General Physics: Electromagnetism, Session 1

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Refresher: Vector Algebra

This section is adapted from the first chapter of *Introduction to Electrodynamics* by **David J. Griffiths**.

Vector operations

In order to describe quantities which have *direction* and *magnitude*, such as displacements, velocity, acceleration, force, etc... one is in need of **vectors** obeying their own arithmetic. In these exercises we will denote "vector A" as $\boldsymbol{A}$ and its magnitude as $|\boldsymbol{A}|$ or also as A.

We can define four vectors operations:

1. **Addition of two vectors.**

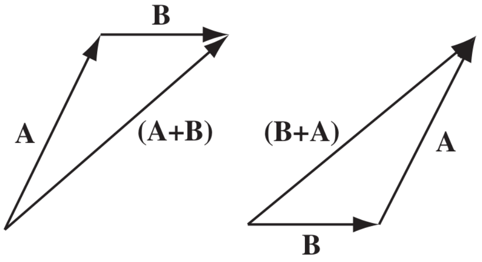

Figure 1: Addition of two vectors

Place the tail of $\boldsymbol{B}$ at the head of $\boldsymbol{A}$; the sum $\boldsymbol{A}+\boldsymbol{B}$ is the vector from the tail of $\boldsymbol{A}$ to the head of $\boldsymbol{B}$ (Fig. 1).

- Addition is *commutative*:

$$\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{B}+\boldsymbol{A}. \tag{1}$$

- Addition is *associative*:

$$(\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C}=\boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C}). \tag{2}$$

2. **Multiplication by a scalar.**
Multiplication of a vector by a scalar a multiplies the *magnitude* but leaves the direction unchanged. Scalar multiplication is *distributive.*

$$a(\boldsymbol{A}+\boldsymbol{B})=a\boldsymbol{A}+a\boldsymbol{B}. \tag{3}$$

3. **Dot product of two vectors.**
The dot product of two vectors is defined as,

$$\boldsymbol{A}\cdot\boldsymbol{B}\equiv AB\cos(\theta), \tag{4}$$

where θ is the angle they form when placed tail-to-tail. Note that $\boldsymbol{A}\cdot\boldsymbol{B}$ is itself a scalar, which is why the dot product is also called **scalar product**. The dot product is *commutative*,

$$\boldsymbol{A}\cdot\boldsymbol{B}=\boldsymbol{B}\cdot\boldsymbol{A} \tag{5}$$

and *distributive*

$$\boldsymbol{A}\cdot(\boldsymbol{B}+\boldsymbol{C})=\boldsymbol{A}\cdot\boldsymbol{B}+\boldsymbol{A}\cdot\boldsymbol{C}. \tag{6}$$

Geometrically, $\boldsymbol{A}\cdot\boldsymbol{B}$ is the product of $|\boldsymbol{A}|$ times the projection of $\boldsymbol{B}$ along $\boldsymbol{A}$. If the two vectors are parallel then, $\boldsymbol{A}\cdot\boldsymbol{B}=AB$. If they are perpendicular, $\boldsymbol{A}\cdot\boldsymbol{B}=0$.

4. **Cross product of two vectors.**
The cross product of two vectors is defined by

$$\boldsymbol{A}\times\boldsymbol{B}\equiv AB\sin(\theta)\hat{\boldsymbol{n}} \tag{7}$$

where $\hat{\boldsymbol{n}}$ is a unit vector pointing perpendicular to the plane $\boldsymbol{A}$ and $\boldsymbol{B}$. The hat denotes unit vectors. There are two directions perpendicular to any plane: "in" and "out". The ambiguity is resolved by the **right-hand rule**: let your index point in the direction of the first vector and your middle finger in the direction of the second vector, then your thumb indicates the direction of $\hat{\boldsymbol{n}}$ (Fig. 2).

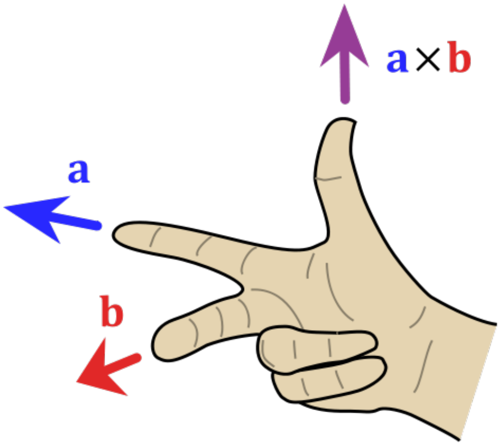

Figure 2: Right-hand rule

$\boldsymbol{A}\times\boldsymbol{B}$ is a vector, which is why the cross product is also called **vector product**. The cross product is *distributive*,

$$\boldsymbol{A}\times(\boldsymbol{B}+\boldsymbol{C})=(\boldsymbol{A}\times\boldsymbol{B})+(\boldsymbol{A}\times\boldsymbol{C}) \tag{8}$$

but not commutative,

$$(\boldsymbol{A}\times\boldsymbol{B})=-(\boldsymbol{A}\times\boldsymbol{B}). \tag{9}$$

Geometrically, $|\boldsymbol{A}\times\boldsymbol{B}|$ is the area of the parallelogram generated by $\boldsymbol{A}$ and $\boldsymbol{B}$. If two vectors are parallel their cross product is zero.

