Final exam
Autumn term 2025

Assignment date: 2025
Due date: 2025

PHYS-314 – Exam – Solutions

Short questions

1. **Basics of the quantum formalism.**

(a)

$$\langle\Psi_{\rm GHZ}|\,|+\rangle\langle+|\otimes\hat{\mathbb{1}}\,|\Psi_{\rm GHZ}\rangle = \frac{1}{2}\left[(\langle+|^{\otimes n}+\langle-|^{\otimes n})\,|+\rangle\langle+|\otimes\hat{\mathbb{1}}(|+\rangle^{\otimes n}+|-\rangle^{\otimes n})\right]$$
$$= \frac{1}{2}\left[\langle+|^{\otimes n}|+\rangle\langle+|\otimes\hat{\mathbb{1}}\,|+\rangle^{\otimes n} + \langle+|^{\otimes n}|+\rangle\langle+|\otimes\hat{\mathbb{1}}\,|-\rangle^{\otimes n}\right.$$
$$\left.+\langle-|^{\otimes n}|+\rangle\langle+|\otimes\hat{\mathbb{1}}\,|+\rangle^{\otimes n} + \langle-|^{\otimes n}|+\rangle\langle+|\otimes\hat{\mathbb{1}}\,|-\rangle^{\otimes n}\right]$$
$$= \frac{1}{2}[1+0+0+0] = \frac{1}{2}$$

(b) If n=1 then we have $|\Psi_{\rm GHZ}\rangle = \frac{1}{\sqrt{2}}(|+\rangle+|-\rangle) = |0\rangle$, then $\langle\Psi_{\rm GHZ}|\,|0\rangle\langle0|\,|\Psi_{\rm GHZ}\rangle = 1$.
In fact it's the last case with n=1.
If n≠1 :

$$\langle\Psi_{\rm GHZ}|\,|0\rangle\langle0|\otimes\hat{\mathbb{1}}\,|\Psi_{\rm GHZ}\rangle$$
$$= \frac{1}{2}\left[\left(\left(\frac{\langle0|+\langle1|}{\sqrt{2}}\right)^{\otimes n} + \left(\frac{\langle0|-\langle1|}{\sqrt{2}}\right)^{\otimes n}\right)|0\rangle\langle0|\otimes\hat{\mathbb{1}}\left(\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)^{\otimes n} + \left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)^{\otimes n}\right)\right]$$
$$= \frac{1}{2}\left[\left(\frac{\langle0|}{\sqrt{2}}\otimes\left(\frac{\langle0|+\langle1|}{\sqrt{2}}\right)^{\otimes(n-1)} + \frac{\langle0|}{\sqrt{2}}\otimes\left(\frac{\langle0|-\langle1|}{\sqrt{2}}\right)^{\otimes(n-1)}\right)\right.$$
$$\left.\left(\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)^{\otimes n} + \left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)^{\otimes n}\right)\right] = \frac{1}{2}\left[\frac{1}{2}+0+0+\frac{1}{2}\right] = \frac{1}{2}$$

(c)

$$\langle\Psi_{\text{GHZ}}|\left(|+\rangle\langle+|\right)^{\otimes 2}\otimes\hat{\mathbb{1}}\,|\Psi_{\text{GHZ}}\rangle=\frac{1}{2}\left[\left(\langle+|^{\otimes n}+\langle-|^{\otimes n}\right)|+\rangle\langle+|^{\otimes 2}\otimes\hat{\mathbb{1}}(|+\rangle^{\otimes n}+|-\rangle^{\otimes n})\right]$$
$$=\frac{1}{2}\left[\langle+|^{\otimes n}\left(|+\rangle\langle+|\right)^{\otimes 2}\otimes\hat{\mathbb{1}}\,|+\rangle^{\otimes n}+\langle+|^{\otimes n}\left(|+\rangle\langle+|\right)^{\otimes 2}\otimes\hat{\mathbb{1}}\,|-\rangle^{\otimes n}\right.$$
$$\left.+\langle-|^{\otimes n}\left(|+\rangle\langle+|\right)^{\otimes 2}\otimes\hat{\mathbb{1}}\,|+\rangle^{\otimes n}+\langle-|^{\otimes n}\left(|+\rangle\langle+|\right)^{\otimes 2}\otimes\hat{\mathbb{1}}\,|-\rangle^{\otimes n}\right]$$
$$=\frac{1}{2}[1+0+0+0]=\frac{1}{2}$$

(d) If n=2 then we have $|\Psi_{\text{GHZ}}\rangle=\frac{1}{\sqrt{2}}(|+\rangle^{\otimes 2}+|-\rangle^{\otimes 2})=\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)$, then $\langle\Psi_{\text{GHZ}}|\,|00\rangle\langle 00|\,|\Psi_{\text{GHZ}}\rangle=\frac{1}{2}$. In fact it's the last case with n=2.
If n≠2 :

$$\langle\Psi_{\text{GHZ}}|\left(|0\rangle\langle 0|\right)^{\otimes 2}\otimes\hat{\mathbb{1}}\,|\Psi_{\text{GHZ}}\rangle$$
$$=\frac{1}{2}\left[\left(\left(\frac{\langle 0|+\langle 1|}{\sqrt{2}}\right)^{\otimes n}+\left(\frac{\langle 0|-\langle 1|}{\sqrt{2}}\right)^{\otimes n}\right)\left(|0\rangle\langle 0|\right)^{\otimes 2}\otimes\hat{\mathbb{1}}\left(\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)^{\otimes n}+\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)^{\otimes n}\right)\right]$$
$$=\frac{1}{2}\left[\left(\frac{\langle 0|}{\sqrt{2}}\otimes\frac{\langle 0|}{\sqrt{2}}\otimes\left(\frac{\langle 0|+\langle 1|}{\sqrt{2}}\right)^{\otimes(n-2)}+\frac{\langle 0|}{\sqrt{2}}\otimes\frac{\langle 0|}{\sqrt{2}}\otimes\left(\frac{\langle 0|-\langle 1|}{\sqrt{2}}\right)^{\otimes(n-2)}\right)\right.$$
$$\left.\left(\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)^{\otimes n}+\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)^{\otimes n}\right)\right]=\frac{1}{2}\left[\frac{1}{4}+0+0+\frac{1}{4}\right]=\frac{1}{4}$$

(e)

$$\langle\Psi_{\text{GHZ}}|\left(|+\rangle\langle+|\right)^{\otimes n}|\Psi_{\text{GHZ}}\rangle=\frac{1}{2}\left[\left(\langle+|^{\otimes n}+\langle-|^{\otimes n}\right)\left(|+\rangle\langle+|\right)^{\otimes n}(|+\rangle^{\otimes n}+|-\rangle^{\otimes n})\right]$$
$$=\frac{1}{2}\left[\langle+|^{\otimes n}\left(|+\rangle\langle+|\right)^{\otimes n}|+\rangle^{\otimes n}+\langle+|^{\otimes n}\left(|+\rangle\langle+|\right)^{\otimes n}|-\rangle^{\otimes n}\right.$$
$$\left.+\langle-|^{\otimes n}\left(|+\rangle\langle+|\right)^{\otimes n}|+\rangle^{\otimes n}+\langle-|^{\otimes n}\left(|+\rangle\langle+|\right)^{\otimes n}|-\rangle^{\otimes n}\right]$$
$$=\frac{1}{2}[1+0+0+0]=\frac{1}{2}$$

