Electromagnetism Review

Let's review the exam information

1 Basic Information

- Date and time: Thursday, January 15 at 9:15am
- Duration: 3 hours
- Location: SwissTech convention center
- You are allowed to bring one A4 cheat sheet (Writing both in the front and back is allowed).

Let’s review the exam information

2 Exam format

- The exam will consist of 35 multiple choice questions and 3 open questions. Each multiple choice question is worth 2 points and each open ended question 10 points (Total of 100 points). Note that NO points are subtracted for wrong answers.
- Questions will test both your conceptual understanding and your ability to to solve problems.

Let’s review the exam information

4 Exam content

All material from Lectures 1 to 23 will be covered the exam. Similarly, you are expected to be able to solve all the types of problems covered in the exercise sheets. You will be tested both on your conceptual understanding and your ability to solve problems carrying out analytical calculations (i.e. you are expected to know how to apply different formulas and do the math). The entire semester we have been careful to define the units of all physical observables discussed (e.g. electric field, potential, magnetic field inductance, capacitance...), you are expected to know them and understand the physical interpretation of these observables with the help of dimensional analysis.

Let’s review the exam information

- Example of what you need to know from Lecture 13:

13. **Magnetostatics I. Lorentz force.**

    - Understand the concept of magnetic force and magnetic field.
    - Ability to use Lorentz force law to calculate the trajectory, velocity or mass of a charged particle in a magnetic field.
    - Understand why magnetic forces do no work.
    - Ability to calculate forces and torques on current loops with diverse geometries in external magnetic fields.

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On moodle week 14 you willl find the document "Electromagnetism_exam_information"

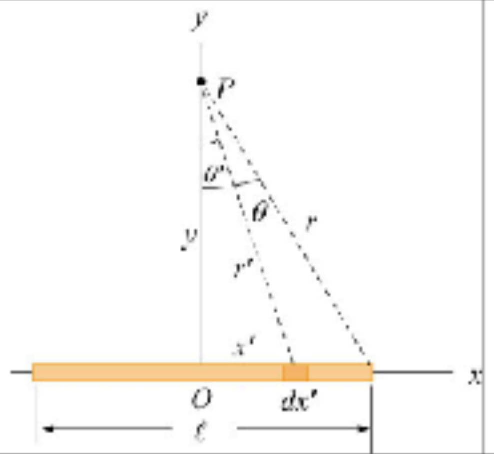
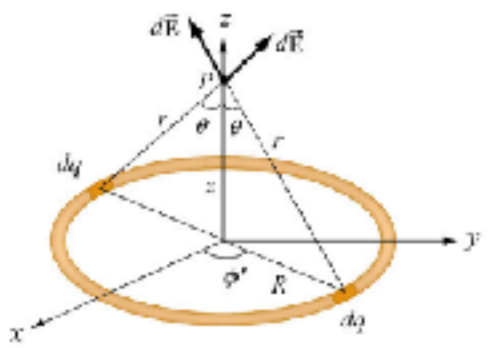
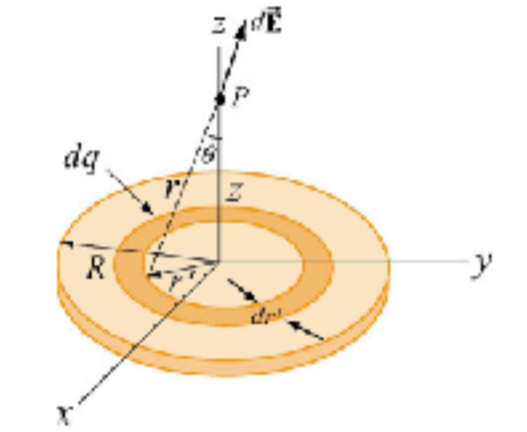
With this information for every lecture of the course

Review of key problem solving abilities in the course

When do we use Coulomb or Gauss to calculate E?

- If the problem has symmetries use Gauss, otherwise use Coulomb.
- Gauss will always be quicker to do calculations.
- If you use Gauss don’t forget to specify the direction of E.
- TIP: Check if your charge distribution is a superposition of two or more distributions with symmetries. Then you can use Gauss and the principle of superposition to calculate E.

Computing the electric field Coulomb

	Line charge	Ring of charge	Uniformly charged disk
Figure			
(2) Express dq in terms of charge density	$dq = \lambda\, dx'$	$dq = \lambda\, d\ell$	$dq = \sigma\, dA$
(3) Write down dE	$dE = k_e \frac{\lambda\, dx'}{r'^2}$	$dE = k_e \frac{\lambda\, dl}{r^2}$	$dE = k_e \frac{\sigma\, dA}{r^2}$
(4) Rewrite r and the differential element in terms of the appropriate coordinates	dx' $\cos\theta = \frac{y}{r'}$ $r' = \sqrt{x'^2 + y^2}$	$d\ell = R\, d\phi'$ $\cos\theta = \frac{z}{r}$ $r = \sqrt{R^2 + z^2}$	$dA = 2\pi r' dr'$ $\cos\theta = \frac{z}{r}$ $r = \sqrt{r'^2 + z^2}$
(5) Apply symmetry argument to identify non-vanishing component(s) of dE	$dE_y = dE\cos\theta$ $= k_e \frac{\lambda y dx'}{(x'^2 + y^2)^{3/2}}$	$dE_z = dE\cos\theta$ $= k_e \frac{\lambda R z\, d\phi'}{(R^2 + z^2)^{3/2}}$	$dE_z = dE\cos\theta$ $= k_e \frac{2\pi\sigma z r' dr'}{(r'^2 + z^2)^{3/2}}$
(6) Integrate to get E	$E_y = k_e \lambda y \int_{-\ell/2}^{+\ell/2} \frac{dx}{(x^2 + y^2)^{3/2}}$ $= \frac{2k_e\lambda}{y} \frac{\ell/2}{\sqrt{(\ell/2)^2 + y^2}}$	$E_z = k_e \frac{R\lambda z}{(R^2 + z^2)^{3/2}} \oint d\phi'$ $= k_e \frac{(2\pi R\lambda) z}{(R^2 + z^2)^{3/2}}$ $= k_e \frac{Qz}{(R^2 + z^2)^{3/2}}$	$E_z = 2\pi\sigma k_e z \int_0^R \frac{r' dr'}{(r'^2 + z^2)^{3/2}}$ $= 2\pi\sigma k_e \left(\frac{z}{\|z\|} - \frac{z}{\sqrt{z^2 + R^2}} \right)$

In this table $k_e = 1/(4\pi\varepsilon_0)$

(1) Start with $d\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0} \frac{dq}{r^2} \hat{\mathbf{r}}$

(2) Rewrite the charge element dq as

$$dq = \begin{cases} \lambda\, d\ell & \text{(length)} \\ \sigma\, dA & \text{(area)} \\ \rho\, dV & \text{(volume)} \end{cases}$$

depending on whether the charge is distributed over a length, an area, or a volume.