Component form

In the previous section, the vector operations have been described using abstract forms, without any references to a coordinate system. In practice, it is easier to set up Cartesian coordinates x, y, z and work with vector components. Let $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{z}}$ be unit vectors parallel to the x, y and z axes, respectively. An arbitrary vector $\boldsymbol{A}$ can be expanded in terms of these basis vectors.

$$\boldsymbol{A}=A_x\hat{\boldsymbol{x}}+A_y\hat{\boldsymbol{y}}+A_z\hat{\boldsymbol{z}}. \tag{10}$$

The number A_x, A_y and A_z, are the components of $\boldsymbol{A}$; geometrically they are the projections of $\boldsymbol{A}$ along the three coordinate axes. We can reformulate the four vector operations as rules for manipulating components:

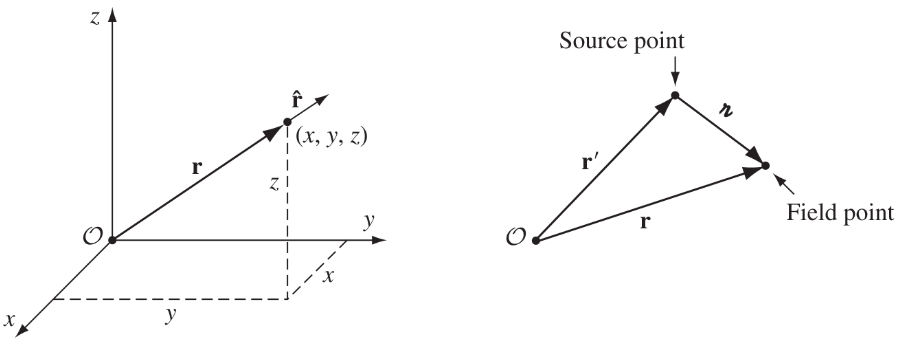

Figure 3: Depiction of position (left) and separation (right) vectors.

1. *To add vectors, add the corresponding components,*

$$\boldsymbol{A} + \boldsymbol{B} = (A_x + B_x)\hat{\boldsymbol{x}} + (A_y + B_y)\hat{\boldsymbol{y}} + (A_z + B_z)\hat{\boldsymbol{z}}. \tag{11}$$

2. *To multiply by a scalar, multiply each component individually,*

$$a\boldsymbol{A} = (aA_x)\hat{\boldsymbol{x}} + (aA_y)\hat{\boldsymbol{y}} + (aA_z)\hat{\boldsymbol{z}}. \tag{12}$$

3. *To calculate the dot product, multiply the corresponding components, and sum them up.*

$$\begin{aligned} \boldsymbol{A} \cdot \boldsymbol{B} &= (A_x\hat{\boldsymbol{x}} + A_y\hat{\boldsymbol{y}} + A_z\hat{\boldsymbol{z}}) \cdot (B_x\hat{\boldsymbol{x}} + B_y\hat{\boldsymbol{y}} + B_z\hat{\boldsymbol{z}}) \qquad (13) \\ &= A_xB_x + A_yB_y + A_zB_z \qquad (14) \end{aligned}$$

4. *To calculate the cross product,* $\boldsymbol{A} \times \boldsymbol{B}$*, form the determinant whose first row is* $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$*, whose second row is* $\boldsymbol{A}$*, and whose third row is* $\boldsymbol{B}$*.*

$$\boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{15}$$

Position, Displacement and Separation vectors

The location of a point in three dimensions can be described by listing its Cartesian coordinates (x, y, z). The vector to that point from the origin $(\mathcal{O})$ is called the **position vector** (Fig. 3(Left)).

$$\boldsymbol{r} = x \cdot \hat{\boldsymbol{x}} + y \cdot \hat{\boldsymbol{y}} + z \cdot \hat{\boldsymbol{z}}. \tag{16}$$

The **infinitesimal displacement vector**, from (x, y, z) to $(x + dx, y + dy, z + dz)$, is

$$d\boldsymbol{r} = dx \cdot \hat{\boldsymbol{x}} + dy \cdot \hat{\boldsymbol{y}} + dz \cdot \hat{\boldsymbol{z}} \tag{17}$$

In electrodynamics, it is common to encounter problems involving two points, a **source point**, $\boldsymbol{r}'$, where an electric charge is located, and a **field point**, $\boldsymbol{r}$, at which you are calculating the electric or magnetic field (Fig. 3(Right)). We define the **separation vector** from the source point to the field point, with the script letter $\boldsymbol{\mathscr{r}}$.

$$\boldsymbol{\mathscr{r}} \equiv \boldsymbol{r} - \boldsymbol{r}' \tag{18}$$

Refresher: Electrostatic

The electrostatic force between two charges is given by

$$\boldsymbol{F} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{\boldsymbol{\mathscr{r}}^2} \hat{\boldsymbol{\mathscr{r}}} \tag{19}$$

where $\boldsymbol{F}$ is the force on a charge q_1 due to the charge q_2 at a distance r.

