General Physics: Electromagnetism, Correction 10

Exercise 1 :

A long straight wire of radius a carries a current that is uniformly distributed over its cross-section. Find the magnetic field both inside and outside the wire.

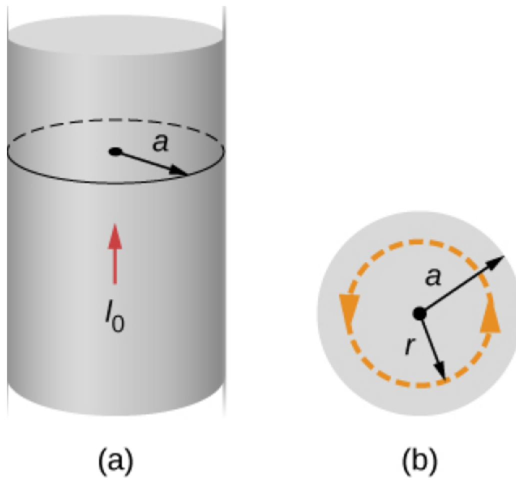

Figure 1: (a) A model of a current-carrying wire of radius a and current I_0 (b) A cross-section of the same wire showing the radius a and the Ampere's loop of radius r.

Solution 1 :

To find the magnetic field we apply the Ampere's law in its full generality

$$\oint_\Gamma d\vec{\ell} \cdot \vec{B} = \mu_0 I_{\text{enc}} = \mu_0 \int_\mathcal{A} d\vec{\sigma} \cdot \vec{J}. \tag{1}$$

We have to distinguish two cases. For $r < a$, we have

$$I_{\text{enc}} = \int_\mathcal{A} d\vec{\sigma} \cdot \vec{J} = \int_0^r dr' 2\pi r' \left(\frac{I_0}{\pi a^2}\right) = \frac{I_0}{a^2} r^2. \tag{2}$$

The Ampere's law gives $2\pi r B_1 = \mu_0 I_0 r^2/a^2$ and then $B_1 = \mu_0 I_0 r/2\pi a^2$.

For $r > a$, we have instead

$$I_{\text{enc}} = \int_\mathcal{A} d\vec{\sigma} \cdot \vec{J} = \int_0^a dr' 2\pi r' \left(\frac{I_0}{\pi a^2}\right) = I_0. \tag{3}$$

The Ampere's law gives $2\pi r B_2 = \mu_0 I_0$ and then $B_2 = \mu_0 I_0/2\pi r$.

Exercise 2 :

Consider a toroid as shown in Figure 2 with a big radius R and a small radius a. The toroid contains N turns of the wire with current I. Suppose that the number of turns N is huge, such that we can consider cylindrical symmetry. Calculate the magnetic field $\overrightarrow{B}$ for:

(a) $r < R - a$;

(b) $R - a < r < R + a$;

(c) $r > R + a$;

and draw the lines of the field. What is the magnitude of the magnetic field in $r = R$ if $I = 500$ A, $N = 100$ and $R = 0.5$ m?

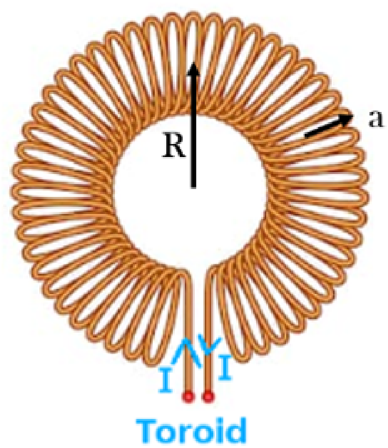

Figure 2: Toroid

Solution 2 :

A toroid can be seen as a solenoid bent in the shape of a toroid. With this assumption as well as the cylindrical symmetry one, one could already deduce that the field is only inside the toroid as seen in theory.