(3) Substitute dq into the expression for $d\vec{\mathbf{E}}$.

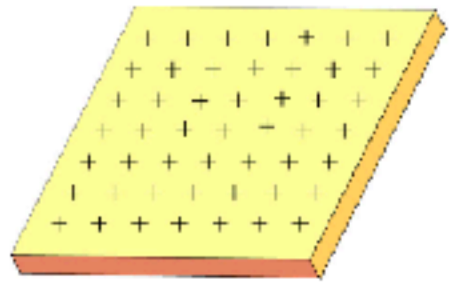
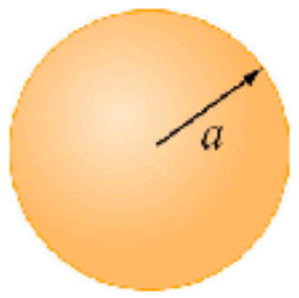
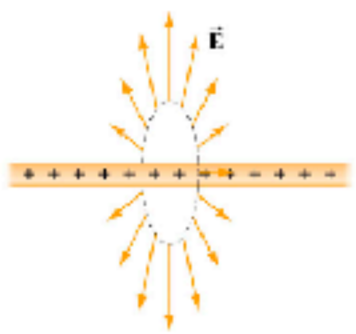
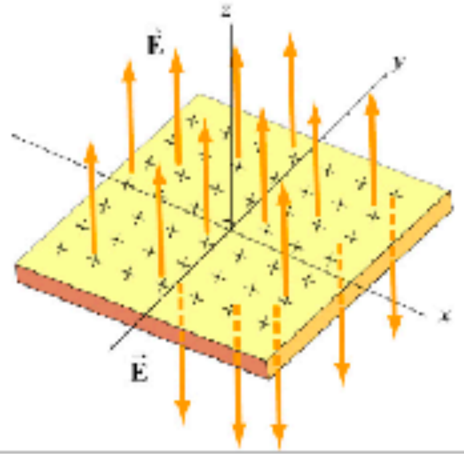
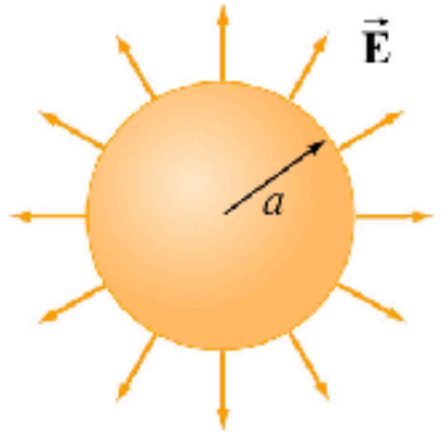
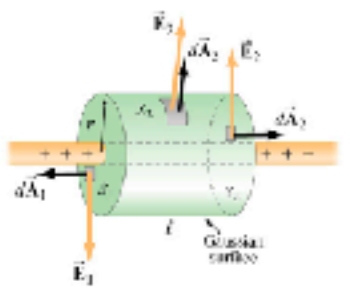
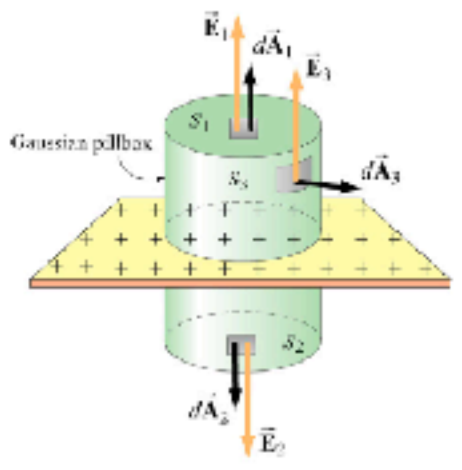
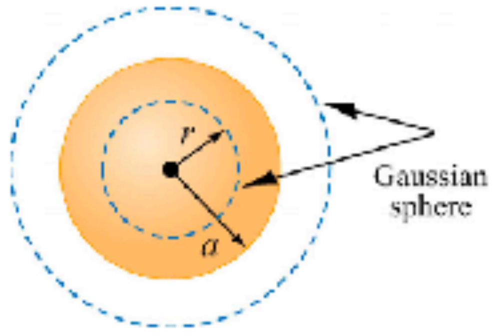
(4) Specify an appropriate coordinate system (Cartesian, cylindrical or spherical) and express the differential element ($d\ell$, dA or dV) and r in terms of the coordinates (see Table 2.1 below for summary.)

	Cartesian (x, y, z)	Cylindrical (ρ, ϕ, z)	Spherical (r, θ, ϕ)
dl	$dx,\ dy,\ dz$	$d\rho,\ \rho d\phi,\ dz$	$dr,\ r\,d\theta,\ r\sin\theta\, d\phi$
dA	$dx\,dy,\ dy\,dz,\ dz\,dx$	$d\rho\,dz,\ \rho\,d\phi\,dz,\ \rho\,d\phi\,d\rho$	$r\,dr\,d\theta,\ r\sin\theta\,dr\,d\phi,\ r^2\sin\theta\,d\theta\,d\phi$
dV	$dx\,dy\,dz$	$\rho\,d\rho\,d\phi\,dz$	$r^2\sin\theta\,dr\,d\theta\,d\phi$

(5) Rewrite $d\vec{\mathbf{E}}$ in terms of the integration variable(s), and apply symmetry argument to identify non-vanishing component(s) of the electric field.

(6) Complete the integration to obtain $\vec{\mathbf{E}}$.

Computing the electric field Gauss

System	Infinite line of charge	Infinite plane of charge	Uniformly charged solid sphere
Figure			
Identify the symmetry	Cylindrical	Planar	Spherical
Determine the direction of $\vec{\mathbf{E}}$			
Divide the space into different regions	$r > 0$	$z > 0$ and $z < 0$	$r \le a$ and $r \ge a$
Choose Gaussian surface	 Coaxial cylinder	 Gaussian pillbox	 Concentric sphere
Calculate electric flux	$\Phi_E = E(2\pi r l)$	$\Phi_E = EA + EA = 2EA$	$\Phi_E = E(4\pi r^2)$
Calculate enclosed charge q_{in}	$q_{enc} = \lambda l$	$q_{enc} = \sigma A$	$q_{enc} = \begin{cases} Q(r/a)^3 & r \le a \\ Q & r \ge a \end{cases}$
Apply Gauss's law $\Phi_E = q_{in}/\varepsilon_0$ to find E	$E = \dfrac{\lambda}{2\pi\varepsilon_0 r}$	$E = \dfrac{\sigma}{2\varepsilon_0}$	$E = \begin{cases} \dfrac{Qr}{4\pi\varepsilon_0 a^3}, & r \le a \\ \dfrac{Q}{4\pi\varepsilon_0 r^2}, & r \ge a \end{cases}$

The following steps may be useful when applying Gauss's law:

(1) Identify the symmetry associated with the charge distribution.