(f)

$$\langle\Psi_{\text{GHZ}}|\left(|0\rangle\langle 0|\right)^{\otimes n}|\Psi_{\text{GHZ}}\rangle$$
$$=\frac{1}{2}\left[\left(\left(\frac{\langle 0|+\langle 1|}{\sqrt{2}}\right)^{\otimes n}+\left(\frac{\langle 0|-\langle 1|}{\sqrt{2}}\right)^{\otimes n}\right)\left(|0\rangle\langle 0|\right)^{\otimes n}\left(\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)^{\otimes n}+\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)^{\otimes n}\right)\right]$$
$$=\frac{1}{2}\left[\left(\left(\frac{\langle 0|}{\sqrt{2}}\right)^{\otimes n}+\left(\frac{\langle 0|}{\sqrt{2}}\right)^{\otimes n}\right)\left(\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)^{\otimes n}+\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)^{\otimes n}\right)\right]$$
$$=\frac{1}{2}\left[\frac{1}{2^n}+\frac{1}{2^n}+\frac{1}{2^n}+\frac{1}{2^n}\right]=\frac{1}{2^{n-1}}$$

1 point for a),c) and e) (0.5 point for the reasoning and/or generalization for c) and e) and 0.5 point for answer) 2 points for b), d) and f) (1.5 points for reasoning (change of basis and action with trensor product) and 0.5 point for answer

2. **Decoherence.**

(a) From the definition of t_d we can write:

$$\frac{\frac{1}{2}e^{-\gamma t_d}}{\frac{1}{2}e^{-\gamma t_0}} = \frac{1}{e} \Rightarrow e^{-\gamma t_d} = e^{-1} \Rightarrow \gamma t_d = 1 \Rightarrow t_d = \frac{1}{\gamma} \tag{1}$$

1 point for the global reasoning and 1 point for the final answer (-0.5 if not using $t_0 = 0$)

(b) First we compute ρ^2 :

$$\rho^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2}e^{-\gamma t} \\ \frac{1}{2}e^{-\gamma t} & \frac{1}{2} \end{bmatrix}^2 = \begin{bmatrix} \frac{1}{4}\left(1+e^{-2\gamma t}\right) & \frac{1}{2}e^{-\gamma t} \\ \frac{1}{2}e^{-\gamma t} & \frac{1}{4}\left(1+e^{-2\gamma t}\right) \end{bmatrix} \tag{2}$$

Then we can compute the trace of ρ^2:

$$\text{Tr}(\rho^2) = \frac{1}{2}\left(1+e^{-2\gamma t}\right) \tag{3}$$

The system goes from the pure state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ at t=0, with $\text{Tr}(\rho(0)) = 1$, to the maximally mixed state $\rho(t \to \infty) = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$ with $\text{Tr}(\rho(0)) = \frac{1}{2}$. This transition is exponentially fast and depend on the strength of the interaction between the system and the environment. As seen on point a), $t_d = \frac{1}{\gamma}$.

1.5 point for $\text{Tr}(\rho^2)$. 1.5 points for the comment on the system transition (0.5 for the pure state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ at t=0 with $\text{Tr}(\rho(0)) = 1$, 0.5 point for maximally mixed state $\rho(t \to \infty) = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$ with $\text{Tr}(\rho(0)) = \frac{1}{2}$) and 0.5 point for the exponentially fast transition

(c)
- The interaction with the environment kills off the off-diagonal 'coherence' terms.
- The interaction with the environment turns the state from a quantum superposition (where we could observe interference effects) to a classical mixture (where we cannot observe interference effects).
- Thus the interaction with the environment leads to a state where measurement results can be explained using classical statistics.
- The stronger the interaction with the environment the quicker this transition.

two marks each for any of the above points, or similar, up to a total of four. If the point was not super clear (eg bullet point 2 : no mention of interference or what implies a classical mixture) only one point is given. Just saying that the system transitions from a pure state to a mixed state is not as it was said in the previous question.

3. **Indistinguishable Particles.**

a) Since we consider bosons, the particles can occupy the same state. We just need to make sure that the state is symmetric w.r.t. permutations of the two particles. The ground-state of this system is obtained when both particles are in the ground-state of the individual harmonic oscillators: $|\psi\rangle_{GS} = |00\rangle$. The energy of that state is given by $E = \hbar\omega(n_1 + n_2 + 1) = \hbar\omega$.
b) The first excited state is given by the state, where one of the particles is in the fist excited state of the individual harmonic oscillator and the other particle remains in the ground state: $|\psi\rangle_1 = |01\rangle$. Since the state should be symmetric we need to sum the combinations of promoting the first or the second particle to the excited state: $|\psi\rangle_1 = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. The energy is again $E = \hbar\omega(n_1 + n_2 + 1) = 2\hbar\omega$.

Addition of two harmonic oscillator energy +1 explanation +1 correct energy in a) (2 points if the energy is correct). In b) 2 points for the state -1 if non symmetric and 1 point for the energy.

4. **Perturbation Theory.**

a) To compute the first order correction of the perturbation to the ground-state of the harmonic oscillator, we need to compute the matrix elements: $\langle 0|\alpha x^4|0\rangle$. We have $x = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a)$. Since $a^\dagger, a$ increase/decrease the quantum state by 1, we only get non-zero contributions if the number of $a^\dagger$ and a is the same. These terms are $\propto$ $(a^\dagger)^2 a^2, a^\dagger a a^\dagger a, a^\dagger a^2 a^\dagger, a(a^\dagger)^2 a, a a^\dagger a a^\dagger, a^2 (a^\dagger)^2$. Additionally terms with an a in the end give zero for the ground-state. We therefore have: $\langle 0|\alpha x^4|0\rangle = \alpha\sqrt{\frac{\hbar}{2m\omega}}^4(\langle 0|a^\dagger a^2 a^\dagger|0\rangle + \langle 0|a a^\dagger a a^\dagger|0\rangle + \langle 0|a^2(a^\dagger)^2|0\rangle) = \alpha(\frac{\hbar}{2m\omega})^2(1+2) = 3\alpha(\frac{\hbar}{2m\omega})^2$.
b) For the correction to the first excited state we can proceed as in a), however this time terms with an a in the end, do not automatically evaluate to 0, however the ones with two a do. We therefore have: $\langle 1|\alpha x^4|1\rangle = \alpha\sqrt{\frac{\hbar}{2m\omega}}^4(\langle 1|a^\dagger a a^\dagger a|1\rangle + \langle 1|a^\dagger a^2 a^\dagger|1\rangle + \langle 1|a(a^\dagger)^2 a|1\rangle + \langle 1|a a^\dagger a a^\dagger|1\rangle + \langle 1|a^2(a^\dagger)^2|1\rangle) = \alpha(\frac{\hbar}{2m\omega})^2(1+2+2+4+6) = 15\alpha(\frac{\hbar}{2m\omega})^2$.

