# Exercise sheet #2

**Problem 1.** We want to make a comparison between the gravitational force and the electrostatic (Coulomb) force.

(a) From a *mathematical* point of view, which is the only qualitative difference between the two forces? From the *physical* point of view, is there another important difference?

(b) Let us now consider two electrons in free space placed at distance $r$: they both feel gravitational attraction because of their mass, and electrostatic repulsion because of their charge. What value the electron mass should have had in order to create the situation where the attraction and the repulsion are balanced and the two electrons do not move?

(c) A charged body with $Q = -1\ \mu C$ is dropped from a height $h = 1\ m$ above the Earth surface. A point charged with $10^{19}$ electrons is placed exactly below it on the surface. By using the concept of potential energy, evaluate the mass $M$ of the body in order to let it float at height $h$ *(remember that the gravitational potential energy at distance $z$ close to the Earth surface can be approximated as $U = Mgz$, with $g = 9.81\ m/s^2$).*

(d) Describe *qualitatively* what happens if the body with mass $M$ as determined in c) placed at $h$ has an initial velocity $v$ directed towards the Earth *(tip: this is best solved using energy diagrams).*

*Solution.* (a) Given the same definition of the distance vector $\vec{r}$, the expressions of the gravitational force $\vec{F}_G$ and the Coulomb force $\vec{F}_C$ have the same structure, apart from their sign: same sign for the mass means gravitational attraction, whereas same sign for the charge means electrostatic repulsion. In Physics, we do observe only positive masses, but both positive and negative charges. Thus $\vec{F}_G$ is only attractive, while $\vec{F}_C$ can be either attractive or repulsive.

(b) In order to balance the two forces $|\vec{F}_G| = G\frac{m_e^2}{r^2}$ and $|\vec{F}_C| = k\frac{q_e^2}{r^2}$ (where $G$ is the gravitational constant and $k = \frac{1}{4\pi\epsilon_0}$ is the Coulomb constant) we would need to set $G\frac{m_e^2}{r^2} = k\frac{q_e^2}{r^2}$ and therefore $m_e = q\sqrt{\frac{k}{G}} = 1.86 \times 10^{-9}\ kg$. Note that this value is $2 \times 10^{21}$ larger than the real value of $m_e$!

(c) The gravitational and Coulomb potential energies of the system are $U_G(z) = Mgz + c_1$ and $U_C(z) = k\frac{Q_1Q_2}{z} + c_2$. The body placed at height $h$ will not move when $-\frac{\partial U}{\partial z} = |\vec{F}| = 0$, with $U = U_G + U_C$. We thus obtain the equation $Mg - k\frac{Q_1Q_2}{z^2} = 0$. Therefore for $z = h$: $M = k\frac{Q_1Q_2}{gh^2}$. By substituting the constants $k$ and $g$ and the values $h = 1\ m$, $Q_1 = 1 \times\ 10^{-6}\ C$ and $Q_2 = 10^{19} \times 1.6 \times\ 10^{-19}\ C$ we obtain $M \approx 1468\ kg$.

(d) Assuming no lateral movements and no friction, the body will oscillate around $h$ indefinitely. At first it will fall from $h$ to $h_1$, where the Coulomb interaction is strong enough to reverse the movement up to $h_2$ where the gravitational interaction becomes stronger, and so on. The values $h_1$ and $h_2$ are determined by $K = \frac{1}{2}mv^2$ as shown in Fig. 1.

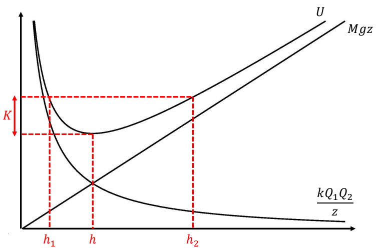



Figure 1: For this plot the potential energy references are chosen so that $c_1 = c_2 = 0$.

□

**Problem 2.** Suppose we loose 1 out of every $10^{12}$ electrons in our body.

(a) Estimate the charge of a typical human body after this event. Make sure to justify the assumptions you need to make and try not to use a calculator.

(b) Estimate the force between two persons at distance of 1 m? How will this change for 1.5 m?

(c) Estimate the acceleration that each person will experience.

