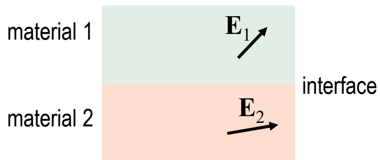
2.6.1 Boundary conditions between dielectrics

Consider two materials put on top of each other. We assume that these are **large plates of two different dielectric materials**. They have been polarized by an electric field. Let's assume two general orientations of the electric vector fields inside the materials as illustrated by the two displayed vectors:



To describe the behavior of the electric fields 1 and 2 at the boundary (interface), we decompose the electric field vectors in their components normal (n) and tangential (t) to the considered interface.

Boundary conditions in two scenarios; scenario A2 is the most general one:

(A1) **No** free charges **in** the interface: $\boxed{\varepsilon_{r1}E_{1n} = \varepsilon_{r2}E_{2n} \quad \text{and} \quad E_{1t} = E_{2t}}$

Conclusion: In the **normal** components of $\vec{E}$ there is a **step-like variation**; **tangential** components **do not vary** across an interface

(A2) **Additional free charges** σ_{free} right at interface: $\varepsilon_0\varepsilon_{r1}E_{1n} - \varepsilon_0\varepsilon_{r2}E_{2n} = \sigma_{free}$ or in short: $D_{1n} - D_{2n} = \sigma_{free}$ and $E_{1t} = E_{2t}$

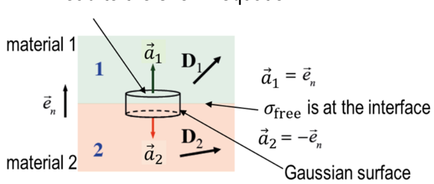
These relations are consistent with the **modified Gauss's law** and $\vec{\nabla} \times \vec{E} = 0$. They fulfill Maxwell's equations. Instructions how to derive them can be found below.

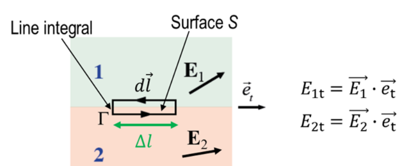
How to verify the boundary conditions?

The boundary conditions stated for an interface between materials 1 and 2 must fulfill the Gauss's law and the Maxwell equation $\vec{\nabla} \times \vec{E} = 0$, otherwise the incorporated fields (field components) would not be correctly chosen.

(1) To verify the validity of the equation for the normal components one uses the modified Gauss's law and a Gaussian surface which is the closed surface of a cylinder. The cylinder contains the interface in its centre (the interface is parallel to its top and bottom surface, see fig. on the left) where there is the assumed added free charge stated in the equation. Assuming displacement vectors $\vec{D}_i = \varepsilon_0\varepsilon_{ri}\vec{E}_i$ only the normal components survive and fulfill Gauss's law according to the given equation.

(2) To verify the validity of the equation for the tangential components one evaluates a line integral which is consistent with $\vec{\nabla} \times \vec{E} = 0$ (right fig.). Here, the tangential components survive and lead to the shown equation.



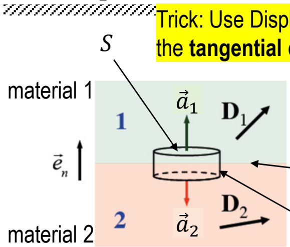


Try to solve (1) and (2). The solution is on the next pages.

$$\oint_\Gamma \vec{E} \cdot d\vec{l} = 0$$

Boundary conditions between dielectrics – normal components

Trick: Use Displacement vectors D and Gaussian surface as a tiny cylinder, the **tangential components cancel out** in the integration over the side walls.



$\vec{a}_1 = \vec{e}_n$

σ_{free} is at the interface

$\vec{a}_2 = -\vec{e}_n$

Gaussian surface

$D_{1\text{n}} = \overrightarrow{D_1} \cdot \overrightarrow{e_\text{n}}$

$D_{2\text{n}} = \overrightarrow{D_2} \cdot \overrightarrow{e_\text{n}}$

We assume uniform $\overrightarrow{D_1}$ in material 1, and uniform $\overrightarrow{D_2}$ in material 2 with orientations as sketched. The areas of top and bottom surfaces is S each.

Symmetry analysis

divergence theorem, and use $\vec{\nabla} \cdot \vec{D} = \rho_{\text{free}}$

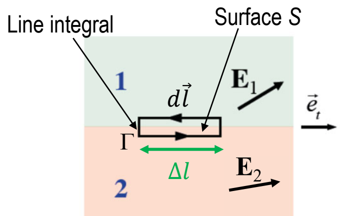
$$\oint\!\!\oint_{\substack{\text{Gaussian}\\ \text{surface}}} \vec{D} \cdot d\vec{a} = (D_{1\text{n}} - D_{2\text{n}}) \cdot S = \iiint \vec{\nabla} \cdot \vec{D} dV = \iiint \rho_{\text{free}} dV = Q_{\text{free}} = \sigma_{\text{free}} S$$

From the comparison one finds the **boundary condition**: $\boxed{D_{1\text{n}} - D_{2\text{n}} = \sigma_{\text{free}}}$

EPFL

Boundary conditions between dielectrics – tangential components

Trick: Use electric field vectors and a closed path integral as sketched (tangential components contribute to the integral, normal ones cancel out). For the line integral the free charges are not part of the equation. This result hence does not depend on them.



$$E_{1t} = \vec{E_1} \cdot \vec{e_t}$$
$$E_{2t} = \vec{E_2} \cdot \vec{e_t}$$

Symmetry analysis — Stokes' theorem — $\vec{\nabla} \times \vec{E} = 0$

$$\oint \vec{E} \cdot d\vec{l} = (E_{2t} - E_{1t}) \cdot \Delta l = \iint_{\substack{\text{surface S} \\ \text{inside path } \Gamma}} \vec{\nabla} \times \vec{E}\, dS = 0 \ \ (\text{conservative force})$$

From the comparison one finds the **boundary condition** between dielectrics: $\boxed{E_{1t} = E_{2t}}$

EPFL

Boundary conditions between dielectrics – case studies

Application of the boundary condition to different cases:

- **No free charge** at boundary ($\sigma_{\text{free}} = 0$): $D_{1n} = D_{2n} \Rightarrow \varepsilon_{r1} E_{1n} = \varepsilon_{r2} E_{2n} \Rightarrow \boxed{E_{2n} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}} E_{1n}}$

  Conclusion: ***D*$_n$ stays constant across an interface,**
  but there is a step in *E*$_n$
  The tangential components E_t do not vary across the interface.

- Material 1 is assumed to be a **metal with $\boldsymbol{\sigma_{free}}$**: $D_1 = 0$ as inside a conductor there is no electric field. Apply boundary condition:

  $$\Rightarrow -D_{2n} = \sigma_{\text{free}} \Rightarrow \boxed{E_{2n} = \frac{-\sigma_{\text{free}}}{\varepsilon_0 \varepsilon_{r2}}}$$

  Conclusion: This is the case consistent with a plate capacitor with a dielectric (Sect. 2.2). In our assumption we assume a surface density of positive free charges $\boldsymbol{\sigma_{free}}$, hence the resulting electrical field must points in $-\vec{e}_n$-direction (downwards) in the sketched dielectric (material 2). And indeed, the component E_{2n} is found to be negative (green box).

  The **modified Gauss's law using *D*** hence provides the correct solution for the electric field inside a capacitor filled with a dielectric!

EPFL