+5 points for correct approach (measure V in the ground state, provide the ground state and establish a correct equation). Can give points if some correct idea is/are provided e.g. +2 to state what we have to compute (e.g. $\langle 0|V|0\rangle$), +1 using annihilator/creator operator, +2 if correctly written. +5 points if correct computation. But, might be only +2 pts if the steps are correct, but there is a mistake and no details provided (e.g. just use a formula for integrals). Only +3 if little mistake in the derivation, but if the details are provided. +1 if the final answer is correct (no need to simplify or no typo). If there is a mistake in the integral but they didn't use a formula from the cheat sheet, it will be more strict than if they do it with creation/annihilation operator. But the idea using creation/anihilation give a bit more points as mentioned before. (Both approaches are the same, one is harder to compute and the other is a better idea but easier to compute so more strict with computation)

5. **Symmetry.**

Consider the 1-dimensional Hamiltonian:

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial \hat{x}^2} + V_0 \cos\left(\frac{2\pi \hat{x}}{a}\right),$$

where $a > 0$. Suppose that the system is confined to a region of width $L = l\,a$, where l is a positive integer, with periodic boundary conditions, i.e. the wavefunction must obey $\psi(L) = \psi(0)$.

(a) Explain why the symmetry group of $\hat{H}$ is $\mathbb{Z}_l$, the cyclic group of order l.

(3 marks)

Most of them wrote text. Any of the following (or similar) I will point max:

- The H commutes with the translation operator T_a.
- Identifying the general eigenfunction, and show that they are also eigenfunctions of T_a.
- Identifying that the potential is periodic in a, and say that this is how $\mathbb{Z}_l$ is defined.

(b) What dimension are the irreducible representations of $\mathbb{Z}_l$? Explain your reasoning.

(2 marks)

This is an Abelian group, thus the irreps are 1-dim. 2 points if they reason it. 0 points if they don't reason it.

I will award one point if they tried a longer proof that makes sense, but that its not complete.

=3+2+2 Any correct proof will be awarded two points.

(c) How many irreducible representations are there of $\mathbb{Z}_l$? Explain your reasoning.

(2 marks)

Burnside lemma. There are l different irreps.

1 point $\rightarrow$ mention the lemma

1 point $\rightarrow$ apply the lemma correctly 1 point.

(d) Write down the irreducible representations of $\mathbb{Z}_l$. Show that they are indeed a representation of $\mathbb{Z}_l$.

(2 marks)

One irreductible irrep is the trivial one. Another is $e_k = e^{-i2\pi k/l}\,|k \in 0,..,l-1$. The other ones are taking $\widetilde{e}_{1,i} = (e_1)^i$ as the representation of the first (non identity) element and then have $\{(\widetilde{e}_{1,i})^k\}_{k\in 0,..,l-1}$, for $i \in \{2,...,l-1\}$.

3 1 point $\rightarrow$ having one non trivial irrep and showing that it is indeed a representation of $\mathbb{Z}_l$. Note I will accept as showing anything that resembles an arguement (else is very harsh no?).

1 point $\rightarrow$ showing they know the others, even if they write it in text.

Extras:

- If they dont get full points, up to 1 point for showing things that are important: trivial rep, general construction $\{(a)^k\}_{k\in 0,..,l-1}$. I did not find more so far.

(e) Consider the following guess for the eigenfunction of $\hat{H}$:

$$\psi_k(x) = e^{i\alpha_k x} u_k(x) \tag{4}$$

What constraints do we need on α_k and $u_k(x)$ to satisfy the symmetry properties and boundary conditions of $\hat{H}$.

(4 marks)

1 point $\rightarrow$ show that we need the property $T_a\psi_k(x) = \lambda\psi_k(x)$

2 point $\rightarrow$ finding $T_a u_k = u_k$

1 point $\rightarrow$ finding condition on α_k imposed by the periodic boundary $e^{-i\alpha_k x} = e^{-i\alpha_k(x+L)}$, $\alpha_k = 2\pi f(k)/L$ such that $f(k) \in \mathbb{Z}$.

(f) Suppose we now turn on an additional perturbation of the form $V_1 \sin(\frac{2\pi\hat{x}}{b})$ where $L = l'\ b$, where l' is a positive integer. Suppose $a = 2$ and $b = 3$. What is the symmetry group of $\hat{H}$ in this case?

(4 marks)

2 point $\rightarrow$ identify that it is still a cyclic group.+2

1 point $\rightarrow$ try to find the period

1 point $\rightarrow$ give the period 6.

Longer questions

Question A - Variational Principle

1) (i) Given our trial, unnormalised wavefunction $\psi(x) = e^{-x^2/\sigma^2}$, we seek σ which minimises

$$\langle E\rangle = \frac{\langle\psi|\hat{H}|\psi\rangle}{\langle\psi|\psi\rangle} \tag{5}$$

where our 1D particle's Hamiltonian is

$$\hat{H} = \hat{T} + \hat{V} = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + r|x|. \tag{6}$$

Write expression for expval of E and the Hamiltonian: 1+1 Points

Recall that ket $|\psi\rangle$ relates to the wavefunction $\psi(x)$ by

$$|\psi\rangle = \int_{-\infty}^{\infty} \psi(x)\,|x\rangle\,\mathrm{d}x, \tag{7}$$

so that expectation values are simply evaluated as

$$\begin{aligned}\langle\psi|\hat{A}|\psi\rangle &= \int_{-\infty}^{\infty} \psi^*(x')\,\langle x'|\,\mathrm{d}x'\,\hat{A}\int_{-\infty}^{\infty}\psi(x)\,\langle x|\,\mathrm{d}x &(8)\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi^*(x')\,\hat{A}\,\psi(x)\ \langle x'|x\rangle\,\mathrm{d}x'\mathrm{d}x &(9)\\ &= \int_{-\infty}^{\infty}\psi^*(x)\,\hat{A}\,\psi(x)\mathrm{d}x. &(10)\end{aligned}$$

This means that the denominator of $\langle E\rangle$ is

$$\langle\psi|\psi\rangle = \langle\psi|\hat{\mathbb{1}}|\psi\rangle = \int_{-\infty}^{\infty}\psi^*(x)\psi(x)\mathrm{d}x = \int_{-\infty}^{\infty}|\psi(x)|^2\mathrm{d}x = \int_{-\infty}^{\infty}|\mathrm{e}^{-x^2/\sigma^2}|^2\mathrm{d}x \tag{11}$$

and since $e^{-x^2\sigma^2}$ is always positive, simplifies to

$$\langle\psi|\psi\rangle = \int_{-\infty}^{\infty} e^{-2x^2/\sigma^2}\mathrm{d}x. \tag{12}$$

We can transform this into our known Gaussian integral using a change of variables. We choose $u = \frac{\sqrt{2}x}{\sigma}$, satisfying $u^2 = \frac{2x^2}{\sigma^2}$ as seen in our integral. Then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\sqrt{2}}{\sigma} \qquad \Longrightarrow \qquad \mathrm{d}x = \frac{\sigma}{\sqrt{2}}\,\mathrm{d}u \tag{13}$$

and $\lim_{x\to\pm\infty} x = \lim_{x\to\pm\infty} u$, so our integration endpoints are unchanged. Ergo

$$\langle\psi|\psi\rangle = \frac{\sigma}{\sqrt{2}}\int_{-\infty}^{\infty} e^{-u^2}\mathrm{d}u = \sigma\sqrt{\frac{\pi}{2}}, \tag{14}$$

as per the standard integral.

