Chapter 5

Identical multi-particle systems

In this section, we discuss the behaviour of identical quantum particles. We will explain how there are two sorts of identical particles distinguished by how their state changes when you swap two particle labels. At least initially in this section, we will switch back to working directly in terms of the wavefunction of particles because i. this is how this topic is conventionally taught, ii. it's good to stay fluent with both sets of notation and iii. I draw in part on Vincenzo Savona's notes here which were written in terms of wavefunctions. However, we could have equally phrased this section entirely in braket notation (or entirely in wavefunction notation).

5.1 Two identical particles

Consider a system with two particles labelled as 1 and 2. Suppose that each one-particle subsystem is described by wave functions $\phi_i(r_i)$ for $i \in \{1, 2\}$. How can you write the wavefunction of the joint system for 1 and 2? The most naive response, which would suggest that the product of one-particle wave functions satisfies the Schrödinger equation, fails in the general case. Indeed, such a solution, on the one hand, assumes that the probabilities of particle presence are entirely independent (which amounts, among other things, to neglecting all interactions between particles), and, on the other hand, potentially violates the linearity of the Schrödinger equation. More generally, for a system of two interacting particles through a potential $\hat{U}(\mathbf{r_1}, \mathbf{r_2})$, writing

$$\left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial \mathbf{r_1}^2} - \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial \mathbf{r_2}^2} + \hat{V}(\mathbf{r_1}) + \hat{V}(\mathbf{r_2}) + \hat{U}(\mathbf{r_1}, \mathbf{r_2})\right)\psi_1(\mathbf{r_1})\psi_2(\mathbf{r_2}) = E\psi_1(\mathbf{r_1})\psi_2(\mathbf{r_2},$$

presupposes that the two-particle Schrödinger equation:

$$\left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial \mathbf{r_1}^2} - \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial \mathbf{r_2}^2} + \hat{V}(\mathbf{r_1}) + \hat{V}(\mathbf{r_2}) + \hat{U}(\mathbf{r_1}, \mathbf{r_2})\right)\psi(\mathbf{r_1}, \mathbf{r_2}) = E\psi(\mathbf{r_1}, \mathbf{r_2}), \tag{5.1}$$

is separable, which is not necessarily true. We must find a way to describe the system using a single wave function that depends on all coordinates.

Suppose the particles are identical. This implies, among other things, that the probability $|\psi(\mathbf{r_1}, \mathbf{r_2})|^2$ of finding one particle at point $\mathbf{r_1}$ and the other at point $\mathbf{r_2}$ must be equal to $|\psi(\mathbf{r_2}, \mathbf{r_1})|^2$. In other words, we must have:

$$\psi(\mathbf{r_2}, \mathbf{r_1}) = e^{i\phi}\psi(\mathbf{r_1}, \mathbf{r_2})$$

Now this equation must hold for any $\mathbf{r_1}$ and $\mathbf{r_2}$ (if it did not it would imply that the space was non-isotropic[1]). This means it also holds in the case that $\mathbf{r_1} \to \mathbf{r_2}$ and $\mathbf{r_2} \to \mathbf{r_1}$ and so we also have

$$\psi(\mathbf{r_1}, \mathbf{r_2}) = e^{i\phi}\psi(\mathbf{r_2}, \mathbf{r_1}) .$$

Combining these two equations then gives

$$\begin{aligned} & \psi(\mathbf{r_2}, \mathbf{r_1}) = e^{i2\phi}\psi(\mathbf{r_2}, \mathbf{r_1}) \\ \implies & e^{i2\phi} = 1 \\ \implies & e^{i\phi} = \pm 1. \end{aligned}$$

Now let's make this a bit more formal by defining $\mathbb{P}_{12}$ be the operator that acts on the system by interchanging particles 1 and 2. By the argument above, we know that for identical particles we have

$$\mathbb{P}_{1,2}\psi(\mathbf{r_1}, \mathbf{r_2}) = \psi(\mathbf{r_2}, \mathbf{r_1}) = \pm\psi(\mathbf{r_1}, \mathbf{r_2}) .$$

Thus the permutation operator $\mathbb{P}_{1,2}$ has eigenvalues ± 1. We note that there is nothing[2] in the above argument that is unique to the vectors $\mathbf{r_1}$ and $\mathbf{r_2}$ being position vectors and so the argument equally applies to arbitrary vectors to the particle variables.

The eigenstates corresponding to the +1 eigenvalue are said to be symmetric under exchange (particles described by these functions are *bosons*) and the particles corresponding to the -1 operator are said to be anti-symmetric under exchange (particles described by these functions are *fermions*). Fermions are half-integer spin particles such as electrons and quarks. Bosons are integer spin particles such as photons or gluons. This correlation with spin can be taken as an empirical fact in standard quantum mechanics.

Fermions. A direct consequence of the wavefunction of fermions being anti-symmetric under the particle permutation operator, i.e. $\mathbb{P}_{1,2}\psi(\mathbf{r_1}, \mathbf{r_2}) = \psi(\mathbf{r_2}, \mathbf{r_1}) = -\psi(\mathbf{r_1}, \mathbf{r_2})$, is that there is zero probability of finding two fermions in precisely the same state. More concretely, if $\psi(\mathbf{r}, \mathbf{r}) = -\psi(\mathbf{r}, \mathbf{r})$ then we must have that $\psi(\mathbf{r}, \mathbf{r}) = 0$.