Exercise 1 :

1. Derive the dot and cross products for the following couples of vectors:
   (a) $\boldsymbol{A} = (6, 2, 1)$ and $\boldsymbol{B} = (8, 9, 2)$
   (b) $\boldsymbol{A} = (8, 1, 7)$ and $\boldsymbol{B} = (9, 6, 9)$
   (c) $\boldsymbol{A} = (5, 2, 5)$ and $\boldsymbol{B} = (-10, -4, -10)$
   (d) $\boldsymbol{A} = (-3, 8, 2)$ and $\boldsymbol{B} = (0, -8, 1)$
2. Find the separation vector $\boldsymbol{\mathscr{r}}$ from the source point $(9, 3, 3)$ to the field point $(6, 1, 7)$. Determine its magnitude, $|\boldsymbol{\mathscr{r}}|$, and construct the unit vector $\hat{\boldsymbol{\mathscr{r}}}$.
3. Find the angle between the body diagonals of a cube.
4. Use the cross product to find the components of the unit vector $\hat{\boldsymbol{n}}$ perpendicular to the shaded plane in Figure 4.

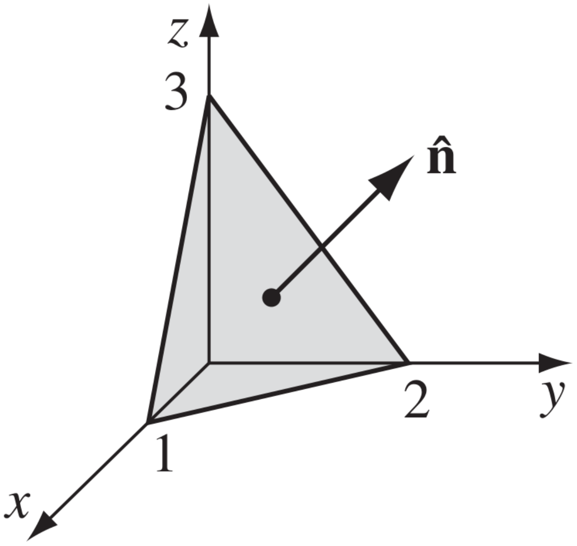

Figure 4: Plane perpendicular to $\hat{\boldsymbol{n}}$.

Exercise 2 :

Two identical water droplets are charged by one extra electron each, such that the electrical force of repulsion compensates for their mutual gravitational force. What is the radius of the droplets ?
Indications: $k_e = 9 \times 10^9\,\mathrm{N\,m^2\,C^{-2}}$, $e = 1.6 \times 10^{-19}\,\mathrm{C}$, $G = 6.7 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$, $\rho_{\mathrm{water}} = 1 \times 10^3\,\mathrm{kg\,m^{-3}}$.

Exercise 3 :

Three small balls with positive charges $+q_A$ and $+q_B = +q_C$ on the balls A, B and C, respectively, can freely move on a ring. What is the ratio of the charges q_A to q_B, if at the equilibrium the angle from the center of the ring to the charges q_B and q_C is $\pi/3$?

Exercise 4 :

Two charges, $-Q_0$ and $-4Q_0$, are at a distance l apart. These two charges are free to move but do not because there is a third charge nearby. What must be the magnitude of the third charge and its placement in order for the first two to be in equilibrium?

Exercise 5 :

A metal spring, put vertically, has the free length of L_0 (see Figure 5). The length becomes L_g when a ball of mass m is put on top of it. Next, the ball is removed, but two massless point charges $+Q$ and $-Q$ are fixed to the opposite ends of the same spring, which is now in horizontal position. Estimate the value of charge Q, if the spring length becomes L_{el} now? The spring obeys the Hook's law: $F_H = -\eta \Delta L$, where ΔL is compression of the spring and η is an unknown constant.

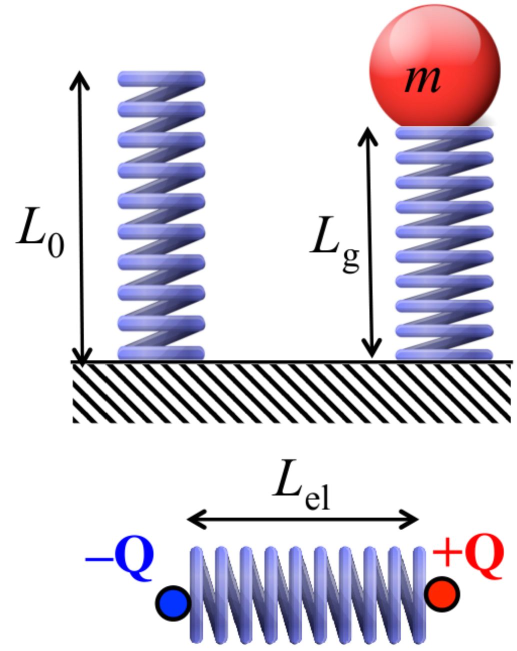

Figure 5: Schematic of the problem in Exercise 5.