*Solution:*

(a) Let us assume that the average human body mostly made of 70 kg of water. There are 10 electrons in a water molecule, and approximately $\frac{70\times10^3}{M_{H_2O}}N_A = \frac{70\times10^3}{18} \times 6,022 \times 10^{23} \approx 2.3 \times 10^{27}$ water molecules in a human body which adds up to around $2.3 \times 10^{28}$ electrons.
If one were to "lose" one out of every $10^{12}$ electrons, their charge would go from neutral up to $Q = \frac{2.3\times10^{28}\times e}{10^{12}} \approx 4 \times 10^{-3}C$

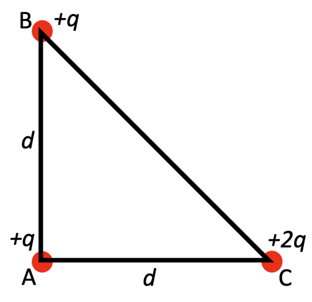
(b) The two bodies, each of charge Q, repel each other (they are both charged positively) with a force inversely proportional to the square of the distance $d$ between them, such that

$$F = \frac{Q^2}{4\Pi\epsilon_0 d^2}$$
$$\approx 9 \times 10^9 \times \frac{(4 \times 10^{-3})^2}{1} \approx 1.4 \times 10^5 N \qquad \text{for } d = 1m$$
$$\approx 9 \times 10^9 \times \frac{(4 \times 10^{-3})^2}{2.25} \approx 6 \times 10^4 N \qquad \text{for } d = 1.5m$$

(c) Newton's second law gives $a = \frac{F}{m} = \frac{1.4\times10^5}{70} \approx 2000\frac{m}{s^2}$

□

**Problem 3.** Consider a $45° - 90°$ triangle of side $d$ and three point charges $+q$, $+q$ and $+2q$ fixed at the three corners $A$, $B$, $C$ respectively as shown in the figure below. Evaluate the Coulomb force vector acting on a positive charge $q_0$ fixed in the midpoint of the hypotenuse.




*Solution.* The Coulomb force acting on a charge $q_0$ fixed in the midpoint of the hypotenuse will be given by the contributions of the three charges in $A$, $B$ and $C$ considerated separately because of the superposition principle. Therefore, we consider a coordinate system as shown in Fig. 2, and evaluate the three forces:

$$\vec{F}_A = k\frac{qq_0}{(d\sqrt{2}/2)^2}\frac{\hat{x}+\hat{y}}{\sqrt{2}} = \frac{\sqrt{2}kqq_0}{d^2}(\hat{x}+\hat{y})$$
$$\vec{F}_B = k\frac{qq_0}{(d\sqrt{2}/2)^2}\frac{\hat{x}-\hat{y}}{\sqrt{2}} = \frac{\sqrt{2}kqq_0}{d^2}(\hat{x}-\hat{y})$$
$$\vec{F}_C = k\frac{2qq_0}{(d\sqrt{2}/2)^2}\frac{-\hat{x}+\hat{y}}{\sqrt{2}} = \frac{2\sqrt{2}kqq_0}{d^2}(-\hat{x}+\hat{y})$$

Therefore we can evaluate the total force $\vec{F} = \vec{F}_A + \vec{F}_B + \vec{F}_C = \frac{2\sqrt{2}kqq_0}{d^2}\hat{y}$. It has to be noted that the calculation would be a bit easier by choosing a coordinate system rotated by 45°. □

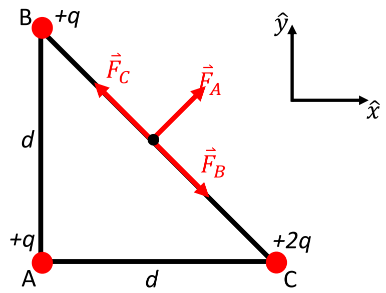



Figure 2

**Problem 4.** Consider two fixed point charges $+q$ and $+2q$ placed at distance $a$.

(a) Find the equilibrium position between the two charges for a positive charge $q_0$.

(b) How does the result change if you consider a negative charge $q_0$?Would they still be able to be in equilibrium?

*Solution.* First of all we need to set a coordinate system. We choose the $x$ axis pointing from the $+q$ charge towards the $+2q$ charge, with the origin set in the $+q$ charge, as shown in Fig. 3.