Of course, a particle can possess more properties than a location. Let $|x\rangle = |m, \mathbf{r}, ...\rangle$ be a single quantum state to denote state dependent properties (e.g. its spin orientation m, position $\mathbf{r}$, ...) of a particle. We can then write a two particle state as

$$|\psi\rangle = \sum_{x,x'} a_{x,x'} |x, x'\rangle . \tag{5.2}$$

Note that now, rather than explicitly labelling the variables as corresponding to system 1 and 2 respectively as in $\psi(\mathbf{r_1}, \mathbf{r_2})$, I am taking the left and the right slots of the ket $|...\rangle|...\rangle$ to correspond to systems 1 and 2 respectively. In this formalism, for fermions we have

$$\mathbb{P}\,|x', x\rangle = |x, x'\rangle = -\,|x', x\rangle \tag{5.3}$$

[1] To see this, imagine $\mathbf{r_1}$ and $\mathbf{r_2}$ are single parameter variables and we place our coordinates such that the origin is midway between them. Now, we're considering $\psi(r, -r)$. Assuming the physics of the universe is invariant under reflections we are free to redefine $-r \leftrightarrow r$ without changing anything physical. Thus if $\psi(r, -r) = e^{i\phi}\psi(-r, r)$ we also have $\psi(-r, r) = e^{i\phi}\psi(r, -r)$. For arbitrary vectors we can also do the same trick of putting the coordinate system midway between the two particles and considering the axis that connects them.

[2] Ok this is where it gets super subtle. Technically my isotropy argument above did rely of these vectors being position vectors. However, for arbitrary variables, we can make an analogous argument saying that the action of the permutation operator should be independent of the variable it acts on. If this is getting too subtle do not worry about it- most discussion seems to gloss over these subtleties anyway.

Or, on the level of the full state, we can see that Eq. (5.3) implies that

$$\begin{aligned}\mathbb{P}|\psi\rangle &= \sum_{x,x'} a_{x,x'}\mathbb{P}|x,x'\rangle \\ &= \sum_{x,x'} -a_{x,x'}|x,x'\rangle = -|\psi\rangle.\end{aligned} \tag{5.4}$$

We now want to understand how the constraint $\mathbb{P}|\psi\rangle = -|\psi\rangle$ effects the allowed $a_{x,x'}$ values. To do this, note that

$$\begin{aligned}|\psi\rangle &= \sum_{x,x'} a_{x,x'}|x,x'\rangle \\ &= \sum_{x,x'} -a_{x,x'}|x',x\rangle \\ &= \sum_{x,x'} -a_{x',x}|x,x'\rangle\end{aligned} \tag{5.5}$$

where in the first line we use Eq. 5.3 and in the second (as we are summing over both x and x') we are free to perform the relabelling $x \to x'$ and $x' \to x$. Thus comparing the first and final line of Eq. (5.5) we see that

$$a_{x,x'} = -a_{x',x} \tag{5.6}$$

and

$$a_{x,x} = 0\,. \tag{5.7}$$

This is the core of what is known as the ***Pauli exclusion principle*** - no two fermions can occupy the same single particle quantum state. Note that while no two fermions with the same spin can occupy the same position, if the fermions spin differ then there can be a non-zero amplitude of finding the two fermions at the same position. That is, electrons in different spin states can be in the same place but electrons in the same spin state avoid one another[3].

Any expansion of a two-fermion state, i.e., Eq. (5.2), involves an even number of terms because from Eq. (5.6) any term of the form $a_{x,x'}$ comes with its negative swapped partner $-a_{x',x}$. The simplest such state of this form corresponds to the case where each electron can take one of two different spin states. Let us label these $|0\rangle$ and $|1\rangle$ and so as $a_{0,1} = -a_{1,0}$ and $a_{0,0} = a_{1,1} = 0$, we obtain

$$|\psi\rangle \propto |0,1\rangle - |1,0\rangle \quad \to \quad |\psi\rangle = \frac{1}{\sqrt{2}}\left(|0,1\rangle - |1,0\rangle\right) := |\Psi_-\rangle\,. \tag{5.8}$$

Thus we see that the simplest possible two particle fermionic state is the singlet Bell state $|\Psi_-\rangle$. To check that this works we note that:

$$\mathbb{P}|\psi\rangle = \frac{1}{\sqrt{2}}\left(\mathbb{P}|0,1\rangle - \mathbb{P}|1,0\rangle\right) = \frac{1}{\sqrt{2}}\left(|1,0\rangle - |0,1\rangle\right) = -\frac{1}{\sqrt{2}}\left(|0,1\rangle - |1,0\rangle\right) = -|\psi\rangle\,. \tag{5.9}$$

Boson. Let us now see what happens if we repeat the same calculation above but suppose the identical particles are bosons. This time we have $\mathbb{P}|x,x'\rangle = |x',x\rangle = |x,x'\rangle$, and so we can write

$$|\psi\rangle = \sum_{x,x'} a_{x,x'}|x,x'\rangle = \sum_{x,x'} a_{x,x'}|x',x\rangle = \sum_{x,x'} a_{x',x}|x,x'\rangle\,, \tag{5.10}$$

where in the second equality we relabel $x \to x'$ and $x' \to x$. It follows that

$$a_{x,x'} = a_{x',x}\,. \tag{5.11}$$

[3]There is a cliched comparison you could make here between electrons and fashionable folk accidentally in the same outfit at a party not wanting to be seen together.