(2) Determine the direction of the electric field, and a "Gaussian surface" on which the magnitude of the electric field is constant over portions of the surface.

(3) Divide the space into different regions associated with the charge distribution. For each region, calculate q_{enc}, the charge enclosed by the Gaussian surface.

(4) Calculate the electric flux Φ_E through the Gaussian surface for each region.

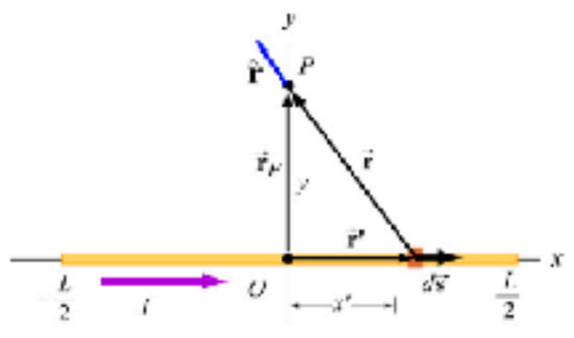
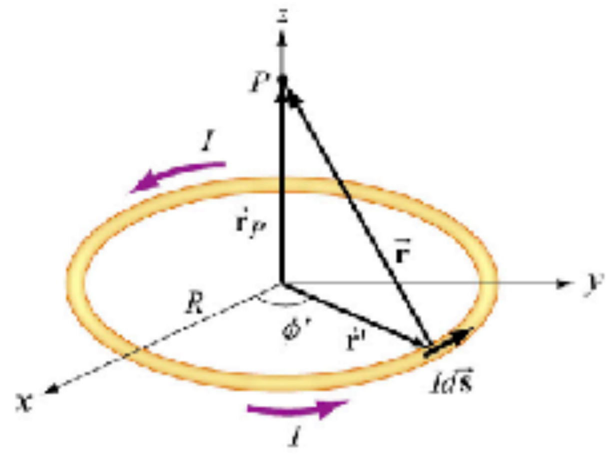
(5) Equate Φ_E with q_{enc}/ε_0, and deduce the magnitude of the electric field.

Don't forget to specify the direction of E

When do we use Biot-Savart or Ampere to calculate B?

- If the problem has symmetries use Ampere, otherwise use Biot-Savart.
- Ampere will always be a quicker to do calculations.
- If you use Ampere don't forget to specify the direction of B.
- TIP: Check if your current distribution is a superposition of two or more distributions with symmetries. Then you can use Ampere and the principle of superposition to calculate B.

Computing the magnetic field Biot-Savart

Current distribution	Finite wire of length L	Circular loop of radius R
Figure		
(1) Source point	$\vec{\mathbf{r}}' = x'\hat{\mathbf{i}}$ $d\vec{\mathbf{s}} = (d\vec{\mathbf{r}}'/dx')dx' = dx'\hat{\mathbf{i}}$	$\vec{\mathbf{r}}' = R(\cos\phi'\hat{\mathbf{i}} + \sin\phi'\hat{\mathbf{j}})$ $d\vec{\mathbf{s}} = (d\vec{\mathbf{r}}'/d\phi')d\phi' = Rd\phi'(-\sin\phi'\hat{\mathbf{i}} + \cos\phi'\hat{\mathbf{j}})$
(2) Field point P	$\vec{\mathbf{r}}_P = y\hat{\mathbf{j}}$	$\vec{\mathbf{r}}_P = z\hat{\mathbf{k}}$
(3) Relative position vector $\vec{\mathbf{r}} = \vec{\mathbf{r}}_P - \vec{\mathbf{r}}'$	$\vec{\mathbf{r}} = y\hat{\mathbf{j}} - x'\hat{\mathbf{i}}$ $r = \lvert\vec{\mathbf{r}}\rvert = \sqrt{x'^2 + y^2}$ $\hat{\mathbf{r}} = \frac{y\hat{\mathbf{j}} - x'\hat{\mathbf{i}}}{\sqrt{x'^2 + y^2}}$	$\vec{\mathbf{r}} = -R\cos\phi'\hat{\mathbf{i}} - R\sin\phi'\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ $r = \lvert\vec{\mathbf{r}}\rvert = \sqrt{R^2 + z^2}$ $\hat{\mathbf{r}} = \frac{-R\cos\phi'\hat{\mathbf{i}} - R\sin\phi'\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{\sqrt{R^2 + z^2}}$
(4) The cross product $d\vec{\mathbf{s}} \times \hat{\mathbf{r}}$	$d\vec{\mathbf{s}} \times \hat{\mathbf{r}} = \frac{y\,dx'\hat{\mathbf{k}}}{\sqrt{y^2 + x'^2}}$	$d\vec{\mathbf{s}} \times \hat{\mathbf{r}} = \frac{R\,d\phi'(z\cos\phi'\hat{\mathbf{i}} + z\sin\phi'\hat{\mathbf{j}} + R\hat{\mathbf{k}})}{\sqrt{R^2 + z^2}}$
(5) Rewrite $d\vec{\mathbf{B}}$	$d\vec{\mathbf{B}} = \frac{\mu_0 I}{4\pi}\frac{y\,dx'\hat{\mathbf{k}}}{(y^2 + x'^2)^{3/2}}$	$d\vec{\mathbf{B}} = \frac{\mu_0 I}{4\pi}\frac{R\,d\phi'(z\cos\phi'\hat{\mathbf{i}} + z\sin\phi'\hat{\mathbf{j}} + R\hat{\mathbf{k}})}{(R^2 + z^2)^{3/2}}$
(6) Integrate to get $\vec{\mathbf{B}}$	$B_x = 0$ $B_y = 0$ $B_z = \frac{\mu_0 I y}{4\pi}\int_{-L/2}^{L/2}\frac{dx'}{(y^2 + x'^2)^{3/2}}$ $= \frac{\mu_0 I}{4\pi}\frac{L}{y\sqrt{y^2 + (L/2)^2}}$	$B_x = \frac{\mu_0 I R z}{4\pi(R^2 + z^2)^{3/2}}\int_0^{2\pi}\cos\phi'\,d\phi' = 0$ $B_y = \frac{\mu_0 I R z}{4\pi(R^2 + z^2)^{3/2}}\int_0^{2\pi}\sin\phi'\,d\phi' = 0$ $B_z = \frac{\mu_0 I R^2}{4\pi(R^2 + z^2)^{3/2}}\int_0^{2\pi}d\phi' = \frac{\mu_0 I R^2}{2(R^2 + z^2)^{3/2}}$