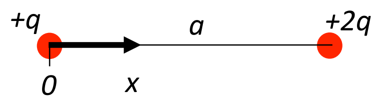



Figure 3

1. The potential energy of the system in the region $0 < x < a$ is given by $U = k\frac{qq_0}{x} + k\frac{2qq_0}{a-x} + k\frac{2q^2}{a}$. The third term, which takes into account the two fixed charges, does not depend on $x$ and will not influence the equilibrium position. In order to find the equilibrium, we look for the point where $0 = |\vec{F}| = -\frac{dU}{dx} = kqq_0\left(\frac{1}{x^2} - \frac{2}{(a-x)^2}\right)$, which leads to the equation $x^2 + 2ax - a^2 = 0$. The only solution within the considered region $0 < x < a$ is $x_0 = (\sqrt{2} - 1)a$.

2. In the case of $q_0 < 0$, since the expression of $|\vec{F}| = -\frac{dU}{dx}$ will be the same but only with opposite sign, the equilibrium position will be the same. However, since also $U$ itself changes sign the type of equilibrium will change from stable to unstable.

□

**Problem 5.** What is the total charge on a sheet with the size $L_x$=60 cm and $L_y$=50 cm, if the charge distribution in $\frac{\text{C}}{\text{m}^2}$ is given by $\sigma(x, y) = x^2y^3 + ln(x)$?
(assume that one of the corners of the sheet is located at (0/0), and it's expanding in positive x- and y-direction)

*Solution.* The total charge on the surface of the sheet is given by

$$Q = \int_0^{L_y} \int_0^{L_x} (x^2y^3 + ln(x))\, dx\, dy$$

We first solve the inner integral,

$$\int_0^{L_x} \left(x^2y^3 + ln(x)\right)\, dx = y^3 \int_0^{L_x} x^2\, dx \;\; + \;\; \int_0^{L_x} ln(x)\, dx$$

for the left side:

$$y^3 \int_0^{L_x} x^2\, dx = y^3 \cdot \frac{1}{3} x^3 \Big|_0^{L_x}$$

for the right side we use integration by parts with $u(x) = ln(x)$, $dv = dx$ and we evaluate $du = 1/x\ dx$ and $v(x) = x$:

$$\int ln(x)\, dx = \int u\, dv = uv - \int v\, du$$

we substitute $u = ln(x)$, $v = x$ and $du = 1/x\ dx$:

$$\begin{aligned}
\int ln(x)\, dx &= uv - \int v\, du \\
&= ln(x)\ x - \int x \cdot \frac{1}{x}\, dx \\
&= ln(x)\ x - \int dx \\
&= ln(x)\ x - x + C
\end{aligned}$$

With boarders:

$$\int_0^{L_x} ln(x)\, dx = ln(x)\ x\Big|_0^{L_x} - x\Big|_0^{L_x}$$

So the inner integral writes as

$$\int_0^{L_x} \left(x^2y^3 + ln(x)\right)\, dx = y^3 \cdot \frac{1}{3} x^3 \Big|_0^{L_x} + ln(x)\ x\Big|_0^{L_x} - x\Big|_0^{L_x}$$

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*Note:* Since $ln(0) = -\infty$ the solution for $x \cdot ln(x)$ is not defined for $x = 0$. However we can still solve the integral in the given borders, since the limit $lim_{x\to 0}\,(x \cdot ln(x)) = 0$ is defined. To prove $lim_{x\to 0}\,(x \cdot ln(x)) = 0$ we use the rule of l'Hopital:

$$\begin{aligned}
&\text{if } lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{0}{0} \text{ or if } lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{\pm\infty}{\pm\infty} \text{ then} \\
&lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = lim_{x\to a}\left(\frac{f'(x)}{g'(x)}\right)
\end{aligned}$$

So we can apply this to our problem:

$$\begin{aligned}
lim_{x\to 0}\,(x \cdot ln(x)) &= lim_{x\to 0}\left(\frac{ln(x)}{\frac{1}{x}}\right) \\
&= lim_{x\to 0}\left(\frac{\frac{1}{x}}{-\frac{1}{x^2}}\right) \\
&= lim_{x\to 0}\,(-x) \\
&= 0
\end{aligned}$$

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Now, we continue with the outer integral

$$\begin{aligned}\int_0^{L_y}\left(y^3\cdot\frac{1}{3}x^3\Big|_0^{L_x}+ln(x)\;x\Big|_0^{L_x}-x\Big|_0^{L_x}\right)dy &= \frac{1}{4}y^4\Big|_0^{L_y}\cdot\frac{1}{3}x^3\Big|_0^{L_x}+y\Big|_0^{L_y}\cdot ln(x)\;x\Big|_0^{L_x}-y\Big|_0^{L_y}\cdot x\Big|_0^{L_x}\\ &= \frac{1}{4}L_y^4\cdot\frac{1}{3}L_x^3+L_y\cdot ln(L_x)\;L_x-L_y\cdot L_x\\ &= L_xL_y\left(\frac{1}{12}L_x^2L_y^3+ln(L_x)-1\right)\end{aligned}$$

substituting the dimensions of the sheet, $L_x = 0.6\ m$ and $L_y = 0.5\ m$, we have