This time we can have non-zero amplitudes for both particles to be in the same state, i.e., have $a_{x,x} \neq 0$. But any amplitude of the form $a_{x,x'}$ comes with an identical amplitude of the form $a_{x',x}$. Consider two photons that can be in the states $|0\rangle$ or $|1\rangle$, the allowed basic states are

$$|0,0\rangle, |1,1\rangle, \frac{1}{\sqrt{2}}\left(|0,1\rangle + |1,0\rangle\right) . \tag{5.12}$$

We can then of course also consider superpositions of these states, eg. $\cos(\theta)\,|0,0\rangle + e^{i\phi}\sin(\theta)\,|1,1\rangle$.

5.2 Multiple identical particles

This reasoning generalizes to systems of n particles, where $n \in \mathbb{N}$. Let $\psi(\mathbf{r_1}, \cdots, \mathbf{r_n})$ be the wave function of the system. First of all, note that exchanging particle j and particle k for $j,\ k \in \{1, \cdots, n\}$ is equivalent to exchanging particle k and particle j, i.e., $\mathbb{P}_{j,k} = \mathbb{P}_{k,j}$. Furthermore,

$$\begin{aligned}\mathbb{P}_{j,k}\left(\mathbb{P}_{j,k}\psi(\mathbf{r_1}, \cdots, \mathbf{r_j}, \cdots, \mathbf{r_k}, \cdots, \mathbf{r_n})\right) &= \mathbb{P}_{j,k}\left(\psi(\mathbf{r_1}, \cdots, \mathbf{r_j}, \cdots, \mathbf{r_k}, \cdots, \mathbf{r_n})\right) \\ &= \psi(\mathbf{r_1}, \cdots, \mathbf{r_j}, \cdots, \mathbf{r_k}, \cdots, \mathbf{r_n}) \\ &= \mathbb{1}\left(\psi(\mathbf{r_1}, \cdots, \mathbf{r_j}, \cdots, \mathbf{r_k}, \cdots, \mathbf{r_n})\right),\end{aligned}$$

so, $\mathbb{P}_{j,k}\mathbb{P}_{j,k} = \mathbb{1}$, and $\mathbb{P}_{j,k}^{-1} = \mathbb{P}_{j,k} = \mathbb{P}_{k,j}$. Finally, the sign of the operator $\mathbb{P}_{j,k}$ must be the same for all $j,\ k \in \{1, \cdots, n\}$. In fact:

$$\mathbb{P}_{j,k} = \mathbb{P}_{1,j}\mathbb{P}_{2,k}\mathbb{P}_{1,2}\mathbb{P}_{2,k}\mathbb{P}_{1,j}.$$

A "permutation operator" is an operator of the form $\mathbb{P} = \prod \mathbb{P}_{j,k}$. It follows that wavefunctions corresponding to eigenvalues of a permutation operator are either symmetric or antisymmetric. This is the *symmetry postulate*, which can be restated as follows:

Symmetrisation Postulate (Cohen-Tannoudji, Diu, Laloe, 1977) : When a system includes several identical particles, only certain kets of its state space can describe its physical state. Physical kets are, depending on the nature of its identical particles, either completely symmetric or completely anti-symmetric with respect to the permutation of these particles. Those particles for which the physical kets are symmetric are called bosons, and those for which they are antisymmetric, fermions.

Notice that this has important consequences in the description of the physics of the system. Consider, for example, an arbitrary observable $\hat{O}$ of the system. Using the above, its average value must satisfy, for all $j,\ k \in \mathbb{N}$:

$$\langle\psi|\hat{O}|\psi\rangle = \langle\psi|\mathbb{P}_{j,k}^{\dagger}\hat{O}\mathbb{P}_{j,k}|\psi\rangle ,$$

which implies $\hat{O} = \mathbb{P}_{j,k}^{\dagger}\hat{O}\mathbb{P}_{j,k}$, and the operator $\mathbb{P}_{j,k}$ commutes with all observables. In particular, if $\hat{H}$ is the system's Hamiltonian, $[\mathbb{P}_{jk}, \hat{H}] = [\hat{H}, \mathbb{P}_{jk}]$ for all $j,\ k \in \mathbb{N}$. Physically, this result is expected: Since the particles are assumed to be identical, there is no reason for the system's Hamiltonian to be modified by the exchange of two particles. As per what was previously discussed, since all $\mathbb{P}_{j,k}$ have the same sign, we can always simultaneously diagonalize $\mathbb{P}$ and $\hat{H}$. In other words, $[\mathbb{P}, \hat{H}] = 0$ for any operator $\mathbb{P}$.

5.2.1 Bosons

Let's now consider the possible basis states for a system of n Bosons. For two Bosons these were:

$$|00\rangle,\ |11\rangle,\ \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) . \tag{5.13}$$

This can equivalently be written as

$$|\psi_{\mathbf{x}}\rangle \propto \sum_{\mathbb{P}\in S_2} \mathbb{P}|x_1, x_2\rangle = \sum_{\mathbb{P}\in S_2} |\mathbf{x}_{\mathbb{P}(1)}\rangle|\mathbf{x}_{\mathbb{P}(2)}\rangle \tag{5.14}$$

where $\mathbf{x} = (x_1, x_2)$ and S_n is the symmetric group on n elements. We will formally define S_n later in term, for now just think of it as the set of all possible permutations of n objects. When $n = 2$ this is just the identity operation and the swap operation. For example, for the case of $x_1 = 0$, $x_2 = 0$ we have

$$|\psi_{\mathbf{x}}\rangle \propto \sum_{\mathbb{P}\in S_2} \mathbb{P}|00\rangle = \mathbb{I}|00\rangle + \mathbb{P}_{12}|00\rangle = |00\rangle + |00\rangle = 2|00\rangle \xrightarrow{\text{normalization}} |00\rangle \tag{5.15}$$

where as for $x_1 = 0$, $x_2 = 1$ we have

$$|\psi_{\mathbf{x}}\rangle \propto \sum_{\mathbb{P}\in S_2} \mathbb{P}|01\rangle = \mathbb{I}|01\rangle + \mathbb{P}_{12}|01\rangle = |01\rangle + |10\rangle \xrightarrow{\text{normalization}} \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)\,. \tag{5.16}$$