The law states that the magnetic field at a point P due to a length element $d\vec{\mathbf{s}}$ carrying a steady current I located at $\vec{\mathbf{r}}$ away is given by

$$d\vec{\mathbf{B}} = \frac{\mu_0 I}{4\pi}\frac{d\vec{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2} = \frac{\mu_0 I}{4\pi}\frac{d\vec{\mathbf{s}} \times \vec{\mathbf{r}}}{r^3}$$

The calculation of the magnetic field may be carried out as follows:

(1) Source point: Choose an appropriate coordinate system and write down an expression for the differential current element $I\,d\vec{\mathbf{s}}$, and the vector $\vec{\mathbf{r}}'$ describing the position of $I\,d\vec{\mathbf{s}}$. The magnitude $r' = \lvert\vec{\mathbf{r}}'\rvert$ is the distance between $I\,d\vec{\mathbf{s}}$ and the origin. Variables with a "prime" are used for the source point.

(2) Field point: The field point P is the point in space where the magnetic field due to the current distribution is to be calculated. Using the same coordinate system, write down the position vector $\vec{\mathbf{r}}_P$ for the field point P. The quantity $r_P = \lvert\vec{\mathbf{r}}_P\rvert$ is the distance between the origin and P.

(3) Relative position vector: The relative position between the source point and the field point is characterized by the relative position vector $\vec{\mathbf{r}} = \vec{\mathbf{r}}_P - \vec{\mathbf{r}}'$. The corresponding unit vector is

$$\hat{\mathbf{r}} = \frac{\vec{\mathbf{r}}}{r} = \frac{\vec{\mathbf{r}}_P - \vec{\mathbf{r}}'}{\lvert\vec{\mathbf{r}}_P - \vec{\mathbf{r}}'\rvert}$$

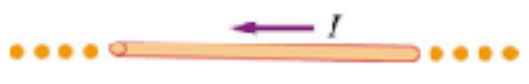
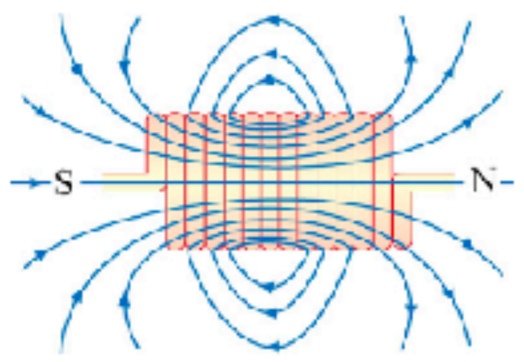
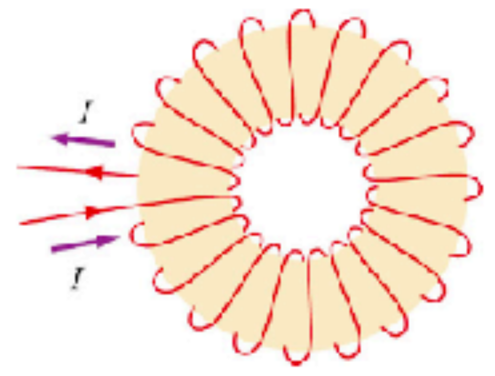
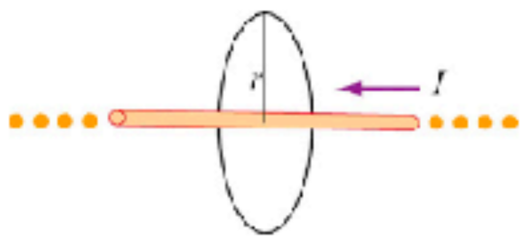
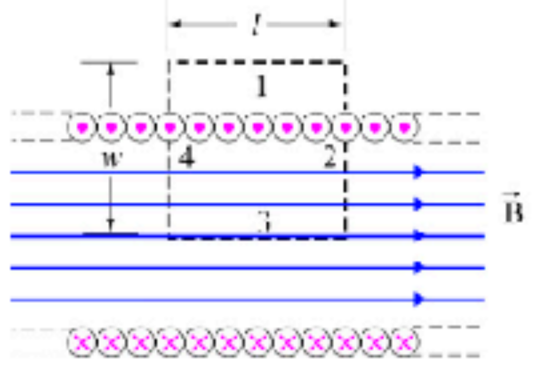
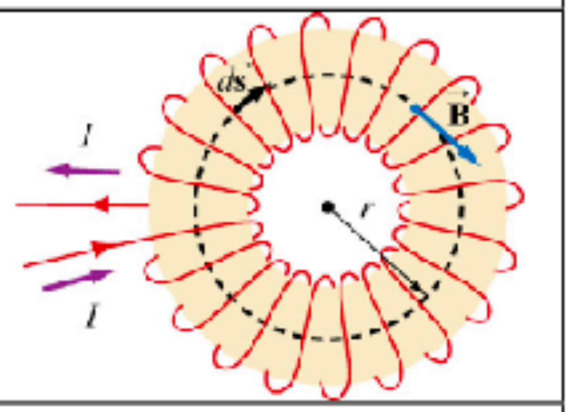
where $r = \lvert\vec{\mathbf{r}}\rvert = \lvert\vec{\mathbf{r}}_P - \vec{\mathbf{r}}'\rvert$ is the distance between the source and the field point P.

(4) Calculate the cross product $d\vec{\mathbf{s}} \times \hat{\mathbf{r}}$ or $d\vec{\mathbf{s}} \times \vec{\mathbf{r}}$. The resultant vector gives the direction of the magnetic field $\vec{\mathbf{B}}$, according to the Biot-Savart law.

(5) Substitute the expressions obtained to $d\vec{\mathbf{B}}$ and simplify as much as possible.

(6) Complete the integration to obtain $\vec{\mathbf{B}}$ if possible. The size or the geometry of the system is reflected in the integration limits. Change of variables sometimes may help to complete the integration.

