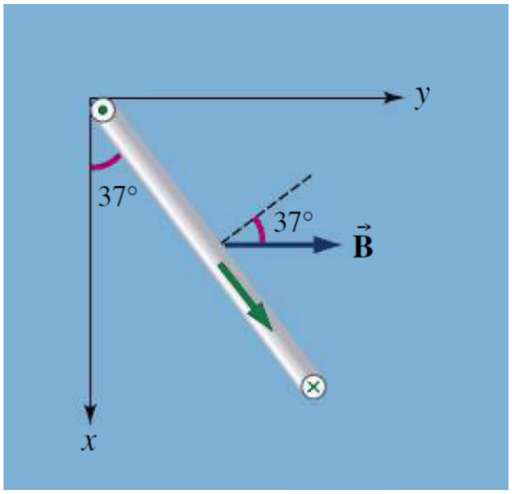
General Physics: Electromagnetism, Correction 9

Exercise 1 :

A rigid wire loop of square shape has sides of length 20 cm. It has fives turns and is carrying a current of 2 A (indicated by green arrow). The normal to the loop makes an angle of 37° with a uniform magnetic field $B = 0.5\hat{y}$ T.



a) Find the magnetic dipole moment.

b) Find the magnitude and direction of the torque acting on the loop.

c) Find the work that an external agent must provide to rotate the frame from its position of minimum energy to the given position.

Solution 1 :

(a) From the figure, we can see that the unit vector normal to the surface of the frame is

$$\hat{u}_n = -\sin(37)\hat{x} + \cos(37)\hat{y} = -0.602\hat{x} + 0.799\hat{y}. \tag{1}$$

The magnetic moment is given by

$$\vec{\mu} = NIA\hat{u}_n = (5)(2)(0.2)^2(-0.602\hat{x} + 0.799\hat{y}) = (-0.241\hat{x} + 0.319\hat{y})\ \text{A m}^2 \tag{2}$$

(b) The torque is given by

$$\vec{\tau} = \vec{\mu} \times \vec{B} = (-0.241\hat{x} + 0.319\hat{y}) \times (0.5\hat{y}) = -0.120\hat{z}\ \mathrm{N\,m} \tag{3}$$

Note that the magnitude can also be calculated with

$$\tau = NIAB\sin\theta = 0.120\ \mathrm{N\,m} \tag{4}$$

(c) The potential energy of a magnetic dipole in a magnetic field is

$$U = -\vec{\mu} \cdot \vec{B} = -\mu B\cos\theta \tag{5}$$

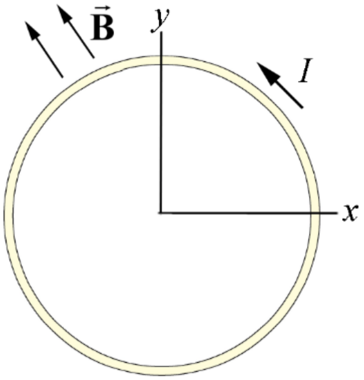
where $\mu = NIA = 0.4\,\mathrm{A\,m^2}$ and $B = 0.5\,\mathrm{T}$ The position with the minimum energy is $\theta = 0$. The work required to turn the frame into the given position is

$$U_f - U_i = (-\mu B\cos(37)) - (-\mu B\cos(0)) = 0.0403\,\mathrm{J} \tag{6}$$

The external work is positive since the dipole moment rotates away from the direction of the field. Therefore, the dipole stores magnetic potential energy.

Exercise 2 :

A wire ring lying in the xy-plane with its center at the origin carries a counterclockwise current I. There is an external uniform magnetic field $B = B_x\hat{i} + B_y\hat{j}$ such that $B_y > 0$ and $B_x < 0$.The magnetic moment vector $\vec{\mu}$ is perpendicular to the plane of the loop and has magnitude $\mu = IA$ and the direction is given by right-hand-rule with respect to the direction of the current. What is the direction and magnitude of the torque on the loop?