Thus we see that Eq. (5.14) gives the correct expression for the basis states up to normalization. This expression generalizes to an n particles system as you would expect:

$$|\psi_{\mathbf{x}}\rangle = \mathcal{N} \sum_{\mathbb{P}\in S_n} \mathbb{P}|x_1, x_2, \ldots, , x_n\rangle \propto \sum_{\mathbb{P}\in S_n} \mathbb{P}|x_1, x_2, \ldots, , x_n\rangle = \sum_{\mathbb{P}\in S_n} |\mathbf{x}_{\mathbb{P}(1)}\rangle|\mathbf{x}_{\mathbb{P}(2)}\rangle \ldots |\mathbf{x}_{\mathbb{P}(n)}\rangle\,. \tag{5.17}$$

where $\mathcal{N}$ is a normalization factor. For example, if we consider a three particle system and $x_1 = 0$, $x_2 = 0$, $x_3 = 1$, as expected we obtain

$$\begin{aligned} |\psi_{\mathbf{x}}\rangle \propto \sum_{\mathbb{P}\in S_2} \mathbb{P}|001\rangle &= \mathbb{I}|001\rangle + \mathbb{P}_{12}|001\rangle + \mathbb{P}_{13}|001\rangle + \mathbb{P}_{23}|001\rangle + \mathbb{P}_{123}|001\rangle + \mathbb{P}_{132}|001\rangle \\ &= |001\rangle + |001\rangle + |100\rangle + |010\rangle + |100\rangle + |010\rangle \\ &= |001\rangle + |010\rangle + |001\rangle \\ &\xrightarrow{\text{normalization}} \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |001\rangle)\,. \end{aligned} \tag{5.18}$$

What about the normalization factor $\mathcal{N}$? Well, there are $n!$ ways of permuting n objects. If the vector $\mathbf{x}$ contains no repeated entries then each of the corresponding states resulting from the permutation are unique and the normalization is simply $\frac{1}{\sqrt{n!}}$. If there are repeated entries however (e.g. as we saw in Eq.(5.15)) you get an extra factor in the numerator that needs to be accounted for. Specifically, if you have n_k repeated entries you have $n_k!$ identical terms in the sum. Hence the normalization factor is

$$\mathcal{N} = \frac{1}{\sqrt{n!}\sqrt{\prod_k n_k!}} \tag{5.19}$$

where $\sum_k n_k = n$. *Exercise: Derive Eq.* (5.19) *for yourself more carefully.*

5.2.2 Fermions

It is also possible to write a general expression for the basis states of a Fermion. In analogy with Eq. (5.17) above, one has

$$|\psi_{\mathbf{x}}\rangle = \frac{1}{\sqrt{n!}} \sum_{\mathbb{P}\in S_n} \text{sign}(\mathbb{P})\mathbb{P}|x_1, x_2, \ldots, , x_n\rangle\,. \tag{5.20}$$

where $\text{sign}(\mathbb{P}) = -1$ if $\mathbb{P}$ involves an odd number of index swaps and $\text{sign}(\mathbb{P}) = 1$ if $\mathbb{P}$ involves an even number of index swaps. We note that given the Pauli exclusion principle, no two Fermions can be in the same state (i.e. $n_k = 1$ for all k), so each state in the sum here is unique and so the normalization is simply $\frac{1}{\sqrt{n!}}$.

Slater determinant form. The expression in Eq. (5.20) can also be written compactly in terms of a *Slater determinant*, which automatically enforces the antisymmetry of fermionic wavefunctions. If the single-particle states are described by spin-orbitals $\{\phi_1, \phi_2, \ldots, \phi_n\}$, then the corresponding n-particle antisymmetric state may be written as

$$|\psi_{\mathbf{x}}\rangle = \frac{1}{\sqrt{n!}} \begin{vmatrix} \phi_1(x_1) & \phi_1(x_2) & \ldots & \phi_1(x_n) \\ \phi_2(x_1) & \phi_2(x_2) & \ldots & \phi_2(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_n(x_1) & \phi_n(x_2) & \ldots & \phi_n(x_n) \end{vmatrix}. \tag{5.21}$$

This determinant form is fully equivalent to the signed permutation sum in Eq. (5.20), since expanding the determinant reproduces the sum over all permutations weighted by their parity.