Computing the magnetic Field Ampere

System	Infinite wire	Ideal solenoid	Toroid
Figure			
(1) Draw the Amperian loop			
(2) Find the current enclosed by the Amperian loop	$I_{enc} = I$	$I_{enc} = NI$	$I_{enc} = NI$
(3) Calculate $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$ along the loop	$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = B(2\pi r)$	$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = Bl$	$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = B(2\pi r)$
(4) Equate $\mu_0 I_{enc}$ with $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$ to obtain $\vec{\mathbf{B}}$	$B = \dfrac{\mu_0 I}{2\pi r}$	$B = \dfrac{\mu_0 NI}{l} = \mu_0 nI$	$B = \dfrac{\mu_0 NI}{2\pi r}$

Ampere's law states that the line integral of $\vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$ around any closed loop is proportional to the total current passing through any surface that is bounded by the closed loop:

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I_{enc}$$

To apply Ampere's law to calculate the magnetic field, we use the following procedure:

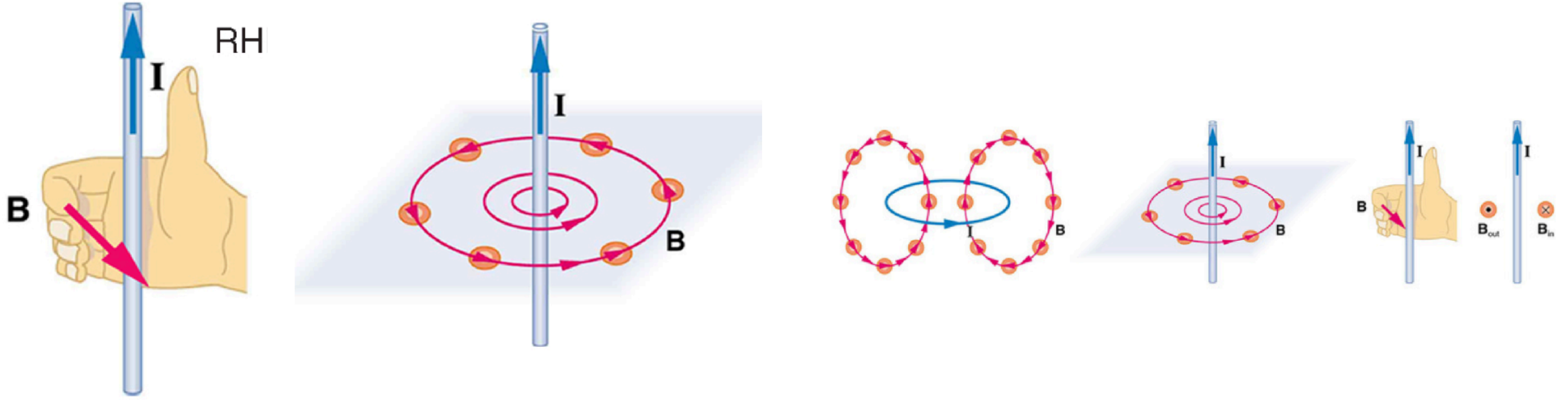
(1) Draw an Amperian loop using symmetry arguments.

(2) Find the current enclosed by the Amperian loop.

(3) Calculate the line integral $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$ around the closed loop.

(4) Equate $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$ with $\mu_0 I_{enc}$ and solve for $\vec{\mathbf{B}}$.

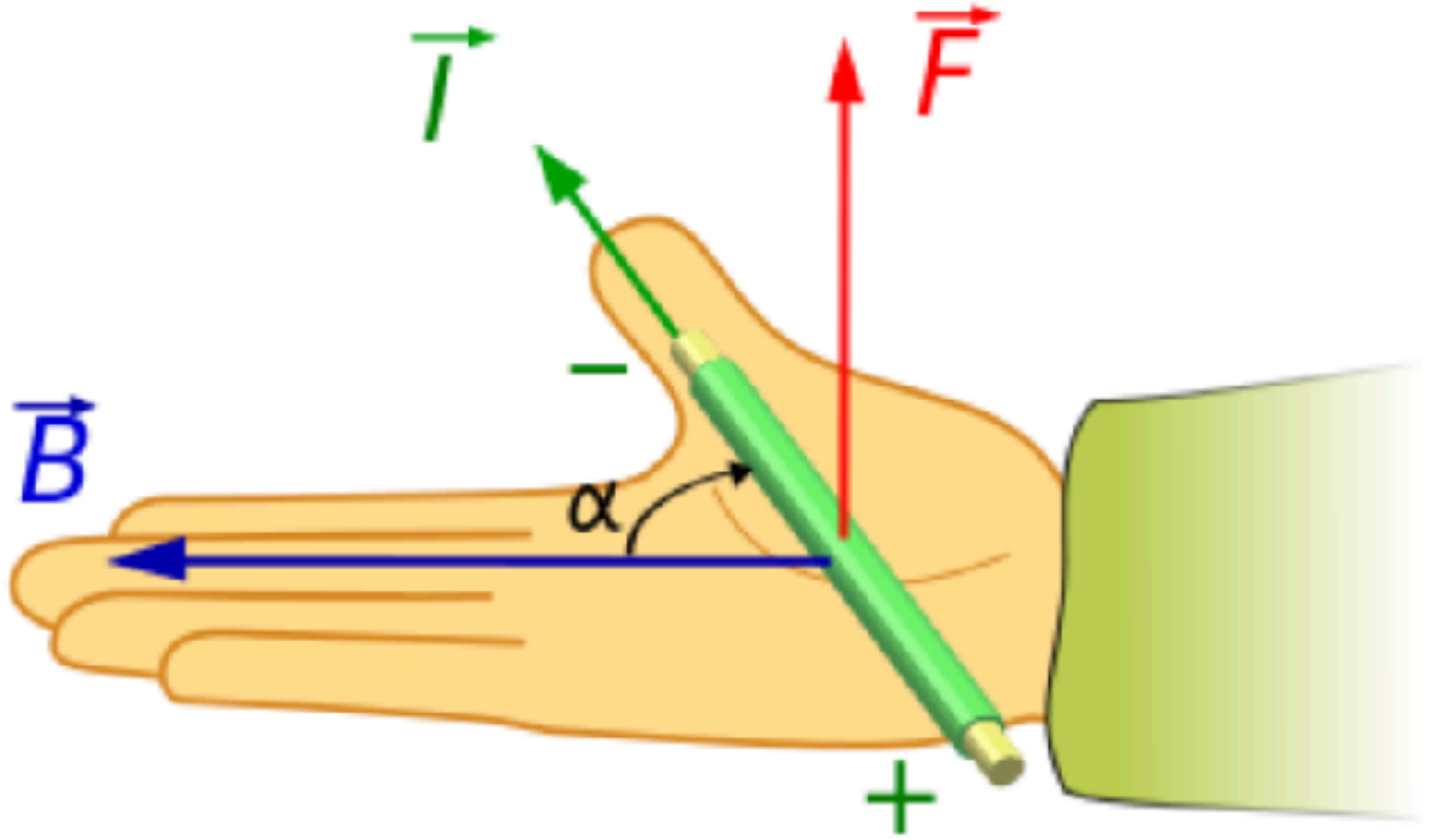
How do you determine the direction of B?