Let's now concretely take the a), b) and c) cases individually and calculate the magnetic field. To better visualize this, a horizontal section of the toroid (simplified) will be considered (Figure 3):

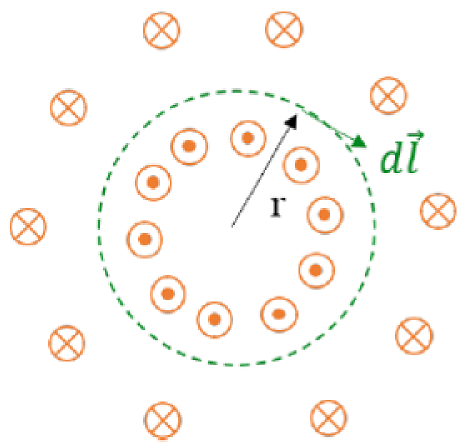

Figure 3: Toroid section

From Ampere's law, stating that magnetic field inside a given area is a result of the enclosed current in this same area, one can conclude that when $r < R - a$ (case a)), their is no enclosed current and thus the magnetic field B is equal to zero as well. At the opposite case, when $r > R + a$ (case c)), outside of the toroid, as for each current entering there is an equal in norm but opposite in direction outgoing current (this can easily be understood from Figure 3), the total current enclosed is also zero and thus magnetic field is also zero. Now, when $R - a < r < R + a$ (case c), which is the case depicted in Figure 3, we have an enclosed current $I_{\text{enclosed}} = NI$ non zero because only the inner part of the toroid and its currents are inside the loop. As these currents are all having the same direction (and intensity), they just add up. This time the Ampere's law has to be concretely applied:

$$\oint_\Gamma d\vec{\ell} \cdot \vec{B} = \mu_0 I_{\text{enclosed}} \tag{4}$$

That we can rewrite due to our two initial assumptions and our formulation for I_{enclosed}:

$$B(r) 2\pi r = \mu_0 N I \tag{5}$$

Thus between $R - a$ and $R + a$, we have a magnetic field $B(r)$ equal to:

$$B(r) = \frac{\mu_0 N I}{2\pi r} \tag{6}$$

The numerical result with the given I, r and N, also knowing the vacuum magnetic permeability $\mu_0 = 4\pi \times 10^{-6}$ N/A^2, is $B \simeq 0.02$ T.

To end up with, the field lines drawing should look like this:

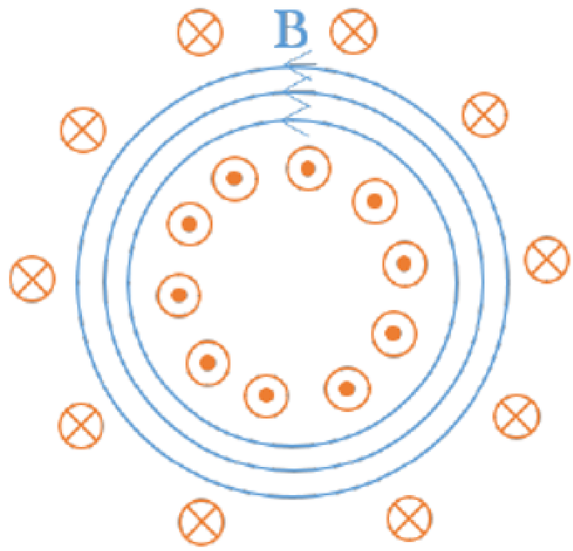

Figure 4: Field lines

Exercise 3 : A solenoid S with a diameter $D = 3.2$ cm has 200 turns/cm and carries a sinusoidal current $I = I_0 \sin(2\pi f t)$ (See Figure 5). In the center, we put a coil C of 130 tight turns with a diameter $d = 2.1$ cm. The amplitude of the current is $I_0 = 1.5$ A and the frequency is $f = 50$ Hz. What is the amplitude of the emf induced in the coil C?