As an explicit example, for three fermions in single-particle states ϕ_1, ϕ_2, ϕ_3, the Slater determinant is

$$|\psi_{x_1,x_2,x_3}\rangle = \frac{1}{\sqrt{6}} \begin{vmatrix} \phi_1(x_1) & \phi_1(x_2) & \phi_1(x_3) \\ \phi_2(x_1) & \phi_2(x_2) & \phi_2(x_3) \\ \phi_3(x_1) & \phi_3(x_2) & \phi_3(x_3) \end{vmatrix}. \tag{5.22}$$

Expanding this determinant yields

$$|\psi_{x_1,x_2,x_3}\rangle = \frac{1}{\sqrt{6}}(\phi_1(x_1)\phi_2(x_2)\phi_3(x_3) - \phi_1(x_1)\phi_2(x_3)\phi_3(x_2) \tag{5.23}$$

$$- \phi_1(x_2)\phi_2(x_1)\phi_3(x_3) + \phi_1(x_2)\phi_2(x_3)\phi_3(x_1) \tag{5.24}$$

$$+ \phi_1(x_3)\phi_2(x_1)\phi_3(x_2) - \phi_1(x_3)\phi_2(x_2)\phi_3(x_1)), \tag{5.25}$$

which matches exactly the permutation-based expression in Eq. (5.20) for $n = 3$. This form makes the antisymmetry under particle exchange manifest and provides a convenient compact representation of fermionic basis states.

5.3 Distinguishing identical particles

At this point, it is perhaps valuable to take a step back and think about how the symmetrisation postulate fits with our understanding of the physics of quantum particles / the world around us more generally.

As the universe is a system containing large numbers of identical particles, the symmetrisation postulate tells us that all identical particles in the universe are in a state with particles of the same type that is symmetric or anti-symmetric under exchange. Either way, as the global phase in quantum mechanics does not correspond to anything physical, this entails that all identical particles of the same type are in a (typically highly entangled!) permutation invariant state. It follows that all identical particles, understood as represented by the indices in the quantum state, share with other particles of the same type both their intrinsic properties and state dependent properties[4].

However, this description of fundamental particles is far from our usual treatment of identical particles. An electron is a fermion but we do not usually think of electrons as being permutable and in exactly the same state as every other particle in the universe. Rather electrons are charge carriers in wires, they are in the shells of atoms, they exist in plasmas and so forth. We take electrons to exist wholly within reasonably well defined finite systems.

[4]Identical particles have the same 'state dependent properties' in the sense that they all have the same reduced density operator (obtained by taking the partial trace).

In practice, we are able to talk about electrons in such reasonably well defined localised roles by identifying stable dynamical properties. These stable dynamical properties enable us to distinguish subsystems of the total symmetrised state of fermions. These stable dynamical properties will typically be spatial. However, they need not be.

Consider a state of two identical particles in the orthogonal states $|\phi\rangle$ and $|\psi\rangle$. The state of the system can be described by:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}[1 + \epsilon\mathbb{P}_{12}]\,|\phi, \psi\rangle \tag{5.26}$$

where $\epsilon = 1$ for bosons and $\epsilon = -1$ for fermions.

Say we are interested in the observable $\hat{Q}$ where $\hat{Q}\,|u_i\rangle = q_i\,|u_i\rangle$. Using the Born rule, the probability amplitude, of obtaining q_i and q_j on measurement, is:

$$1/2\,\langle u_i, u_j|\,[1 + \epsilon\mathbb{P}_{21}^{\dagger}][1 + \epsilon\mathbb{P}_{21}]\,|\phi, \psi\rangle$$

$$= \langle u_i\phi\rangle\langle u_j\psi\rangle + \epsilon\langle u_i\psi\rangle\langle u_j\phi\rangle\,.$$

The first term is known as the direct integral and the second is the exchange integral.

The state of a pair of non-identical (i.e. non-permutable) particles in the orthogonal states $|\phi\rangle$ and $|\psi\rangle$ respectively is written $|\psi\rangle\otimes|\phi\rangle$. In this case, the probability of measuring q_i, q_j is simply $|\langle u_i\psi\rangle\langle u_j\phi\rangle|^2$. This suggests the following operational claim: Particle permutation between a pair of particles can be ignored when either the direct or exchange integral between that pair of particles vanishes. Otherwise, the symmetry postulate entails that permutation must be taken into account.

One way in which one of the integrals can disappear is if both the particles and the measuring devices are spatially separated. Say, the wavepackets of the identical particles are well localised and spatially separated such that $\langle x|\psi\rangle = 0$ if x is in the region R and $\langle x|\phi\rangle = 0$ if x is in the region L. Similarly, suppose the measuring device wavepackets are spatially separated such that $\langle x|u_i\rangle = 0$ if x is in the region R and $\langle x|u_j\rangle = 0$ if x is in the region of L. Thus we have

$$\langle u_j|\psi\rangle = \sum_x \langle u_j|x\rangle\langle x|\psi\rangle = 0 \tag{5.27}$$

and as such the exchange integral disappears[5]. When this is the case we can identify each particle by its well defined positions and we say things like 'the particle on the left is in state ψ' and 'the particle on the right is in state ϕ' and write $|\psi, \phi\rangle$ where the left and right slots correspond to the left and right electrons respectively.

We make use of the vanishing exchange integral between pairs of spatially separated systems and measuring devices when we wish to consider a particular subset of particles in the universe. We can consider a pair of electrons in a shell of helium and treat these two electrons as permutable, without needing to consider permutations between these two electrons and all other electrons in the universe. It is also the reason why, combined with the fact that the position and spin operators commute, we do not have to take into account the symmetric spatial part of the wavefunction in Bell type experimental setups.

[5]There is nothing significant about the exchange rather than direct integral disappearing. I could have swapped the location of the measurement devices for the converse.