- For a detailed explanation of the direction of the magnetic field of the most common current geometries please read Purcell 6.4,6.5, 6.6 pages 296-306.

- Please read and understand these sections, it will greatly help you during the exam.

How do you determine the direction of the Lorentz force?



Force on a current carrying wire

How do you do calculations in different coordinate systems?

F.1 Vector operators

F.1.1 Cartesian coordinates

$$ds = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}},$$

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z},$$

$$\nabla f = \frac{\partial f}{\partial x}\,\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\,\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\,\hat{\mathbf{z}},$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z},$$

$$\nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{z}},$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}. \qquad \text{(F.1)}$$

F.1.2 Cylindrical coordinates

$$ds = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + dz\,\hat{\mathbf{z}},$$

$$\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\mathbf{z}}\frac{\partial}{\partial z},$$

$$\nabla f = \frac{\partial f}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\hat{\boldsymbol{\theta}} + \frac{\partial f}{\partial z}\,\hat{\mathbf{z}},$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z},$$

$$\nabla \times \mathbf{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{\partial (rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right)\hat{\mathbf{z}},$$

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}. \qquad \text{(F.2)}$$

F.1.3 Spherical coordinates

$$ds = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}},$$

$$\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi},$$

$$\nabla f = \frac{\partial f}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\,\hat{\boldsymbol{\phi}},$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial (r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (A_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi},$$

$$\nabla \times \mathbf{A} = \frac{1}{r\sin\theta}\left(\frac{\partial (A_\phi \sin\theta)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi}\right)\hat{\mathbf{r}} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial (rA_\phi)}{\partial r}\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{\partial (rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right)\hat{\boldsymbol{\phi}},$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}. \qquad \text{(F.3)}$$

F.3 Divergence

The divergence produces a scalar from a vector. The divergence of a vector function was defined in Eq. (2.47) as the net flux out of a given small volume, divided by the volume. In Section 2.10 we derived the form of the divergence in Cartesian coordinates, and it turned out to be the dot product of the ∇ operator with a vector **A**, that is, $\nabla \cdot \mathbf{A}$. We use the same method here to derive the form in cylindrical coordinates. We then give a second, more mechanical, derivation. A third derivation is left for Exercise F.2.

F.3.1 Cylindrical divergence, first method

Consider the small volume that is generated by taking the region in the r-θ plane shown in Fig. F.2 and sweeping it through a span of z values from a particular z up to $z + \Delta z$ (the $\hat{\mathbf{z}}$ axis points out of the page). Let's first look at the flux of a vector field **A** through the two faces perpendicular to the $\hat{\mathbf{z}}$ direction. As in Section 2.10, only the z component of **A** is relevant to the flux through these faces. In the limit of a small volume, the area of these faces is $r\,\Delta r\,\Delta\theta$. The inward flux through the bottom face equals $A_z(z)\,r\,\Delta r\,\Delta\theta$, and the outward flux through the top face equals $A_z(z+\Delta z)\,r\,\Delta r\,\Delta\theta$. We have suppressed the r and θ arguments of A_z for simplicity, and we have chosen points at the midpoints of the faces, as in Fig. 2.22. The net outward flux is therefore

$$\begin{aligned}\Phi_{z\ \text{faces}} &= A_z(z+\Delta z)\,r\,\Delta r\,\Delta\theta - A_z(z)\,r\,\Delta r\,\Delta\theta \\ &= \left(\frac{A_z(z+\Delta z) - A_z(z)}{\Delta z}\right) r\,\Delta r\,\Delta\theta\,\Delta z \\ &= \frac{\partial A_z}{\partial z}\,r\,\Delta r\,\Delta\theta\,\Delta z. \qquad \text{(F.8)}\end{aligned}$$

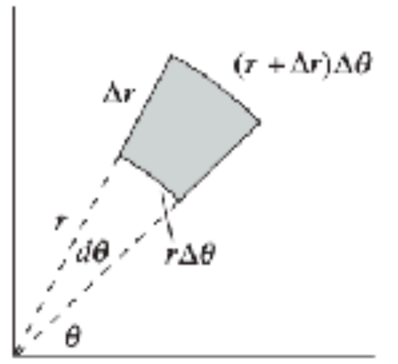
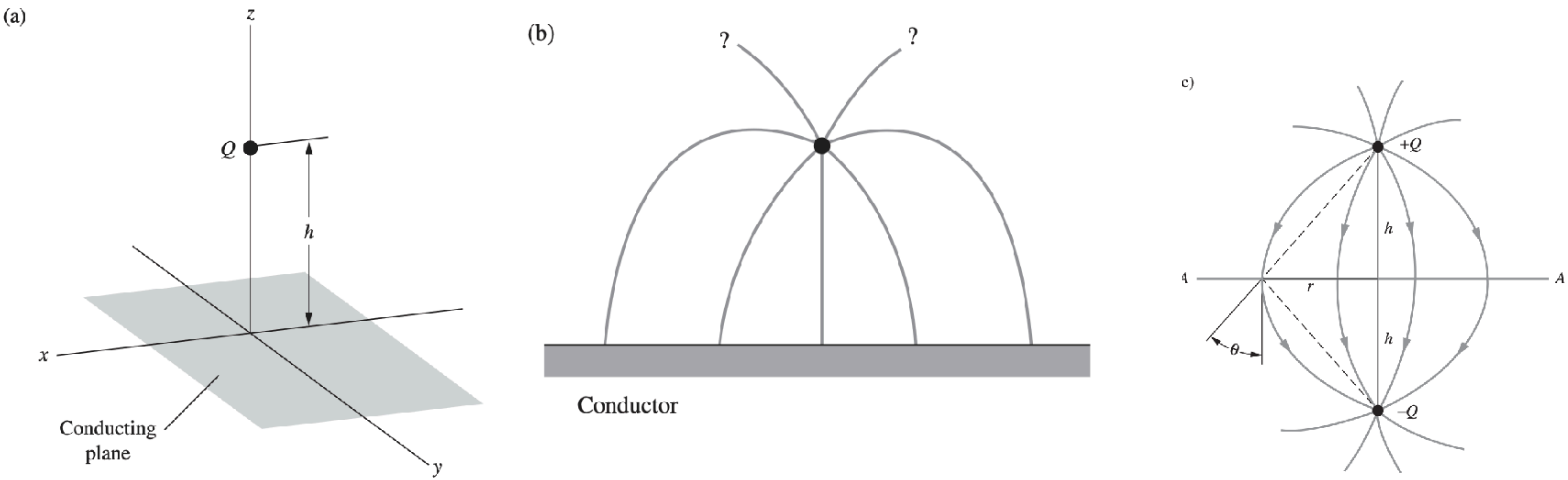


Figure F.2.
A small region in the r-θ plane.