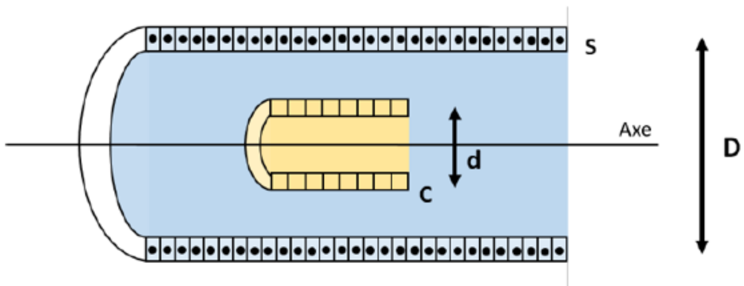

Figure 5: Solenoid S with sinusoidal current containing a coil C.

Solution 3 :

Since the coil C is placed inside the solenoid, it feels the magnetic field produced by the current I flowing in the solenoid S. We indicate the magnetic flux in the coil C as Φ_B^C. When Φ_B^C changes, a emf ε is induced in the coil C thanks to Faraday's law. Since the coil C has N turns, the Faraday's law can be written as:

$$\varepsilon = -\frac{d\Phi_B^C}{dt} = -N\frac{d\Phi_B}{dt}, \tag{7}$$

where $N = 130$ and Φ_B is the magnetic flux across a single turn of C. Since $\vec{B}$ is uniform and has direction perpendicular to the surface A, the flux through each turn of the coil C is $\Phi_B = BA$. The magnitude of the magnetic field B inside the solenoid depends on its current I and on its number of turns per meter n: $B = \mu_0 I n = \mu_0 I_0 n \sin(2\pi f t)$. In this case, $A = \frac{1}{4}\pi d^2$ and $n = 20000$ turns/m. Therefore, we have:

$$\Phi_B = BA = \mu_0 I_0 n \frac{1}{4}\pi d^2 \sin(2\pi f t). \tag{8}$$

and then we can write:

$$\frac{d\Phi_B}{dt} = \frac{\pi^2}{2}\mu_0 I_0 n d^2 f \cos(2\pi f t) = 4.52 \times 10^{-3} \cos(2\pi f t). \tag{9}$$

So, finally, the emf amplitude is:

$$|\varepsilon| = N\frac{\pi^2}{2}\mu_0 I_0 n d^2 f = 0.58 \text{ V} \tag{10}$$

Exercise 4 :

A conductive rod of length l, mass m and resistance R slides down on a vertical conductive frame from the height $H \gg l$ in the presence of homogeneous magnetic field B which is perpendicular to the frame (Figure 1). Estimate the Kinetic energy of the rod, when it hits the grass.

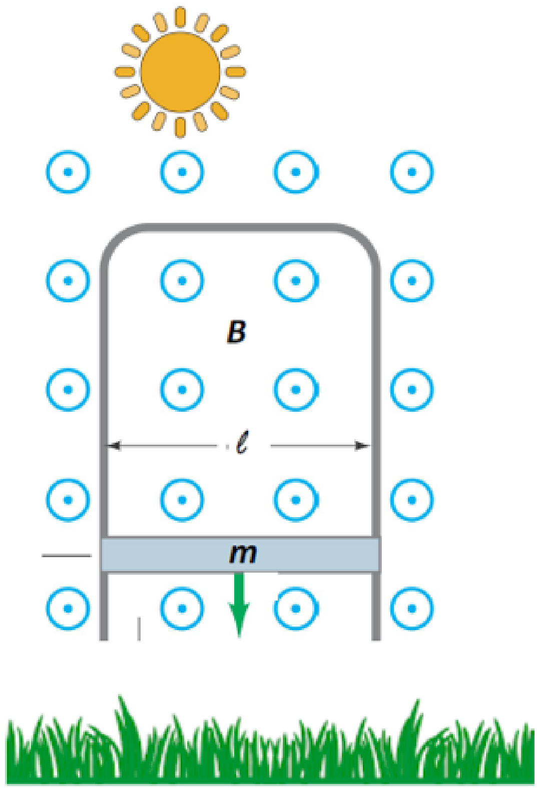

Figure 6: Conductive rod sliding down a conductive frame in a homogeneous magnetic field.

Solution 4 :

a) The frame and the rod together make a conductive loop. Moreover, the rod is falling down with a velocity $v(t)$ due to gravitational force. As a result, when the rod falls down, the loop area increases.