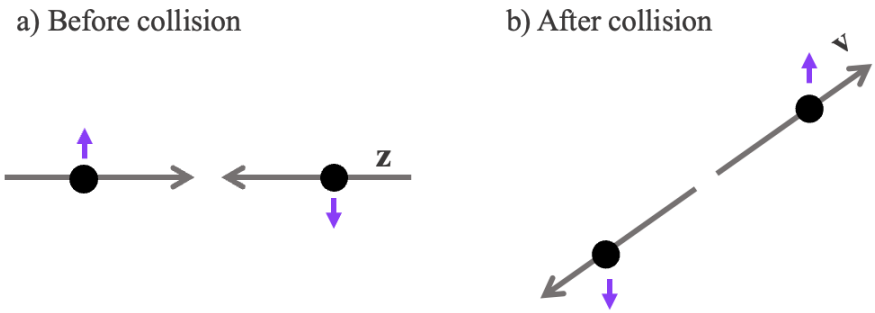

Figure 5.1: Diagram of a collision experiment with no spin interaction ((Cohen-Tannoudji, Diu, Laloe, 1977).)

However, there is nothing fundamentally special about position. We could equally have used a different stable dynamical property to distinguish the particles. For example, if spin dependent interactions are negligible in a scattering experiment then the different spin alignments of a pair of particles can be used to treat the two particles that scatter as non-permutable. Consider the following initial state for the collision problem sketched in Fig. 5.1,

$$|\Psi_{\text{initial}}\rangle = \frac{1}{\sqrt{2}}[1 + \epsilon\mathbb{P}_{12}]\,|p_{\mathbf{z}}, +, -p_{\mathbf{z}}, -\rangle\,. \tag{5.28}$$

Here $|p_{\mathbf{z}}, +\rangle$ denotes the state of the particle with momentum in the positive z direction and spin z of $+\frac{1}{2}$ (and conversely for $|-p_{\mathbf{z}}, -\rangle$). Say we are interested in knowing the probability that the system is in the final state

$$|\Psi_{\text{final}}\rangle = \frac{1}{\sqrt{2}}[1 + \epsilon\mathbb{P}_{12}]\,|p_{\mathbf{v}}, +, -p_{\mathbf{v}}, -\rangle \tag{5.29}$$

where $\pm p_{\mathbf{v}}$ denotes momentum in the plus and minus $\mathbf{v}$ directions sketched in Fig. 5.1b).

The evolution operator responsible for the collision, $\hat{U}(t, t_0)$, commutes with the permutation operator. Thus we have

$$\hat{U}(t, t_0)\,|\Psi_{\text{initial}}\rangle = \frac{1}{\sqrt{2}}(\hat{U}(t, t_0)\,|p_{\mathbf{z}}, +, -p_{\mathbf{z}}, -\rangle + \epsilon\hat{U}(t, t_0)\,|-p_{\mathbf{z}}, -, p_{\mathbf{z}}, +\rangle \tag{5.30}$$

and so

$$\langle\Psi_{\text{final}}|\,\hat{U}(t, t_0)\,|\Psi_{\text{initial}}\rangle \propto \langle p_{\mathbf{v}}, +, -p_{\mathbf{v}}, -|\,\hat{U}(t, t_0)\,|p_{\mathbf{z}}, +, -p_{\mathbf{z}}, -\rangle + \epsilon\,\langle p_{\mathbf{v}}+, -p_{\mathbf{v}}-|\,\hat{U}(t, t_0)\,|-p_{\mathbf{z}}, -, p_{\mathbf{z}}, +\rangle \tag{5.31}$$

Now we are interested in the case where $\hat{U}(t, t_0)$ does not affect spin interactions. As such, the exchange term is sandwiched between two orthogonal states and vanishes, and so we are left with

$$\langle\Psi_{\text{final}}|\,\hat{U}(t, t_0)\,|\Psi_{\text{initial}}\rangle \propto \langle p_{\mathbf{v}}, +, -p_{\mathbf{v}}, -|\,\hat{U}(t, t_0)\,|p_{\mathbf{z}}, +, -p_{\mathbf{z}}, -\rangle\,. \tag{5.32}$$

That is, we are left with the probability associated with two non-permutable particles.

In both the case of the spatially separated particles and the particle denoted by its spin, operationally we are free to work directly with states labelled according to their distinguishing properties:

$$\frac{1}{\sqrt{2}}(1 + \epsilon\mathbb{P}_{12})\,|\phi, \psi\rangle \to |\phi\rangle_L \otimes |\psi\rangle_R \tag{5.33}$$

$$\frac{1}{\sqrt{2}}(1+\epsilon\mathbb{P}_{12})\,|p_{\mathbf{z}},+,-p_{\mathbf{z}},-\rangle \to |p_{\mathbf{z}}\rangle_+|-p_{\mathbf{z}}\rangle_- \tag{5.34}$$

What do we conclude from these examples? The symmetrisation postulate is a fundamental theorem in quantum mechanics that implies that all identical fermions are in an anti-symmetric entangled state. However, this does not mean that we need to consider this state in practise most of the time. If there are stable dynamical properties to distinguish quantum two electrons over time, we can label those electrons by those properties and just those two properties (i.e. the electron on the left/the electron on the right or the spin up electron/spin down electron). In practise, this treatment of permutable particles is empirically successful and what we end up working with most of the time.

5.4 Second Quantization:

Second quantization is an approach used to represent systems composed of multiple particles. We consider a situation where the number of particles can potentially change, noting that a particle's state is entirely determined by the one-particle functions in the basis of $\mathcal{H}_1$. We construct the *Fock space* where kets indicate the number of times a wave function is involved.