**a)**

$$\begin{aligned}Q &= \int_0^{L_y}\int_0^{L_x}\left(x^2y^3+ln(x)\right)dx\,dy\\ &= \left[0.6\cdot 0.5\left(\frac{1}{12}0.6^2 0.5^3+ln(0.6)-1\right)\right]\\ &= -0.4521\ C\end{aligned}$$

□

**Problem 6.** A disc shows a radial charge distribution given by $\sigma(r) = e^r$. What is the mean surface charge density as a function of the disc radius R?

*Solution.* For a disk of radius R, the mean charge density on its surface can be computed by

$$\overline{\sigma_{disk}} = \frac{Total\ charge\ on\ disk}{Disk\ area} = \frac{Q(R)}{\pi\cdot R^2}$$

The total charge on the disk, in the polar coordinate system, is given by the expression

$$Q = \int_0^{2\pi}\int_0^{R} r\cdot\sigma(r,\theta)\,dr\,d\theta$$

where $\sigma$ is the charge distribution of the disk.

Then, Q needs to be computed in order to find $\overline{\sigma_{disk}}(R)$. The double integral to solve is the following

$$Q(R) = \int_0^{2\pi}\int_0^{R} r\cdot e^r\,dr\,d\theta$$

The inner most integral can be solved by parts, using the following substitutions: $u = r$, $du = dr, v = e^r$ and $dv = e^r\,dr$. That is

$$\begin{aligned}\int u\,dv &= u\,v-\int v\,du\\ \int r\,e^r\,dr &= r\,e^r-\int e^r\,dr\\ &= e^r\,(r-1)\end{aligned}$$

Now, the outer integral can be solved

$$\int_0^{2\pi} \Big[ e^r \, (r-1) \Big]_0^R d\theta = \int_0^{2\pi} (\, e^R \, (R-1) + 1 \,) \, d\theta$$
$$= (\, e^R \, (R-1) + 1 \,) \cdot \theta \Big|_0^{2\pi}$$
$$= 2\pi \, (\, e^R \, (R-1) + 1 \,)$$

The mean charge density is:
$\overline{\sigma_{disk}}(R) = \frac{2\pi \, (\, e^R \, (R-1)+1 \,)}{\pi \, R^2} = \frac{2}{R^2} \Big( e^R \, (R-1) + 1 \Big) \; \frac{C}{m^2}$

□

**Problem 7.** In atomic Physics, one of the first attempts to describe the atomic structure is the *Bohr model*. According to this, an atom is depicted as a small positively charged nucleus surrounded by electrons which travel around it in circular orbits. The Hydrogen atom has only one electron and a nucleus made by only one proton (the charge of which has the same magnitude as the electron charge). The radius of the orbit is known as *Bohr radius* and its value is $a_0 = 5.29 \times 10^{-11} \; m$.

(a) Evaluate the potential energy stored in the Hydrogen atom.

(b) Evaluate the kinetic energy of the electron as a function of its charge (Remember the expression for the centripetal force is $F = mv^2/r$).

(c) Evaluate the ionization energy of the Hydrogen atom, that is the energy required to extract the electron from it.

*Solution.*

(a) The potential energy is: $U = -k\frac{q_e^2}{a_0} \approx -4.34 \times 10^{-18} \; J = -4.34 \; aJ$.

(b) Since the magnitude of the Coulomb force $|\vec{F}| = k\frac{q_e^2}{a_0^2}$ corresponds to a centripetal force $|\vec{F}| = \frac{mv^2}{a_0}$, the kinetic energy of the electron can be expressed as: $K = \frac{1}{2}mv^2 = \frac{|\vec{F}|r}{2} = k\frac{q_e^2}{2a_0} \approx 2.17 \; aJ$.

(c) The total energy is $E = U + K = -k\frac{q_e^2}{2a_0} \approx -2.17 \; aJ$. The fact that $E$ is negative means that the electron is bound to the nucleus. In order to extract the electron from the Hydrogen atom therefore one has to supply an ionization energy $I = -E = 2.17 \; aJ$.

□