The magnetic flux through the loop increases as well and, by Faraday's law, an e.m.f. ϵ and a current I_{ind} will be induced in the loop.

$$I_{\text{ind}} = \frac{\epsilon}{R} \tag{11}$$

It creates a magnetic field which is going into the sheet inside the loop. Since now there is a current flowing in the rod, it will experience a Lorenz force because of the static field $\mathbf{B}$. This force F_m is directed upward and it is going in the opposite direction of the gravitational force.

Since $H \gg l$, we can safely assume that before the rod hits the surface, the magnetic force has time to increase and to equilibrate the gravitational force. When this happens, the rod will finish its fall with a constant velocity v_f.

The gravitational force F_{gr} is given by the equation 12.

$$F_{gr} = m \cdot g \tag{12}$$

And the magnetic force on a straight wire is given by the formula 13.

$$F_m = B \cdot I \cdot l \tag{13}$$

As the two previous forces equilibrate, we have:

$$F_{gr} = -F_m \tag{14}$$

$$m \cdot g = -B \cdot I_{ind} \cdot l \tag{15}$$

Using Lenz's law, we get :

$$I_{\text{ind}} = \frac{\epsilon}{R} = -\frac{1}{R} \cdot \frac{d\phi_B}{dt} = -\frac{1}{R} \cdot \frac{d}{dt}(B \cdot v_f \cdot t \cdot l) = -\frac{B \cdot v_f \cdot l}{R} \tag{16}$$

Where we used that the area in which the magnetic field goes (inside the loop) is :

$$A = v_f \cdot t \cdot l \tag{17}$$

Therefore,

$$m \cdot g = \frac{B^2 \cdot v_f \cdot l^2}{R} \Leftrightarrow v_f = \frac{m \cdot g \cdot R}{B^2 \cdot l^2} \tag{18}$$

When the rod hits the ground, its kinetic energy is :

$$E_{kin} = \frac{1}{2} \cdot m \cdot v_f^2 = \frac{m^3 \cdot g^2 \cdot R^2}{2 \cdot B^4 \cdot l^4} \tag{19}$$

Exercise 5 :

A square $b \times b = 5 \times 5$ cm conductive frame is moved by an external force with constant velocity $v = 1$ m/s through the area of width $d = 20$ cm of homogeneous magnetic field $B = 1$ T, which is orthogonal to v (see Figure 7). The external work required to pass the field was $W = 2.5 \cdot 10^{-3}$ J. What is the resistance R of the frame?

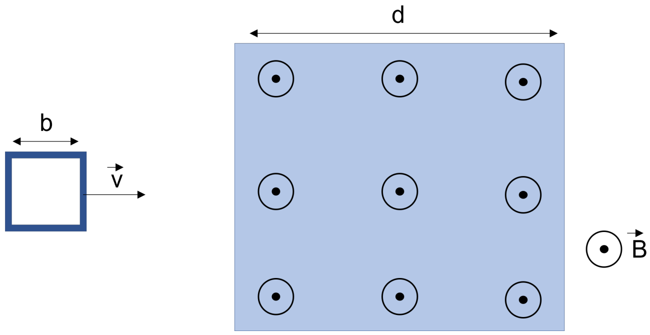

Figure 7: Conductive frame moved by an external force with constant velocity in a homogeneous magnetic field.

Solution 5 :

Let's start by considering that the frame must move at a constant speed; from Newton's law, the resulting force that acts on the frame at any time must be equal to 0:

$$\sum \vec{F} = m\vec{a} = 0 \tag{20}$$

By Lenz's law there's a magnetic force which acts on the frame when the flux of magnetic field going through the frame area changes: this happens only while the frame goes in and out from the magnetic field, in both cases for a length x = b. Pay attention that while the frame is completely inside the magnetic field, there will be a flux of magnetic field, but this will not generate any force as this value is constant.