For example, the transformation from 1st quantisation (what we have been discussing so far in this chapter) to second quantisation looks like

$$\begin{aligned}&\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \to |11\rangle\\ &|\uparrow\uparrow\rangle \to |20\rangle\\ &|\downarrow\downarrow\rangle \to |02\rangle\,.\end{aligned} \tag{5.35}$$

Here the left and right slots in the Fock basis indicate the number of Bosons in the $\uparrow$ and $\downarrow$ states respectively. Similarly, for Fermions we could have

$$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \to |11\rangle\,. \tag{5.36}$$

It's worth noting that for bosons, the n_i appearing in $|n_1, n_2, \cdots\rangle$ can be arbitrary, while for fermions, they can only take the values 0 or 1 due to the Pauli exclusion principle. Also note that it's important once in the second quantisation to know whether the state you are looking at is a Fermionic or Bosonic state as, for example, a state of the form $|11\rangle$ could refer to either but behaves differently in the two cases.

We introduce creation and annihilation operators to increase or decrease the number of particles.

- The Bosonic case is entirely analogous with the case of a simple harmonic oscillator which you should be familiar with from Quantum Physics 1. Specifically we have:

$$\begin{cases}\hat{c}_i^\dagger\,|n_1,\cdots,n_i,\cdots\rangle = \sqrt{n_i+1}\,|n_1,\cdots,n_i+1,\cdots\rangle\,,\\ \hat{c}_i\,|n_1,\cdots,n_i,\cdots\rangle = \sqrt{n_i}\,|n_1,\cdots,n_i-1,\cdots\rangle\,,\end{cases}$$

  It follows that (*check this!*) that creation and annihilation operators in the bosonic case satisfy:

  - $[\hat{c}_i,\hat{c}_j] = [\hat{c}_i^\dagger,\hat{c}_j^\dagger] = 0$
  - $[\hat{c}_i,\hat{c}_j^\dagger] = \delta_{ij}$.

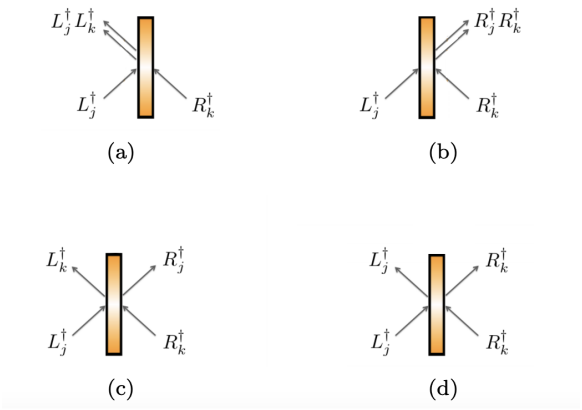

Figure 5.2: The Hong-Ou-Mandel effect and Bosonic bunching.

- The Fermionic case is much more subtle. In this case we need to ensure that the resulting states are antisymmetric under exchange. This can be achieved by defining the creation and annihilation operators as follows:

$$\begin{cases} \hat{c}_i^\dagger |n_1, \cdots, n_i, \cdots\rangle = (-1)^{n_1+\cdots+n_{i-1}} (1-n_i) |n_1, \cdots, n_i+1, \cdots\rangle, \\ \hat{c}_i |n_1, \cdots, n_i, \cdots\rangle = (-1)^{n_1+\cdots+n_{i-1}} n_i |n_1, \cdots, n_i-1, \cdots\rangle, \end{cases}$$

  To get a sense of the form of these expressions first notice that the $(1-n_i)$ factor ensures that you cannot create Fermionic states with more than one particle in the same state.

  The factor of $(-1)^{n_1+\cdots+n_{i-1}}$ then ensures the antisymmetrisation. For example, we require that $\hat{c}_0^\dagger \hat{c}_1^\dagger |00\rangle = -\hat{c}_1^\dagger \hat{c}_0^\dagger |00\rangle$. We indeed have this as $\hat{c}_0^\dagger \hat{c}_1^\dagger |00\rangle = \hat{c}_0^\dagger (-1)^0 |01\rangle = |11\rangle$ and $\hat{c}_1^\dagger \hat{c}_0^\dagger |00\rangle = \hat{c}_1^\dagger |10\rangle = (-1)^1 |11\rangle = -|11\rangle$. The general case can be understood by iterating this argument.

  It is straightforward to verify (*check this!*) that the creation and annihilation operators in the fermionic case satisfy:

  – $\{\hat{c}_i, \hat{c}_j\} = \{\hat{c}_i^\dagger, \hat{c}_j^\dagger\} = 0$

  – $\{\hat{c}_i, \hat{c}_j^\dagger\} = \delta_{ij}$

  where $\{A, B\} = AB + BA$.

5.5 The Hong-Ou-Mandel Effect and Bosonic Bunching

To get a bit of practise of working in the second quantisation, and as another illustration of the difference between fermions and bosons, we'll end this chapter by presenting something called the Hong-Ou-Mandel (HOM) effect. The HOM effect describes what happens when two identical photons hit a beamsplitter. It shows that while fermions have a tendency to avoid each other, bosons have a tendency to clump together.