Solution 2 :

The torque on a current loop in a uniform field is given by

$$\vec{\tau} = \vec{\mu} \times \vec{B} \tag{7}$$

where $\vec{\mu}$ is the magnetic dipole moment with magnitude $\mu = IA$ and points perpendicular to the plane of the loop and right-handed with respect to the direction of current flow. The magnetic dipole moment is given by

$$\vec{\mu} = I\vec{A} = I\pi R^2\hat{\mathbf{k}} \tag{8}$$

Therefore,

$$\vec{\tau} = \vec{\mu} \times \vec{B} = (I\pi R^2)\hat{\mathbf{k}} \times (B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}}) = I\pi R^2(B_x\hat{\mathbf{j}} - B_y\hat{\mathbf{i}}) \tag{9}$$

Then the direction of the torque is given by

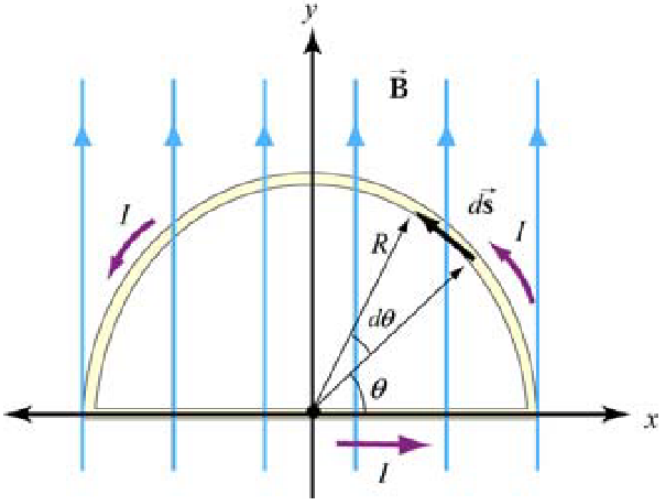
$$\theta = \tan(-B_x/B_y) > 0 \tag{10}$$

The magnitude is

$$|\vec{\tau}| = I\pi R^2\sqrt{B_x^2 + B_y^2} \tag{11}$$

Exercise 3 :

Consider a closed semi-circular loop lying in the xy plane carrying a current I in the counter-clockwise direction, as shown in the figure below. A uniform magnetic field pointing in the $+y$ direction is applied. Find the magnetic force acting on the straight segment and the semicircular arc.



Solution 3 :

We have $\vec{B} = B\hat{\mathbf{j}}$ and $\vec{F}_1$ and $\vec{F}_2$ the forces acting on the straight segment and the semicircular parts, respectively.

Since the length of the straight segment is $2R$, we can calculate the magnetic force on this segment using $\vec{F} = I\vec{l} \times \vec{B}$.

$$\vec{F}_1 = I(2R\hat{\mathbf{i}}) \times (B\hat{\mathbf{j}}) = 2IRB\hat{\mathbf{k}} \tag{12}$$

with $\hat{\mathbf{k}}$ the direction out of the page.

To calculate $\vec{F}_2$, we need to use a similar principle, but we need to define the differential length element $d\vec{s}$ on the semicircle. This can be written as $d\vec{s} = Rd\theta(-\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}})$. The force acting on the small element $d\vec{s}$ is then

$$d\vec{F}_2 = Id\vec{s} \times \vec{B} = IRd\theta(-\sin\theta\hat{\mathbf{i}} \times \cos\theta\hat{\mathbf{j}}) \times (B\hat{\mathbf{j}}) = -IBR\sin\theta d\theta\hat{\mathbf{k}} \tag{13}$$

Integrating over the entire semi-circular arc, we have

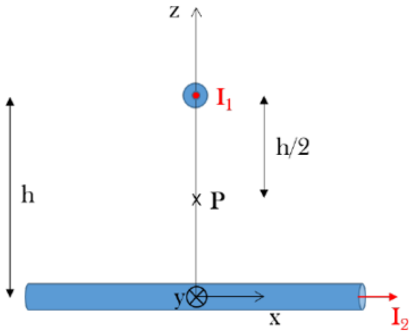
$$\vec{F}_2 = -IBR\hat{\mathbf{k}} \int_0^{\pi} \sin\theta d\theta = -2IBR\hat{\mathbf{k}} \tag{14}$$

Thus, the net force acting over the entire semi-circular wire is

$$\vec{F}_{\text{net}} = \vec{F}_1 + \vec{F}_2 \tag{15}$$

Exercise 4 :

Two long wires of radius a are perpendicularly oriented as shown in figure below. The upper wire has a current I_1 in the $\hat{y}$ direction and the lower cable has a current I_2 in the $\hat{x}$ direction.