Let's now see how we can compute this magnetic force and let's think on its direction. The induced ϵ will reduce the change of magnetic flux. Intuitively, we can conclude that while the frame gets in the magnetic field, the force will push the frame out, pointing to the left in the previous drawing. While the frame goes out, the force will push it inside, pointing always in the left direction. In both cases the force is opposite to the movement of the frame. The induced ϵ will generate a current according to the Ohm's law:

$$I = -\frac{1}{R}\frac{d\phi_B}{dt} \tag{21}$$

$$\phi_B = \int d\vec{S}\vec{B} = BA(t)cos(\theta) = Bbx = Bb(vt) \tag{22}$$

$$I = -\frac{1}{R}Bbv \tag{23}$$

where the magnetic field is constant over time as the angle between the magnetic field and the area ($\theta = 0$). The area in this case is the quantity which changes over time, and x is the amount of frame which is inside the magnetic field (in the direction parallel to the velocity).

The direction of this current is such to generate an induced magnetic field (generated by the wire itself) which will try to reduce the external one (again as a consequence of the Lenz law! this is just another way of seeing it). While the wire gets in, the magnetic field generated by the wire must point down in the previous drawing, thus the current in the wire must flow clockwise.

From previous lectures, we know that a wire where current flows inserted in a magnetic field will experience a force (as a generalization of the Lorentz Force):

$$\vec{F}_{magn} = I\vec{l} \times \vec{B} \tag{24}$$

In our case three wires are inserted in the magnetic field as the force act (one of length b and two of length x). By using the right hand rule and considering the direction of the current, the forces on the two x-wires will cancel out. Only the force on the b-wire will act, pointing in the left direction. This is the same result we have obtained in the intuitive picture above! On the other hand, when the frame goes out the direction of the current will be anticlockwise, as it will try to increase the external magnetic field as the flux is decreasing. Again the force on the wire will point to the left. In the end, the force in these two cases is:

$$\vec{F}_{magn} = -\frac{1}{R}v(Bb)^2 \tag{25}$$

Thus, we have that in this region where the magnetic flux changes over time (2 times b), an external force must balance the magnetic force and this will create a work. As in both cases the external force is in the same direction of the movement ($\alpha = 0$, is the angle between the external force and the movement), the two contributes sum up:

$$\begin{gathered} F_{magn} = -F_{ext} \\ W = W_{in} + W_{out} = \vec{F}_{ext}\dot{2}\vec{b} = -F_{magn}2b\cos(\alpha) = -F_{magn}2b \\ R = \frac{2vB^2b^3}{W} = 0.1\Omega \end{gathered} \tag{26}$$

Exercise 6 :

Consider a conductive rod of mass m and resistance R that can freely slide along a horizontal solid conductive frame of width w and length l, which is much longer than the width (Figure 8). The homogeneous magnetic field B is orthogonal to the frame. The field is changing with time t as $B = B_0t$. In which direction and by what distance Δl will the rod be displaced from its initial position at $t = 0$ after a short time $t = T$, assuming that the displacement is small compared to l?

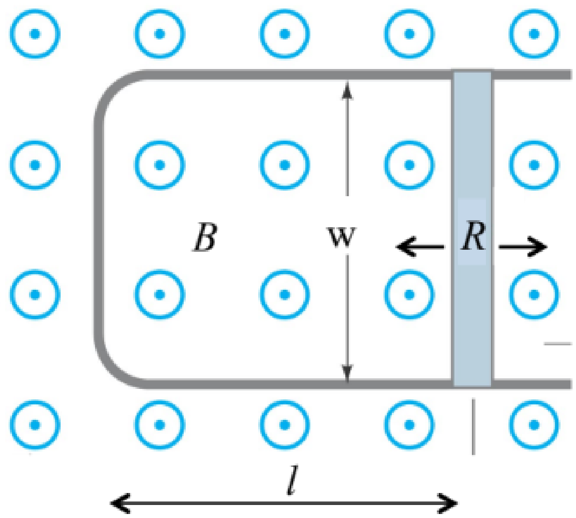

Figure 8: Conductive rod sliding along a conductive frame in a homogeneous magnetic field.

