopic realism and non-invasive measurability imply that $P(Q_1, Q_2, Q_3)$ is well defined, show that

   $$K_3 = \sum_{Q_1,Q_2,Q_3} P(Q_1, Q_2, Q_3)\,(Q_1Q_2 + Q_2Q_3 - Q_1Q_3)\,. \tag{5}$$

   (1 mark)

3. Hence, show that macroscopic realism and non-invasive measurability imply that

   $$K_3 = 1 - 4(P(+1, -1, +1) + P(-1, +1, -1))\,.$$

   Hint: Use the fact that $Q(t)$ takes values ± 1 to determine what values $Q_1Q_2 + Q_2Q_3 - Q_1Q_3$ can take.

   (6 marks)

4. Hence explain why macroscopic realism and non-invasive measurability imply that

$$K_3 \leq 1\,.$$

(1 mark)

This is known as the *Leggett-Garg inequality.*

In a quantum formalism the two-time correlation can be computed as

$$C_{ij} = \langle\psi(0)|\hat{Q}(t_i)\hat{Q}(t_j)|\psi(0)\rangle$$

where $\hat{Q}$ is the Hermitian observable being measured and $|\psi(0)\rangle$ is the initial state.

Consider a quantum two-level system evolving under the Hamiltonian:

$$\hat{H} = \frac{1}{2}\Omega\,\hat{\sigma}_x\,, \qquad \Omega \in \mathbb{R}$$

and suppose that $|\psi(0)\rangle = |0\rangle$ and $\hat{Q} = \hat{\sigma}_z$.

5. By computing an expression for the Heisenberg time evolved observable,

$$\hat{\sigma}_z(t) = e^{i\hat{H}t}\,\hat{\sigma}_z\,e^{-i\hat{H}t}\,, \tag{6}$$

show that $C_{ij} = \cos\left[\Omega(t_i - t_j)\right]$.

To simplify your solutions you may use the following trigonometric identities:

$$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 \qquad \sin(2x) = 2\sin(x)\cos(x)$$
$$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$$

(10 marks)

6. Now assume equally spaced time intervals τ, such that $t_{m+1} - t_m = \tau$. Show that in this case

$$K_3 = 2\cos(\Omega\tau) - \cos(2\Omega\tau),$$

(2 marks)

7. Determine the values of $\Omega\tau$ for which $K_3 > 1$, signifying a violation of the Leggett–Garg inequality.

(2 marks)

8. What does this violation of the Leggett–Garg inequality tell us about quantum physics?

(2 marks)

Question C - Lie Algebras and Angular Momentum

The special orthogonal group $SO(3)$ is the Lie group of rotations in 3 dimensions. An orthogonal matrix is the real analogue of a unitary matrix and is defined by the properties $\Re[M] = M$ and $MM^T = M^T M = I$. For an orthogonal matrix to be a rotation matrix we also require that $\det(M) = 1$.

The Lie algebra is formally defined as the tangent space to the Lie group at the identity element. Any element $A \in G$ can be written as $A(t) = e^{tX}, X \in g$, where g is the Lie algebra associated with the Lie group G. Therefore one can access the generators, X, by looking at: $\frac{d}{dt}A(t)|_{t=0}$.

1. From the definition of $SO(3)$ given above, compute the 3-dimensional generators X of the Lie algebra of $SO(3)$.

   (3 marks)

2. Define $J = -iX$ with X the generators of the Lie algebra of $SO(3)$. Calculate the commutators of all the J's and identify their corresponding structure constants.

   (2 marks)

3. Find a 4-dimensional irreducible representation the Lie algebra of $SO(3)$.

   Hint: look at the action of the ladder operators $\hat{J}_+ = \hat{J}_x + i\hat{J}_y$ and $\hat{J}_- = \hat{J}_x - i\hat{J}_y$ on a spin 3/2 system.

   (10 marks)

The total angular momentum operators in the composite system can be written as $\hat{J}_{i,\text{total}} = \hat{J}_{i,1} \otimes \hat{\mathbb{1}}_2 + \hat{\mathbb{1}}_1 \otimes \hat{J}_{i,2}$ and the total angular momentum eigenstates $|j, m\rangle$ of the composite system can be written in terms of the eigenstates of the individual subsystems as follows:

$$|j, m\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |j_1, m_1, j_2, m_2\rangle\langle j_1, m_1, j_2, m_2|j, m\rangle \,. \tag{7}$$

The coefficients $\langle j_1, m_1, j_2, m_2|j, m\rangle$ are known as the Clebsch-Gordan coefficients.

4. Show that $\frac{3}{2} \otimes 1 = \frac{5}{2} \oplus \frac{3}{2} \oplus \frac{1}{2}$ and calculate the Clebsch-Gordan coefficients for the composite system composed of a $j_1 = 3/2$ and a $j_2 = 1$ subsystem.

   (10 marks)

You will find the following relations helpful.

$$\begin{aligned} \hat{J}_+ |m\rangle &= \sqrt{(j+1+m)(j-m)}\, |m+1\rangle \\ \hat{J}_- |m\rangle &= \sqrt{(j+1-m)(j+m)}\, |m-1\rangle \\ \hat{J}_z |m\rangle &= m\, |m\rangle \end{aligned} \tag{8}$$