(a) Find the magnetic field along the z axis, between $z = a$ and $z = h$.

(b) For $I_1 = 100$ A and $I_2 = 150$ A, with the distance $h = 2.5$ cm, what is the magnitude of the magnetic field at point P?

(c) Describe the direction of the compass needle placed in point P.
Hint: The magnetic field of Earth is about $5 \cdot 10^{-5}$ T.

Solution 4 :

(a) each wire will generate a magnetic field. For $a < z < h - a$ (i.e the space between the two wires), along the z axis, the magnetic field of each wire is:

$$B_1 = \frac{\mu_0 I_1}{2\pi(h-z)} \tag{16}$$

$$B_2 = \frac{\mu_0 I_2}{2\pi z} \tag{17}$$

With the right hand rule, we can determine the direction of each magnetic field along the z axis:

$$\vec{B}_1 = B_1 \hat{e}_x \tag{18}$$

$$\vec{B}_2 = -B_2 \hat{e}_y \tag{19}$$

Hence, the total magnetic field along the z axis is:

$$\vec{B} = \vec{B}_1 + \vec{B}_2 = \frac{\mu_0}{2\pi}\left(\frac{I_1}{h-z}\hat{e}_x - \frac{I_2}{z}\hat{e}_y\right) \tag{20}$$

(b) Point P is found at $z = h/2$, between the two wires. Therefore, the magnetic field is found by inserting $z = h/2$:

$$\vec{B} = \frac{\mu_0}{\pi h}(I_1\hat{e}_x - I_2\hat{e}_y) \tag{21}$$

Finally, the magnitude of the magnetic field is:

$$\left|\vec{B}\right| = \frac{\mu_0}{\pi h}\sqrt{I_1^2 + I_2^2} = 2.88 \cdot 10^{-3}\ T \tag{22}$$

(c) Earth's magnetic field ($5{\cdot}10^{-5}\ T$) is around 100 times weaker than the magnetic field generated by the wires at point P ($2.88 \cdot 10^{-3}\ T$). The compass needle will therefore follow the magnetic field generated by the wires rather than Earth's magnetic field. At point P, the needle will be oriented at an angle $\theta = \arctan(B_y/B_x) = -56°$ with respect to the x axis in the x-y plane.

Exercise 5 :

Consider two infinitely long wires carrying currents are in the x direction.

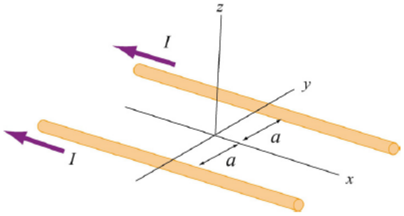


Figure 1: Non-uniform current density

(a) Draw a schematic of the magnetic field pattern in the yz-plane.

(b) Find the distance d along the z-axis where the magnetic field is maximum.

Solution 5 :

(a) The magnetic field lines are shown in Figure 4. Notice that the directions of both currents are into the page.

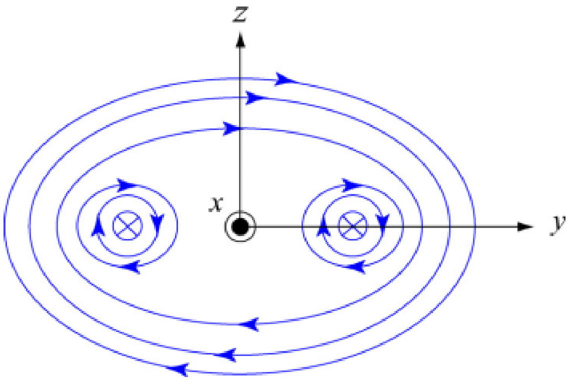


Figure 2: Magnetic field lines of two wires carrying current in the same direction.