Meanwhile, the numerator of $\langle E \rangle$ is

$$\langle\psi|\hat{H}|\psi\rangle = \int_{-\infty}^{\infty} \psi^*(x)\hat{H}\psi(x)\mathrm{d}x = \int_{-\infty}^{\infty} e^{-\frac{x^2}{\sigma^2}} \left(-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + r|x|\right) e^{-\frac{x^2}{\sigma^2}}\mathrm{d}x, \tag{15}$$

where, using the chain and product differentiation rules,

$$\frac{\mathrm{d}}{\mathrm{d}x}\, e^{-\frac{x^2}{\sigma^2}} = -\frac{2x}{\sigma^2}e^{-\frac{x^2}{\sigma^2}}, \tag{16}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\, e^{-\frac{x^2}{\sigma^2}} = -\frac{2}{\sigma^2}e^{-\frac{x^2}{\sigma^2}} + \frac{4x^2}{\sigma^4}e^{-\frac{x^2}{\sigma^2}}, \tag{17}$$

we obtain

$$\langle\psi|\hat{H}|\psi\rangle = \int_{-\infty}^{\infty} e^{-\frac{x^2}{\sigma^2}} \left(\frac{-\hbar^2}{2m}\right)\left(-\frac{2}{\sigma^2}e^{-\frac{x^2}{\sigma^2}} + \frac{4x^2}{\sigma^4}e^{-\frac{x^2}{\sigma^2}}\right)\mathrm{d}x + \int_{-\infty}^{\infty} r|x|e^{-2\frac{x^2}{\sigma^2}}\mathrm{d}x \tag{18}$$

$$= \frac{\hbar^2}{m\sigma^2}\int_{-\infty}^{\infty} e^{-2\frac{x^2}{\sigma^2}}\mathrm{d}x - \frac{2\hbar^2}{m\sigma^4}\int_{-\infty}^{\infty} x^2 e^{-2\frac{x^2}{\sigma^2}}\mathrm{d}x + 2\,r\int_{0}^{\infty} x\,e^{-2\frac{x^2}{\sigma^2}}\mathrm{d}x, \tag{19}$$

where the last integral was changed by observing $|x|e^{-x^2}$ is an even function. We will now evaluate these integrals in-turn. We saw already (when evaluating $\langle\psi|\psi\rangle$) that the first integral is

$$\int_{-\infty}^{\infty} e^{-2\frac{x^2}{\sigma^2}}\mathrm{d}x = \sigma\sqrt{\frac{\pi}{2}}. \tag{20}$$

The second integral does not yet match the problem's given Gaussian integral $\int x^2 e^{-x^2}\mathrm{d}x$ due to the $2\sigma^2$ coefficient of the exponent, so we perform another change of variables. We can in fact use the same substitution when calculating $\langle\psi|\psi\rangle$, whereby $u = \frac{\sqrt{2}\,x}{\sigma}$ and $\mathrm{d}x = \frac{\sigma}{\sqrt{2}}\mathrm{d}u$, such that $x^2 = \frac{\sigma^2}{2}u^2$ and

$$\int_{-\infty}^{\infty} x^2 e^{-2\frac{x^2}{\sigma^2}}\mathrm{d}x = \frac{\sigma^2}{2}\frac{\sigma}{\sqrt{2}}\int_{-\infty}^{\infty} u^2 e^{-u^2}\mathrm{d}u = \frac{\sigma^3}{2\sqrt{2}}\frac{\sqrt{\pi}}{2} = \frac{\sigma^3}{4}\sqrt{\frac{\pi}{2}}. \tag{21}$$

The third integral is recognised to be of the form $\propto \int f'(x)e^{f(x)}\mathrm{d}x = e^{f(x)}$.

$$\int_{0}^{\infty} x\,e^{-2\frac{x^2}{\sigma^2}} = -\frac{\sigma^2}{4}\int_{0}^{\infty} -\frac{4}{\sigma^2}\,x\,e^{-2\frac{x^2}{\sigma^2}} = -\frac{\sigma^2}{4}\left[e^{-2\frac{x^2}{\sigma^2}}\right]_0^{\infty} = \frac{\sigma^2}{4}. \tag{22}$$

Thus

$$\langle\psi|\hat{H}|\psi\rangle = \frac{\hbar^2}{m\sigma^2}\left(\sigma\sqrt{\frac{\pi}{2}}\right) - \frac{2\hbar^2}{m\sigma^4}\left(\frac{\sigma^3}{4}\sqrt{\frac{\pi}{2}}.\right) + 2\,r\left(\frac{\sigma^2}{4}\right) \tag{23}$$

$$= \frac{\hbar^2\sqrt{2\pi}}{4m\sigma} + r\frac{\sigma^2}{2}. \tag{24}$$

There are 3 integrals to solve. 1) Normalization: 2 Points. 2) Pot. Energy: 2 Points 3) Kinetic energy: 4 points Our parameterised expectation value is therefore

$$\langle E\rangle = \frac{\langle\psi|\hat{H}|\psi\rangle}{\langle\psi|\psi\rangle} = \left(\frac{\hbar^2\sqrt{2\pi}}{4m\sigma} + r\frac{\sigma^2}{2}\right)\frac{\sqrt{2}}{\sigma\sqrt{\pi}} \tag{25}$$

$$= \frac{\hbar^2}{2m\sigma^2} + \frac{r\sigma}{\sqrt{2\pi}}. \tag{26}$$

We seek σ_0 which minimises this energy, which satisfies

$$0 = \frac{\mathrm{d}}{\mathrm{d}\sigma_0}\langle E\rangle|_{\sigma=\sigma_0} = -\frac{\hbar^2}{m\sigma^3} + \frac{r}{\sqrt{2\pi}} \tag{27}$$

$$\implies \sigma_0{}^3 = \frac{\hbar^2\sqrt{2\pi}}{m\,r} \tag{28}$$

$$\implies \sigma_0 = \left(\frac{\hbar^2\sqrt{2\pi}}{m\,r}\right)^{1/3}, \tag{29}$$

Writing derivative = 0: 1 Point, compute correct σ_0: 1 Point
where we have chosen the real and positive root, consistent with $\sigma > 0$. We can check this value corresponds to the *minimum* of $\langle E\rangle$, as opposed to a maximum or inflection point, by checking that $\frac{\mathrm{d}^2}{\mathrm{d}\sigma^2}\langle E\rangle > 0$ at this value, which is evidently satisfied for all σ.
Substituting σ_0 into $\langle E\rangle$ produces our estimate of the ground state energy.

$$E_0 = \frac{\hbar^2}{2m\sigma_0{}^2} + \frac{r\sigma_0}{\sqrt{2\pi}} \tag{30}$$

$$= \frac{\hbar^2}{2m}\left(\frac{m\,r}{\hbar^2\sqrt{2\pi}}\right)^{2/3} + \frac{r}{\sqrt{2\pi}}\left(\frac{\hbar^2\sqrt{2\pi}}{m\,r}\right)^{1/3} \tag{31}$$

$$= \frac{1}{2}\left(\frac{\hbar^2 r^2}{2\pi\,m}\right)^{1/3} + \left(\frac{\hbar^2 r^2}{2\pi\,m}\right)^{1/3} \tag{32}$$

$$= \frac{3}{2}\left(\frac{\hbar^2 r^2}{2\pi\,m}\right)^{1/3}. \tag{33}$$

(ii) An estimate to the ground state wavefunction is similarly found by substituting σ_0 into the Gaussian trial wavefunction, normalised.

$$\psi_0(x) = \frac{\psi(x)}{\langle\psi|\psi\rangle}\bigg|_{\sigma=\sigma_0} = \frac{\sqrt{2}}{\sigma_0\sqrt{\pi}}e^{-x^2/\sigma_0{}^2} \tag{34}$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}}\left(\frac{m\,r}{\hbar^2\sqrt{2\pi}}\right)^{1/3} e^{-x^2\left(\frac{m\,r}{\hbar^2\sqrt{2\pi}}\right)^{2/3}} \tag{35}$$

$$= \left(\frac{2\,m\,r}{\hbar^2\pi^2}\right)^{1/3} e^{-x^2\left(\frac{m^2\,r^2}{2\pi\,\hbar^4}\right)^{1/3}}. \tag{36}$$

Plugging optimal value of σ into energy and wave-function 1+1 points.