When working in the second quantisation it is often helpful to work in the Heisenberg picture and consider the action of any unitary process on the creation and annihilation operators rather

than on a given state directly. Suppose we have a photon impinge on a 50-50 beamsplitter as shown in Fig. 5.3. Let $\hat{L}_H$, $\hat{L}_V$, $\hat{R}_H$, $\hat{R}_V$ denote the annihilation operators for horizontally and vertically polarised photons on the left and right hand side of the beamsplitter. The action of this beamsplitter can be modelled in the Heisenberg picture as

$$\begin{aligned} \hat{L}_k^\dagger &\to \frac{1}{\sqrt{2}}(\hat{L}_k^\dagger + \hat{R}_k^\dagger) \\ \hat{R}_k^\dagger &\to \frac{1}{\sqrt{2}}(\hat{L}_k^\dagger - \hat{R}_k^\dagger) \end{aligned} \tag{5.37}$$

for $k = H$ and $k = V$ and where the minus sign in the second line above is to ensure unitarity.

When an H photon and a V photon (i.e. two perfectly distinguishable photons) impinge on opposite sides of a beamsplitter simultaneously we have

$$\begin{aligned} |1\rangle_{LH}|0\rangle_{LV}|0\rangle_{RH}|1\rangle_{RV} = \hat{L}_H^\dagger \hat{R}_V^\dagger |0000\rangle &\to \frac{1}{2}(\hat{L}_H^\dagger + \hat{R}_H^\dagger)(\hat{L}_V^\dagger - \hat{R}_V^\dagger)|0000\rangle \\ &= \frac{1}{2}(-|1\rangle_{LH}|0\rangle_{LV}|0\rangle_{RH}|1\rangle_{RV} + |1\rangle_{LH}|1\rangle_{LV}|0\rangle_{RH}|0\rangle_{RV} \\ &\quad - |0\rangle_{LH}|0\rangle_{LV}|1\rangle_{RH}|1\rangle_{RV} + |0\rangle_{LH}|1\rangle_{LV}|1\rangle_{RH}|0\rangle_{RV}) \end{aligned} \tag{5.38}$$

That is, there are four equally probable outcomes as sketched in Fig. 5.3:

(a) the photon from the right is transmitted and the photon from the left is reflected,

(b) the photon from the left is transmitted and the photon from the right is reflected,

(c) both photons are transmitted,

(d) both photons are reflected.

However, when the two photons are indistinguishable, something intriguing happens. Suppose both photons are horizontally polarized (and the same frequency etc). In this case (dropping the unchanged vacuum V modes for simplicity) we have

$$\begin{aligned} |1\rangle_{LH}|1\rangle_{RH} = \hat{L}_H^\dagger \hat{R}_H^\dagger |00\rangle &\to \frac{1}{2}(\hat{L}_H^\dagger + \hat{R}_H^\dagger)(\hat{L}_H^\dagger - \hat{R}_H^\dagger)|00\rangle \\ &= \frac{1}{2}\left(\hat{L}_H^{\dagger\,2} - \hat{R}_H^{\dagger\,2}\right)|00\rangle \\ &= \frac{1}{\sqrt{2}}(|2\rangle_{LH}|0\rangle_{RH} - |0\rangle_{LH}|2\rangle_{RH}) \end{aligned} \tag{5.39}$$

The amplitude for both photons to be reflected by the BS and the amplitude for both photons to be transmitted through the BS have destructively interfered, and thus the probability for the photons to exit the beamsplitter through opposite sides vanishes. Indistinguishable photons are therefore guaranteed to leave a beamsplitter in the same mode, a phenomenon known as 'bosonic bunching'.

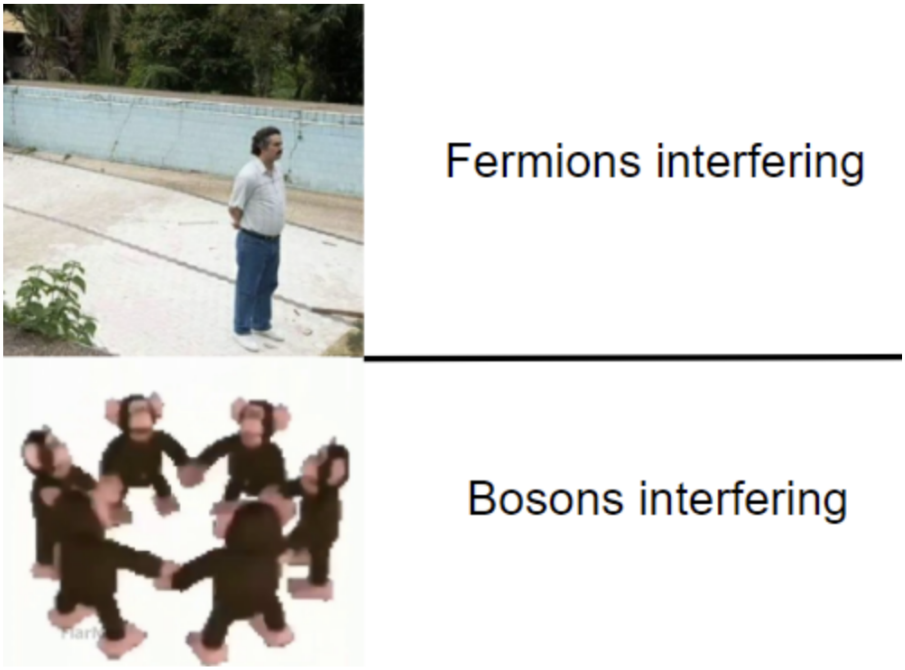

Figure 5.3: Credit: Nicolas Emile Bourquin