- Check appendix F of Purcell. It has all the formulas for operations with the del operator in different coordinate systems.
- This appendix also has some explanations so you get some intuition on what these operations are geometrically and physically. An it includes the derivations of these equations if you want to understand them.

Review of questions posted on ED

How do we use the method of images?

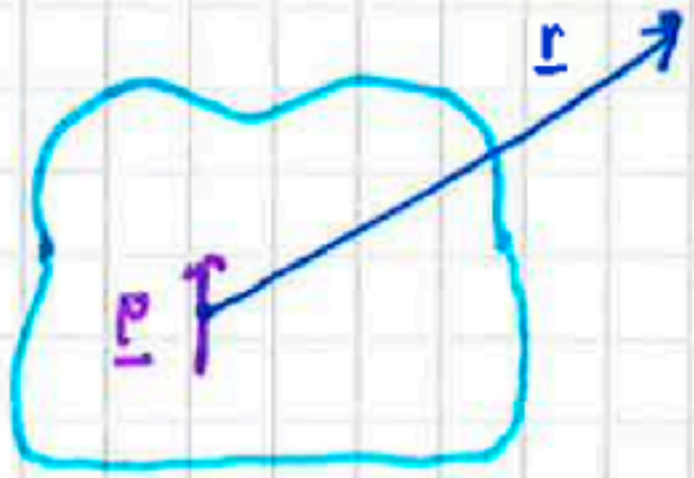


- Basics: Find an image charge located outside of the row such that the BC for the potential remain the same

- This is explained in detail section 3.4 Purcell (page136)

What are bound charges?

Potential due to a polarized object



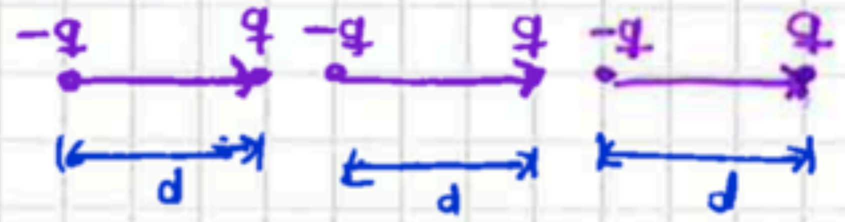
We have a piece of polarized material. What is the potential generated by it?

We know it's full of little dipoles so we'll consider the potential of single dipole & then add them up:

Physical interpretation of bound charge

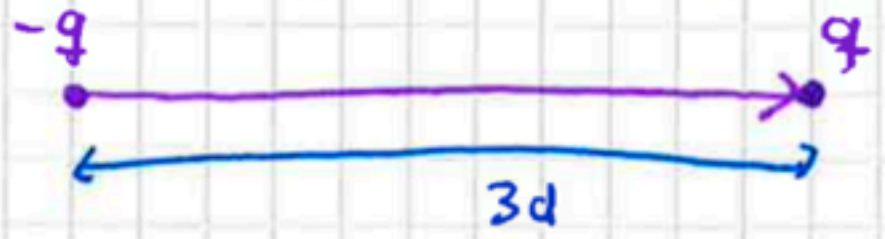
bound charges are not mathematical artifacts, they actually ∃.

Consider the following example:

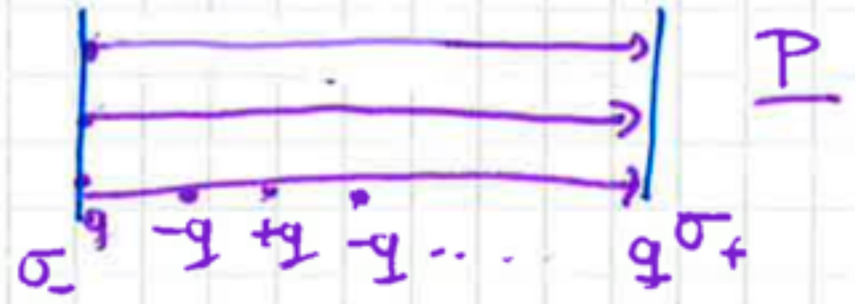


we have 3 dipoles

if they are aligned like in the drawing (lengthwise) the charges that are not at the edges cancel each other so we get a system that looks like a single dipole with dipole moment 3dq:



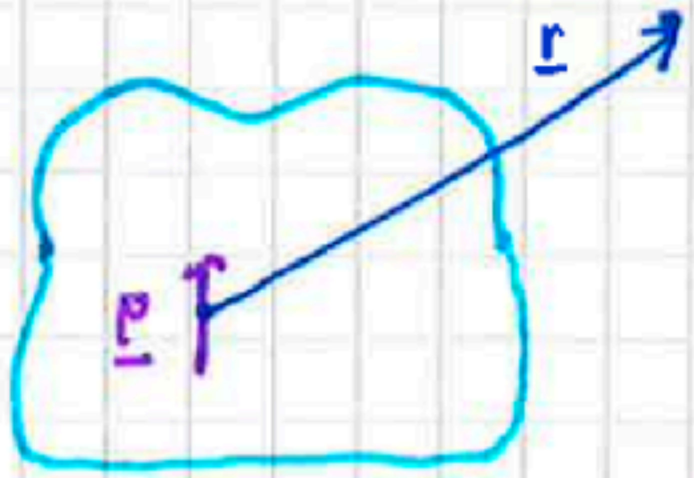
Intuitively if we have a piece of material & a uniform polarization then:



we get charge accumulation at the ends, which corresponds to bound charge.

What are bound charges?

Potential due to a polarized object



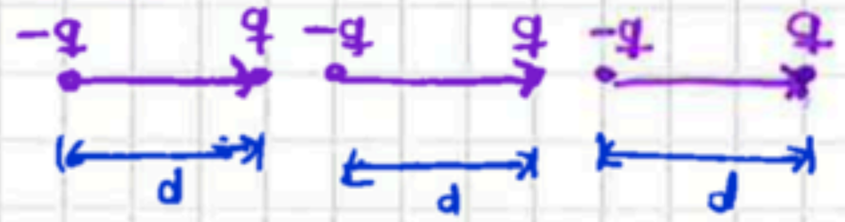
We have a piece of polarized material. What is the potential generated by it?