Now suppose the potential is modified to

$$V(x) = r\left(\frac{|x|}{L}\right)^p \tag{37}$$

where $r > 0$, $L > 0$ and $p > 0$.

2. Sketch $V(x)$ for:

   i) $p = 0$

   ii) $p \to \infty$

   (2 marks)

   1 mark for $p = 0$, 1 mark for $p \to \infty$.

   Solution. For $p = 0$ $V(x) = r$, for $p \to \infty$, $V(x) = 0$ for $|x| < L$ and $V(x) = \infty$ for $|x| > L$.

3. Explicitly state (or, if you prefer, carefully sketch) the exact ground state wavefunction in each of the following limits:

   i) $p = 0$

   ii) $p = 2$

   iii) $p \to \infty$

   (4 marks)

   The non-normalizable state turned out confusing for many. For this reason, the allocation of points has been: 1 point for i), 1.5 points for ii), 1.5 points for iii). In i) 0.5/0.75 points if it was mentioned that the solution is a "scattering state" or a sinusoidal wave, but without noting that the momentum had to be zero. -0.5 points if they mentioned a plane wave or a delta function. 0 points if they said that the solution is $\psi = 0$. In iii) 0.75 points for sinusoidal wavefunctions but with wrong wavelengths, without boundary conditions. No penalty for wrong normalization factors, because the question could have been answered by just a sketch, so this is more fair to those who took the analytical approach.

   Solution. For $p = 0$ one has a non-normalizable ground state, corresponding to the zero-momentum plane-wave state. For $p = 2$, the problem is a harmonic oscillator and the exact solution is a Gaussian wavefunction:

$$\psi(x) = (2/(\pi\sigma^2))^{1/4} e^{-x^2/\sigma^2} \ , \tag{38}$$

where $\sigma^2 = 2\hbar L/\sqrt{2mr}$. The variational wavefunction is, in the case $p = 2$, an exact solution of the ground state. For $p \to \infty$, the problem reduces to a particle in a box, and the ground-state wavefunction becomes

$$\psi(x) = (1/\sqrt{L}) \cos(\pi x/(2L)) \; . \tag{39}$$

4. What do you expect is the optimal value of σ in the Gaussian trial function to make the variational energy as low as possible in the following regimes? (*You do not need to perform explicit integrals*)

   i) $p \to 0$

   ii) $p = 2$

   iii) $p \to \infty$

   1 point for i), 1 point for ii), 1 point for iii). First two questions where graded in a quite sharp way: with basically two values correct/incorrect. Very synthetic solutions were not penalized, even if some of the students only reported the results, because the text of the problem did not imply automatically that a detailed explanation was required. In iii), 0.5 points if the answer was $\sigma \approx L$. This is wrong, but not unreasonable.

   In ii), instead, $\sigma \approx L$ was considered to be wrong. (In the harmonic oscillator the spread of the wavefunction should contain $\hbar$, the mass m). No point was given to an answer $\sigma \approx L$ in this case.

   0.5 points for reasonings if not wrong.

   Solution. For $p \to 0$, it does not pay to make the packet more focused, as the problem has only kinetic energy. Then it is convenient to spread the packet and make σ large. Thus one expects $\sigma \to \infty$ for $p \to 0$. For $p = 2$, the optimal value of σ is its exact value, $\sigma^2 = 2L/\sqrt{2mr}$. When $p \to \infty$, the problem becomes a particle in a box. The Gaussian function has always tails that fall within the forbidden region. Thus one must have $\sigma \to 0$ in the limit $p \to \infty$.

   In fact, one may even derive these results exactly. For the normalized wavefunction $\psi(x) = (2/(\pi\sigma^2))^{1/4} e^{-x^2/\sigma^2}$, the variational energy can be calculated for arbitrary p and reads:

$$E(\sigma) = \frac{\hbar^2}{2m\sigma^2} + \frac{r}{\sqrt{\pi}L^p} \left(\frac{\sigma}{\sqrt{2}}\right)^p \Gamma((1+p)/2) \; . \tag{40}$$

   Minimizing gives for the width of the variational solution:

$$0 = -\frac{\hbar^2}{m\sigma^2} + \frac{pr}{\sqrt{\pi}L^p} \left(\frac{\sigma}{\sqrt{2}}\right)^p \Gamma((1+p)/2) \; . \tag{41}$$

   The solution is, as it should, proportional to $(\hbar^2 L^p/(mr))^{1/(p+2)}$. This must be because this is the only length scale present in the problem (the only which depends on r and L only via the ratio r/L^p, which defines the potential energy). However, the expression also shows the presence of the numerical prefactor $p\Gamma((1+p)/2)$ which tends to zero for $p \to 0$ and diverges very quickly to ∞ for $p \to \infty$. This implies that the variational wavepacket will be infinitely spread for $p \to 0$ and infinitely narrow for $p \to \infty$.

(3 marks)

5. Comment on features of the Gaussian trial wavefunction that make it suitable and/or unsuitable for the regimes

   i) $0 < p \ll 1$

   ii) $p = 2$

   iii) large p

   (2 marks)

   2 points allocated on the basis of the overall solution.

   Here, a comment was explicitly asked, in a clear way, in the question. For this reason, short answers not providing motivations and physical explanations were penalized.

   Solution. For $p = 2$, as seen before, the Gaussian is perfectly suitable: provided that σ is adjusted correctly, it gives an exact solution of the ground state. For large p, the tails of the Gaussian function, although they are rapidly decreasing, remain "too fat", and incorrectly penetrate within the region $|x| > L$ which becomes forbidden by the large L. This makes the Gaussian wavefunction unsuitable: although the exact solution has a finite spread (of order L), the best variational wavefunction of Gaussian form must have $\sigma \to 0$ for p large as shown above. The case $0 < p \ll 1$ is the most complex to analyze qualitatively. If $p > 0$, the potential is confining and the ground state must have a normalizable wavefunction which decreases at large x. The energy of the ground state must be positive and finite (an upper bound being given by any Gaussian wave function). From this one expects a kind of bell-shaped wavefunction which is concave near 0 and becomes convex at some x, thus entering a "tail" region in which it decays to zero.

You may find the following Gaussian integrals helpful:

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}, \quad \int_{-\infty}^{+\infty} x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

Question B - Leggett–Garg Inequality

1) [1 marks total]

- *Macroscopic realism* prescribes that $Q_i \in \{-1, 1\} \forall i$, such that we can formulate probabilities $P(Q_i = 1)$ and $P(Q_i = -1)$ where Q_i are random variables.