(b) We have to compute the total magnetic field at the point $(0,0,z)$. Let's start from wire 1, which is placed on the left. We can use the Ampere's law to find the magnitude of the magntic field

$$B_1 = \frac{\mu_0 I}{2\pi r} = \frac{\mu_0 I}{2\pi\sqrt{a^2+z^2}}. \tag{23}$$

To find the direction, we have to compute the following cross product

$$(-\hat{e}_x) \times \vec{r}_1 = (-\hat{e}_x) \times r(\hat{e}_y \cos\theta + \hat{e}_z \sin\theta) = \hat{e}_y \sin\theta - \hat{e}_z \cos\theta, \tag{24}$$

with θ being the angle between the wire and the point $(0,0,z)$, and $\vec{r}_1 = r(\hat{e}_y \cos\theta + \hat{e}_z \sin\theta)$ the vector connecting the two points. Thus,

$$\vec{B}_1 = \frac{\mu_0 I}{2\pi\sqrt{a^2+z^2}}(\hat{e}_y \sin\theta - \hat{e}_z \cos\theta). \tag{25}$$

With the same procedure, we can find the magnetic field on $(0,0,z)$ generated by the second wire

$$\vec{B}_2 = \frac{\mu_0 I}{2\pi\sqrt{a^2+z^2}}(\hat{e}_y \sin\theta + \hat{e}_z \cos\theta). \tag{26}$$

The total magnetic field reads

$$\vec{B} = \vec{B}_1 + \vec{B}_2 = \frac{\mu_0 I \sin\theta}{\pi\sqrt{a^2+z^2}}\hat{e}_y = \frac{\mu_0 I z}{\pi(a^2+z^2)}\hat{e}_y = B(z)\hat{e}_y, \tag{27}$$

since $\sin\theta = z/\sqrt{a^2+z^2}$.

To compute the distance d along the z-axis where the magnetic field is maximum we take the derivative of B with respect to z

$$0 = \frac{\partial}{\partial z}B(z) = \frac{\mu_0 I}{\pi}\frac{a^2-z^2}{(a^2+z^2)^2}, \tag{28}$$

so that $z = a$ and the magnetic field is $B_{\max} = \mu_0 I/2\pi a$.

Exercise 6 :

Find the minimum diameter of the wires d that can transmit $P = 225$ MW of electricity with only a 2.0% loss. Their length is $l = 185$ km. Assume there are two wires to make a complete circuit (the length is thus doubled). The wires are to be made of aluminum ($\rho = 2.6 \cdot 10^{-8}\ \Omega \cdot m$) and the voltage is $V = 660$ kV.

Solution 6 :

The minimum diameter is the one which gives, from the 2nd Ohm's law, a value of resistance R which introduces 2.0% of losses:

$$R = \rho \frac{l_{\text{tot}}}{A} = \rho \frac{2l}{\pi (d/2)^2}, \tag{29}$$

since, as specified in the text, the two wires form a loop of length $2l$.
The dissipated power is $P_{\text{dis}} = RI^2 = \alpha P$, where $P = VI$ is the transmitted power and $\alpha = 0.02$ is the loss factor. So, we can write:

$$P_{dis} = \alpha P = RI^2 = R\frac{P^2}{V^2}, \tag{30}$$

which gives:

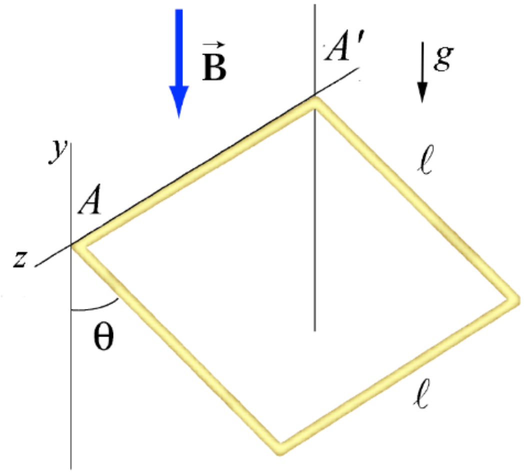
$$\alpha P = R\frac{P^2}{V^2} \to R = \frac{\alpha V^2}{P}. \tag{31}$$