Solution 6 :

Without doing any calculation we can immediately guess the direction of the magnetic force by using the Lenz law: the induced magnetic force will have the effect of decreasing the magnetic flux due to the external magnetic field B. We can think to this action in two ways: by decreasing the loop area or by generating a magnetic field in the opposite direction to the external one. In both the scenarios we conclude that the magnetic force must points to the left (and the induced current, for the right-hand rule, must flow downward).

A more quantitative description of the process, valid for small displacements Δl, can be achieved by exploiting Faraday's law:

$$\varepsilon(t) = -\frac{d}{dt}\Phi(t) \tag{27}$$

Where $\varepsilon(t)$ is the induced emf and $\Phi(t)$ is the time-varying magnetic flux. Since the magnetic field is uniform and orthogonal to the covered area, the magnetic flux is simply given by $\Phi = AB$. The most general expression for the magnetic flux in our problem is:

$$\Phi(t) = wl(t)B(t) \tag{28}$$

Where $B = B_0 t$ is the time-varying magnetic field and $l(t)$ is the unknown displacement after a time t. The crucial point is the information that the displacement is small compared to l, then $\Delta l \ll l$. Therefore, at zero-th order, we can approximate the covered area as constant in time. Then the magnetic flux is simply:

$$\Phi(t) = wlB_0 t \tag{29}$$

The induced emf is then:

$$\varepsilon(t) = -\frac{d}{dt} wlB_0 t = -wlB_0 \tag{30}$$

The induced current is given by Ohm's law:

$$I = \frac{\varepsilon}{R} = -\frac{wlB_0}{R} \tag{31}$$

The "-" sign indicates that the current flows downward, according to what we deduced with Lenz law. The magnetic force is simply the Lorentz force:

$$F(t) = IwB(t) = -\frac{B_0^2 w^2 l}{R} t \tag{32}$$

The "-" sign confirms again what we deduced at the beginning, using the physical interpretation of Lenz's law, since it means that the force is pointing to the left. In order to conclude the exercise, we use Newton's second law, the acceleration is given by $F_{tot} = ma$:

$$a = -\frac{B_0^2 w^2 l}{mR} t \tag{33}$$

The velocity is the integral of the acceleration:

$$v = \int dt\ a(t) = -\frac{B_0^2 w^2 l}{2mR} t^2 \tag{34}$$

Finally, Δl is the integral of the velocity between 0 and the final time T (notice that the integral of the acceleration is an indefinite integral, because we need to know the velocity at each instant of time, so it is wrong to compute the velocity between 0 and T and perform a further integration in order to find the displacement Δl):

$$\Delta l = \int_0^T dt\ v(t) = -\frac{B_0^2 w^2 l}{6mR} T^3 \tag{35}$$

REMARK: An exact description of the process can be obtained by writing down a differential equation for the variable $l(t)$, indeed by using the time dependent magnetic flux:

$$\Phi(t) = wl(t)B_0 t \tag{36}$$

One obtain the emf:

$$\varepsilon(t) = -w[\dot{l}(t)B_0 t + l(t)B_0] \tag{37}$$

This leads to the magnetic force $F(t) = I(t)wB(t)$ where $I(t) = \varepsilon(t)/R$. By using Newton's second law $F_{tot} = m\ddot{l}(t)$ we obtain a differential equation for $l(t)$:

$$\ddot{l}(t) = -\frac{w^2 B_0^2}{mR}[\dot{l}(t)t^2 + l(t)t] \tag{38}$$

This equation describes with great accuracy the motion of the rod, unfortunately is a highly non-trivial differential equation. If we neglect the term with $\dot{l}t^2$ we get:

$$\ddot{l}(t) = -\frac{w^2 B_0^2}{mR} l(t) t \tag{39}$$

Which is still hard to solve. In the limit of small displacements $l(t)$ is constant and $\ddot{l}(t)$ is simply the acceleration of the rod a, then:

$$a = -\frac{w^2 B_0^2 l}{mR} t \tag{40}$$

Which is exactly what we obtained using immediately the information $\Delta l \ll l$.