  [1/2 marks]

- *Non-invasive measurability* permits us to formulate these probabilities exclusively in terms of the outcomes Q_i without any additional state (like a flag to indicate whether a prior measurement was made).

  [1/2 marks]

2) [1 marks total]

We are given

$$C_{ij} = \sum_{Q_i, Q_j} Q_i Q_j P(Q_i, Q_j) \tag{42}$$

and

$$\begin{aligned} K_3 &= C_{12} + C_{23} - C_{13} && (43) \\ &= \sum_{Q_1,Q_2} Q_1 Q_2 P(Q_1, Q_2) + \sum_{Q_2,Q_3} Q_2 Q_3 P(Q_2, Q_3) + \sum_{Q_1,Q_3} Q_1 Q_3 P(Q_1, Q_3) && (44) \end{aligned}$$

[1/4 marks]

As well-defined random variables, the law of total probability states $P(A) = \sum_B P(A, B)$, and ergo that $\{Q_i\}$ obey

$$P(Q_i, Q_j) = \sum_{Q_k} P(Q_i, Q_j, Q_k) \quad \forall \{i, j, k\}, \tag{45}$$

[1/4 marks]

This can be equivalently expressed as e.g.

$$\begin{aligned} 1 &= \sum_{Q_3} P(Q_3|Q_1, Q_2) \\ \implies P(Q_1, Q_2) &= P(Q_1, Q_2) \sum_{Q_3} P(Q_3|Q_1, Q_2) \\ &= \sum_{Q_3} P(Q_1, Q_2) P(Q_3|Q_1, Q_2) \\ &= \sum_{Q_3} P(Q_1, Q_2, Q_3) \end{aligned}$$

via conditional probability.

Ergo

$$\begin{aligned} K_3 = & \sum_{Q_1,Q_2} \sum_{Q_3} Q_1 Q_2 P(Q_1, Q_2, Q_3) + \sum_{Q_2,Q_3} \sum_{Q_1} Q_2 Q_3 P(Q_1, Q_2, Q_3) && (46) \\ & + \sum_{Q_1,Q_3} \sum_{Q_2} Q_1 Q_3 P(Q_1, Q_2, Q_3) && (47) \\ & = \sum_{Q_1,Q_2,Q_3} P(Q_1, Q_2, Q_3)(Q_1 Q_2 + Q_2 Q_3 - Q_1 Q_3) && (48) \end{aligned}$$

[1/2 marks]

alternative accepted solutions:

- expansion of K_3 into (± 1)-weighted sum of $P(\ldots)$ (e.g. $P(1,1,-1)$ compared to a similar expansion of the target form (ew)!

partial credit for:

- recognising the form of $E[Q_1 Q_2]$ (1/2 mark)

3) [6 marks total]

Let $X = Q_1 Q_2 + Q_2 Q_3 - Q_1 Q_3$. Given $Q_i = \pm 1$, we can enumerate all possible values of X:

Q_1	Q_2	Q_3	X
1	1	1	1
1	1	-1	1
1	-1	1	-3
1	-1	-1	1
-1	1	1	1
-1	1	-1	-3
-1	-1	1	1
-1	-1	-1	1

[4 marks]

Notice only two are non-unity, with value $X = -3$, occurring when $(Q_1, Q_2, Q_3) = (1, -1, 1)$ and $(-1, 1, -1)$. Therefore

$$K_3 = -3\left(P(1,-1,1) + P(-1,1,-1)\right) + P' \tag{49}$$

where P' is the sum of the probabilities of the remaining scenarios.

[1 marks]

Because the probabilities must sum to one, i.e.

$$1 = \sum_{Q_i, Q_j, Q_k} P(Q_i, Q_j, Q_k) = P(1,-1,1) + P(-1,1,-1) + P', \tag{50}$$

[1/2 marks]

we know that

$$P' = 1 - P(1,-1,1) - P(-1,1,-1) \tag{51}$$

and ergo

$$K_3 = 1 - 4\left(P(1,-1,1) + P(-1,1,-1)\right) \tag{52}$$

[1/2 marks]

4) [1 marks total]

Macroscopic realism and non-invasive measurability together imply the above form of K_3. Since $P(\dots) \geq 0$ (since it is a probability), it is clear that

$$K_3 = 1 - 4(P(\dots) + P(\dots)) \leq 1 \tag{53}$$

[(accepting *any* algebraic demo) 1 marks]

alternate solutions:

- ≥ 1 correlation "stronger than classical" influence (shaky but ok)
- 1/2 mark for misinterpreting above by asserting $K_3 \geq 1$ implies "positive correlation"

5) [10 marks total]

[the simplifications can appear in any order, even deferred until the final expression, so the marks here are indicative of logical steps, not exact expressions]

[instantiating and evaluating the operators as matrices is also acceptable, which avoids needing to use any Pauli commutators. For example, evaluating $Q(t_i)Q(t_j)$ are a matrix, noticing we only need the top-left element because of $\langle 0| \circ |0\rangle$, and stopping. It is ergo valid to populate only sub-matrices of the multiplied matrices which inform the top-left element]

[it is also acceptable to express $|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and use the spectral theorem to effect $\exp(\sigma_x)$ upon its eigenstates, etc]

$$\begin{aligned}
\sigma_z(t) &= e^{i\hat{H}t}\sigma_z e^{-i\hat{H}t} && (54)\\
&= e^{i\frac{1}{2}\Omega\sigma_x t}\sigma_z e^{-i\frac{1}{2}\Omega\sigma_x t} && (55)\\
&= \left(\cos(\frac{1}{2}\Omega t)I + i\sin(\frac{1}{2}\Omega t)\sigma_x\right)\sigma_z\left(\cos(\frac{1}{2}\Omega t)I - i\sin(\frac{1}{2}\Omega t)\sigma_x\right) && (56)\\
&= \cos^2\ \sigma_z + i\sin\cos\ \sigma_x\sigma_z - i\sin\cos\ \sigma_z\sigma_x + \sin^2\ \sigma_x\sigma_z\sigma_x && (57)
\end{aligned}$$

taking care to maintain the order of the Pauli operators.