By solving Eq. (29) for d and inserting R from Eq. (31), we get

$$d = \sqrt{\frac{8\rho l}{\pi R}} = \sqrt{\frac{8\rho l P}{\pi \alpha V^2}} = \frac{1}{V}\sqrt{\frac{8\rho l P}{\pi \alpha}} = 18 \text{ cm}. \tag{32}$$

Exercise 7 :

A square loop of wire, of length l on each side, and mass m, pivots about an axis AA' that corresponds to a horizontal side of the square, as shown in the figure on the left below. The external magnetic field $\vec{B}$ of magnitude B is directed vertically downward, and uniformly fills the region in the vicinity of the loop. A current I flows around the loop. The gravitational torque on the loop and the magnetic torque on the loop sum to zero when the loop makes an angle θ with the z -axis. The magnitude of the gravitational field is $g = 9.8\ \text{m s}^{-2}$.



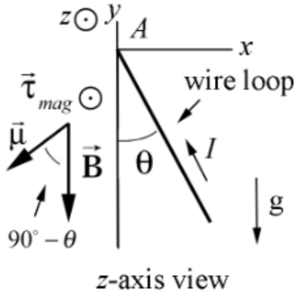
a) In what direction does the current need to flow in order that the magnetic torque acts in an opposite direction from the gravitational torque?

b) Calculate the magnitude of the magnetic torque on this loop of wire in terms of the quantities given.

c) Suppose that the mass of the loop $m = 0.4$ kg and the length of a side is $l = 1.0$ m . Suppose that when current in the loop is $I = 2.0$ A , the torques on the loop balance when $\theta = 45°$. What is the magnitude of the magnetic field?

Solution 7 :

1. Since the gravitational torque acts in the negative z direction, the magnetic torque must act in the positive z direction. The magnetic torque is given by the expression $\vec{\tau} = \vec{\mu} \times \vec{B}$. The magnetic field $\vec{B} = -\vec{B}\hat{\mathbf{j}}$ and so the magnetic dipole moment $\vec{\mu} = \mu_x\hat{\mathbf{i}} + \mu_y\hat{\mathbf{j}}$ must point towards the lower left with both negative x and y components, i.e. $\mu_x < 0$ and $\mu_y < 0$. Therefore, the current must flow in the clockwise direction as seen from above.



2. We know that $\vec{\tau}_{mag} = \vec{\mu} \times \vec{B}$ and that the magnitude of the magnetic moment of the loop is $\mu = IA = Il^2$. Note that the angle between the magnetic field the magnetic moment is $90^\circ - \theta$, therefore the magnitude of the torque is

$$\tau_{mag} = Il^2 B \sin(90^\circ - \theta) = Il^2 B \cos(\theta) \tag{33}$$

3. Since the loop is pivoted, we need to consider the gravitational torque on the bottom leg and two side legs. The gravitational torque is

$$\vec{\tau}_{grav} = (m/4)gl\sin(\theta)(-\hat{\mathbf{k}}) + 2(m/4)g(l/2)\sin(\theta)(-\hat{\mathbf{k}}) = (m/2)gl\sin(\theta)(-\hat{\mathbf{k}}) \tag{34}$$

   The magnetic torque is $\vec{\tau}_{mag} = Il^2 B\cos(\theta)(\hat{\mathbf{k}})$ and so the torques balance when

$$Il^2 B\cos(\theta) = (m/2)gl\sin(\theta) \tag{35}$$

   Therefore, the magnitude of the magnetic field is

$$B = \frac{mgl\sin(\theta)}{2Il^2\cos(\theta)} = \frac{mg\tan(\theta)}{2Il} = 1.0\text{T} \tag{36}$$