[2 marks]

To proceed, we recognise that

$$\sigma_x\sigma_z = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = -i\sigma_y, \qquad \sigma_z\sigma_x = (\sigma_x\sigma_z)^\dagger = i\sigma_y,$$
$$\sigma_x\sigma_z\sigma_x = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} = -\sigma_z$$

[1 marks]

and ergo

$$\sigma_z(t_j) = (\cos^2 - \sin^2)\sigma_z + i\sin\cos(-i\sigma_y - i\sigma_y) \tag{58}$$

[1 marks]

By then invoking $\cos(2x) = \cos^2(x) - \sin^2(x)$ and $\sin(2x) = 2\sin(x)\cos(x)$, this simplifies to

$$\sigma_z(t_j) = \cos(\Omega t)\sigma_z + \sin(\Omega t)\sigma_y. \tag{59}$$

[1/2 marks]

The two-time correlation is therefore

$$\begin{aligned}
C_{ij} &= \langle\psi(0)|\, Q(t_i)Q(t_j)\, |\psi(0)\rangle && (60)\\
&= \langle 0|\, \sigma_z(t_i)\, \sigma_z(t_j)\, |0\rangle && (61)\\
&= \langle 0|\, (\cos(\Omega t_i)\sigma_z + \sin(\Omega t_i)\sigma_y)\, (\cos(\Omega t_j)\sigma_z + \sin(\Omega t_j)\sigma_y)\, |0\rangle && (62)\\
&= \langle 0|\, \big(\cos(\Omega t_i)\cos(\Omega t_j){\sigma_z}^2 + \cos(\Omega t_i)\sin(\Omega t_j)\sigma_z\sigma_y && (63)\\
&\quad + \sin(\Omega t_i)\cos(\Omega t_j)\sigma_y\sigma_z + \sin(\Omega t_i)\sin(\Omega t_j){\sigma_y}^2\big)\, |0\rangle && (64)
\end{aligned}$$

[3 marks]

(or partial 2 marks given if formulation is correct but is not completed or contains an algebraic mistake)

We next use that

$$\sigma_z\sigma_y = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix} = -i\sigma_x, \qquad \sigma_y\sigma_z = (\sigma_z\sigma_y)^\dagger = i\,\sigma_x \tag{65}$$

[1 marks]

The middle two terms of C_{ij} both become $\propto \langle 0|\,\sigma_x\,|0\rangle = \langle 0|1\rangle = 0$, so we retain only the outer two terms. And since ${\sigma_z}^2 = {\sigma_y}^2 = I$, and $\langle 0|\, xI\, |0\rangle = x$ for scalar x, we have

$$C_{ij} = \cos(\Omega t_i)\cos(\Omega t_j) + \sin(\Omega t_i)\sin(\Omega t_j). \tag{66}$$

[1 marks]

Lastly, we recognise the cosine-difference formula

$$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b), \tag{67}$$

to conclude

$$C_{ij} = \cos(\Omega(t_i - t_j)). \tag{68}$$

[1/2 marks]

6) [2 marks total]

When measurements occur at equally spaced τ intervals, then

$$C_{ij} = \cos(\Omega\tau(i-j)) \tag{69}$$

[1/2 marks]

and ergo

$$\begin{aligned} K_3 &= C_{12} + C_{23} - C_{13} && (70)\\ &= \cos(\Omega\tau) + \cos(\Omega\tau) - \cos(2\Omega\tau) && (71)\\ &= 2\cos(\Omega\tau) - \cos(2\Omega\tau) && (72) \end{aligned}$$

[3/2 marks]

7) [2 marks total]

To solve when $K_3 > 1$, we first find where $K_3 = 1$, i.e.

$$1 = 2\cos(\Omega\tau) - \cos(2\Omega\tau). \tag{73}$$

[1/2 marks]

We substitute double-angle formula $\cos(2x) = 2\cos^2(x) - 1$ in order to obtain

$$0 = 2x - 2x^2 = 2x(1-x) \tag{74}$$

where $x = \cos(\Omega\tau)$. This has solutions $x = 0$ and $x = 1$.

[1/2 marks]

Since the coefficient of x^2 is negative, we know the polynomial is above zero between the axis intercepts at $x = 0$ and $x = 1$. Therefore, $K_3 > 1$ when

$$0 < \cos(\Omega\tau) < 1 \tag{75}$$

[1/2 marks]

This is the right half of the trigonometric circle, but importantly *excludes zero*. Therefore

$$\Omega\tau \in \left(-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right) \quad \Longrightarrow K_3 > 1 \tag{76}$$

By periodicity of cos, the same domain shifted by an integer multiple of 2π is also a solution. Ergo

$$K_3 > 1 \quad \Longrightarrow \quad \Omega\tau \in \bigcup_{n\in\mathbb{Z}} \left(-\frac{\pi}{2} + 2n\pi, 2n\pi\right) \cup \left(2n\pi, \frac{\pi}{2} + 2n\pi\right) \tag{77}$$

[1/2 marks]

We entirely forgive:

(a) failure to recognise periodicity

(b) failure to recognise exclusion of zero

(c) any awkward presentation/notation of the domain

1/2 mark awarded for:

(a) recognising immediately $0 < \Omega\tau < \pi/2 \implies K_3 < 1$

1/4 mark deducted for:

(a) including $\pm\pi/2$ boundaries (should be excluded)

8) [2 marks total]

The violation suggests that quantum physics is inconsistent with one or both of the principles of macroscopic realism and non-invasive measurability.

[2 marks]

We know in fact that both are violated; quantum physics permits states which are not fully described by / consistent with each possible observable outcome, and that measurement does affect the state (and ergo its evolution).

(above wording not necessary; instead, e.g. "this means that...")

(a) partial 1/4 mark for vaguely suggesting it implies quantum physics is not explainable by classical laws

Question C - Lie Algebras and Angular Momentum

1) We know that $SO(3)$ is the group of unitary matrices with determinant equal to 1. It means that for any $M \in SO(3)$, it has the following conditions:

$$M^T M = M M^T = \mathbb{1} \tag{78}$$

$$\det(M) = 1 \tag{79}$$

Using these conditions and the rotation of the infinitesimal angle form $M \simeq \mathbb{1} + A$, A should be an antisymmetric matrix. And from this condition, we can find the generators that make an arbitrary 3×3 antisymmetric matrix as follows.

$$\mathcal{J}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \tag{80}$$

$$\mathcal{J}_y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{81}$$

$$\mathcal{J}_z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{82}$$

You can just as well use only the 3 dimensional representations of the matrices for rotations and take their derivatives, as hinted in the beginning of the question.

(1 mark for final answer, 2 marks for the derivation)

From these generators, any 3×3 antisymmetric matrix can be written as

$$A = \theta_x \mathcal{J}_x + \theta_y \mathcal{J}_y + \theta_z \mathcal{J}_z \tag{83}$$

And also we can multiply the generators by some imaginary number in order to make them hermitean. And define the new set of generators $\{J_x, J_y, J_z\}$ and continue working with them.

$$J_x \equiv -i\mathcal{J}_x \tag{84}$$

$$J_y \equiv -i\mathcal{J}_y \tag{85}$$

$$J_z \equiv -i\mathcal{J}_z \tag{86}$$

2) The commutation relation of the generators is the following.

$$[J_x, J_y] = iJ_z \tag{87}$$

$$[J_y, J_z] = iJ_x \tag{88}$$

$$[J_z, J_x] = iJ_y \tag{89}$$

These relations can be simplified as $[J_i, J_j] = i\varepsilon_{ijk}J_k$, so the structure constant is equal to the Levi-Cevita symbol.

(1 mark for the commutators and 1 mark for the structure constant)

3) In this part we will find a four-dimensional irreducible representation of this algebra by finding the ladder operators of a spin $j = 3/2$. For this spin the eigenvectors of the spin z are $\{|\frac{3}{2}\rangle, |\frac{1}{2}\rangle, |-\frac{1}{2}\rangle, |-\frac{3}{2}\rangle\}$. We can find ladder operators by finding the effect of them on each basis.

$$J_+ |\frac{3}{2}\rangle = 0 \tag{90}$$

$$J_+ |\frac{1}{2}\rangle = \sqrt{3} |\frac{3}{2}\rangle \tag{91}$$

$$J_+ |-\frac{1}{2}\rangle = 2 |\frac{1}{2}\rangle \tag{92}$$

$$J_+ |-\frac{3}{2}\rangle = \sqrt{3} |-\frac{1}{2}\rangle \tag{93}$$

And the same effects for J_-.

$$J_- |\frac{3}{2}\rangle = \sqrt{3} |\frac{1}{2}\rangle \tag{94}$$

$$J_- |\frac{1}{2}\rangle = 2 |-\frac{1}{2}\rangle \tag{95}$$

$$J_- |-\frac{1}{2}\rangle = \sqrt{3} |-\frac{3}{2}\rangle \tag{96}$$

$$J_- |-\frac{3}{2}\rangle = 0 \tag{97}$$

Using these equations, we can write the representation of J_+, J_-, and J_z.

$$J_+ = \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{98}$$

$$J_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} \tag{99}$$

$$J_z = \begin{pmatrix} 3/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & -3/2 \end{pmatrix} \tag{100}$$

And since we know that $J_x = \frac{1}{2}(J_+ + J_-)$ and $J_y = \frac{1}{2i}(J_+ - J_-)$, the 4-dimensional irreducible representation of J_x and J_y can be written as follows.

$$J_x = \frac{1}{2}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} \tag{101}$$

$$J_y = \frac{1}{2i}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ -\sqrt{3} & 0 & 2 & 0 \\ 0 & -2 & 0 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 0 \end{pmatrix} \tag{102}$$

(2 marks for J_x, J_y, J_z. 4 marks for calculating J_+. 4 marks for calculating J_-.)

4) We have the basis for the whole system as $|j_1, j_2; m_1, m_2\rangle$ which is $|j_1, m_1\rangle \otimes |j_2, m_2\rangle$, but we want the basis that we can know the total angular momentum from that. Here we have the addition of two angular momentum $j_1 = 3/2$ and $j_2 = 1$. We define a new basis $|j, m\rangle$ where it is the eigenvector of $\mathcal{J}_z$, $\mathcal{J}^2$, J_1^2, and J_2^2. So we should find the transformation between these two sets of bases. Since $\mathcal{J}_z = J_{1z} + J_{2z}$ we have $m = m_1 + m_2$. The easiest eigenvector to find is the maximum one. The maximum value for m_1 and m_2 is j_1 and j_2 so we have $m_{max} = j_1 + j_2$ and since $j \geq |m|$ we know that $j_{max} = j_1 + j_2$. And since the total angular momentum comes from the addition of two angular momentum vectors we know that $j_{min} = |j_1 - j_2|$. So here we have $j = \{5/2, 3/2, 1/2\}$ and for each j we have $2j+1$ values for m. We start with the maximum.

Case $j = 5/2$**:**

We can start with $|j, m\rangle = |\frac{5}{2}, \frac{5}{2}\rangle$.

$$|\frac{5}{2}, \frac{5}{2}\rangle = |j_1 = \frac{3}{2}, j_2 = 1; m_1 = \frac{3}{2}, m_2 = 1\rangle = |\frac{3}{2}, 1\rangle \tag{103}$$

to find $|5/2, 3/2\rangle$ we can apply $\mathcal{J}_-$ on $|5/2, 5/2\rangle$.

$$\mathcal{J}_- |\frac{5}{2}, \frac{5}{2}\rangle = \sqrt{5}\,|\frac{5}{2}, \frac{3}{2}\rangle \tag{104}$$

$$= \mathcal{J}_- |\frac{3}{2}, 1\rangle = \sqrt{3}\,|\frac{1}{2}, 1\rangle + \sqrt{2}\,|\frac{3}{2}, 0\rangle \tag{105}$$

So we have the following.

$$|\frac{5}{2}, \frac{3}{2}\rangle = \sqrt{\frac{3}{5}}\,|\frac{1}{2}, 1\rangle + \sqrt{\frac{2}{5}}\,|\frac{3}{2}, 0\rangle \tag{106}$$

We can continue applying $\mathcal{J}_-$ to find other eigenvectors with $j = 5/2$.

$$|\frac{5}{2},\frac{1}{2}\rangle = \sqrt{\frac{3}{10}}\,|-\frac{1}{2},1\rangle + \sqrt{\frac{3}{5}}\,|\frac{1}{2},0\rangle + \sqrt{\frac{1}{10}}\,|\frac{3}{2},-1\rangle \tag{107}$$

$$|\frac{5}{2},-\frac{1}{2}\rangle = \sqrt{\frac{3}{10}}\,|\frac{1}{2},-1\rangle + \sqrt{\frac{3}{5}}\,|-\frac{1}{2},0\rangle + \sqrt{\frac{1}{10}}\,|-\frac{3}{2},1\rangle \tag{108}$$

$$|\frac{5}{2},-\frac{3}{2}\rangle = \sqrt{\frac{3}{5}}\,|-\frac{1}{2},-1\rangle + \sqrt{\frac{2}{5}}\,|-\frac{3}{2},0\rangle \tag{109}$$

$$|\frac{5}{2},-\frac{5}{2}\rangle = |-\frac{3}{2},-1\rangle \tag{110}$$

Case $j = 3/2$: For the first vector of $j = 3/2$ we can use the fact that $|5/2, 3/2\rangle$ and $|3/2, 3/2\rangle$ are orthogonal.

$$|\frac{3}{2},\frac{3}{2}\rangle = -\sqrt{\frac{2}{5}}\,|\frac{1}{2},1\rangle + \sqrt{\frac{3}{5}}\,|\frac{3}{2},0\rangle \tag{111}$$

Then we can apply $\mathcal{J}_-$ to find other eigenvectors.

$$|\frac{3}{2},\frac{1}{2}\rangle = -\sqrt{\frac{8}{15}}\,|-\frac{1}{2},1\rangle + \sqrt{\frac{1}{15}}\,|\frac{1}{2},0\rangle + \sqrt{\frac{2}{5}}\,|\frac{3}{2},-1\rangle \tag{112}$$

$$|\frac{3}{2},-\frac{1}{2}\rangle = \sqrt{\frac{8}{15}}\,|\frac{1}{2},-1\rangle - \sqrt{\frac{1}{15}}\,|-\frac{1}{2},0\rangle - \sqrt{\frac{2}{5}}\,|-\frac{3}{2},1\rangle \tag{113}$$

$$|\frac{3}{2},-\frac{3}{2}\rangle = \sqrt{\frac{2}{5}}\,|-\frac{1}{2},-1\rangle - \sqrt{\frac{3}{5}}\,|-\frac{3}{2},0\rangle \tag{114}$$

Case $j = 1/2$:

$$|\frac{1}{2},\frac{1}{2}\rangle = \sqrt{\frac{1}{2}}\,|\frac{3}{2},-1\rangle - \sqrt{\frac{1}{3}}\,|\frac{1}{2},0\rangle + \sqrt{\frac{1}{6}}\,|-\frac{1}{2},1\rangle \tag{115}$$

$$|\frac{1}{2},-\frac{1}{2}\rangle = \sqrt{\frac{1}{6}}\,|\frac{1}{2},-1\rangle - \sqrt{\frac{1}{3}}\,|-\frac{1}{2},0\rangle + \sqrt{\frac{1}{2}}\,|-\frac{3}{2},1\rangle \tag{116}$$

(4 marks for 5/2, 3 marks for 3/2, 3 marks for 1/2)