We know it's full of little dipoles so we'll consider the potential of single dipole & then add them up:

Physical interpretation of bound charge

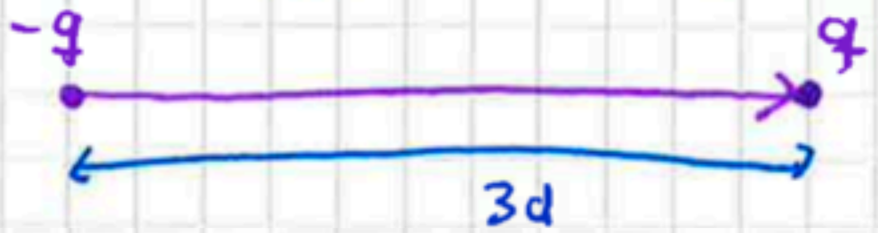
bound charges are not mathematical artifacts, they actually ∃.

Consider the following example:

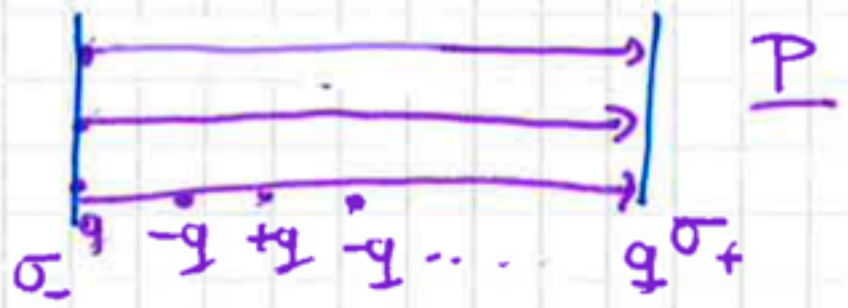


we have 3 dipoles

if they are aligned like in the drawing (lengthwise) the charges that are not at the edges cancel each other so we get a system that looks like a single dipole with dipole moment 3dq:



Intuitively if we have a piece of material & a uniform polarization then:



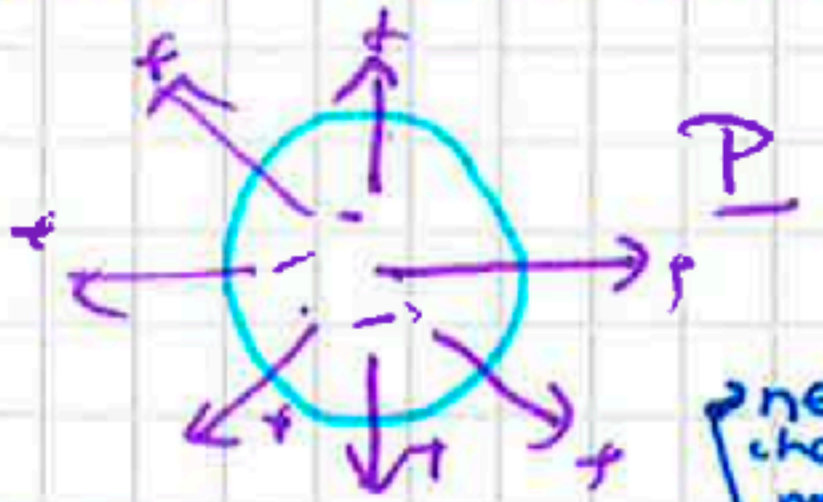
we get charge accumulation at the ends, which corresponds to bound charge

More generally if we have an oblique cut:

$$\sigma_b = \frac{q}{A_{end}} = P\cos\theta = \mathbf{P}\cdot\hat{n}$$

What are bound charges?

Now what happens in the case where the polarization is non-uniform? From our expression for the potential we can see that volume bound charge comes from a divergence → non uniform P:



net charge on material must be zero.

non uniform P leads to an accumulation of bound charge within the material, as well as the surface.

$$\int_V \rho_b \, dV = -\oint_S \underline{P} \cdot d\underline{a} = -\int_V (\nabla \cdot \underline{P}) \, dV$$

Flux of P

The net bound charge on a volume is equal to the amount that has been 'pushed' through a surface (remember flux idea?) but has opposite sign.

since this is true for any volume:

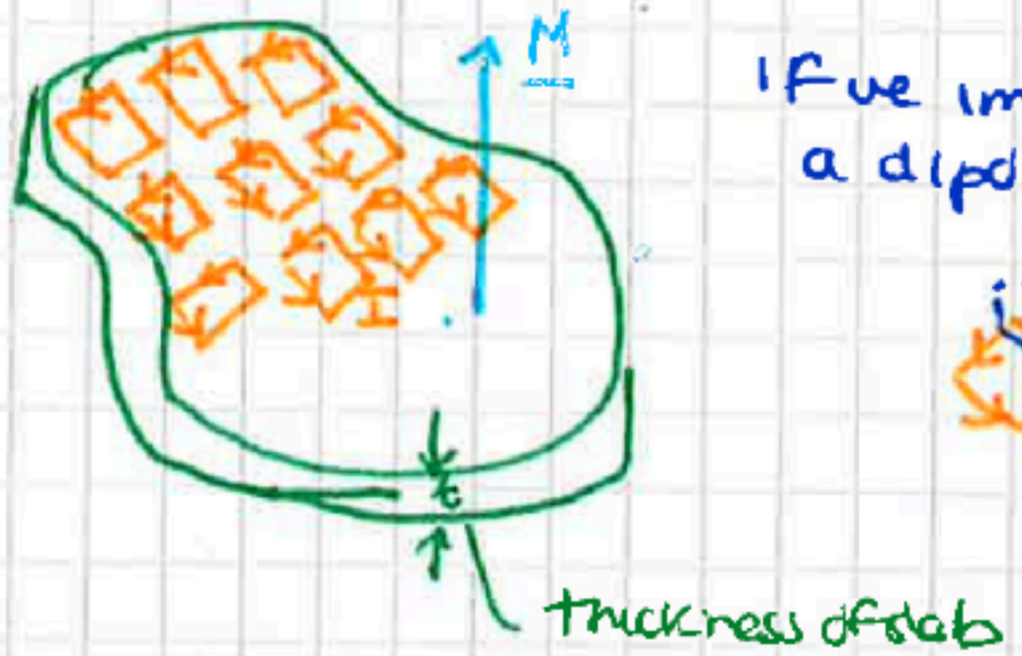
$$\rho_b = -\nabla \cdot \underline{P}$$

so we have the same results as we derived rigorously.

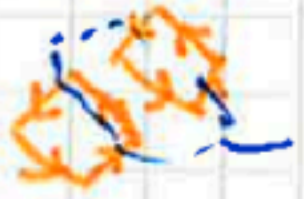
$$\int_A \sigma_b \, da + \int_V \rho_b \, dV = 0$$

What are bound currents?

Let's consider a slab of uniformly magnetized material:

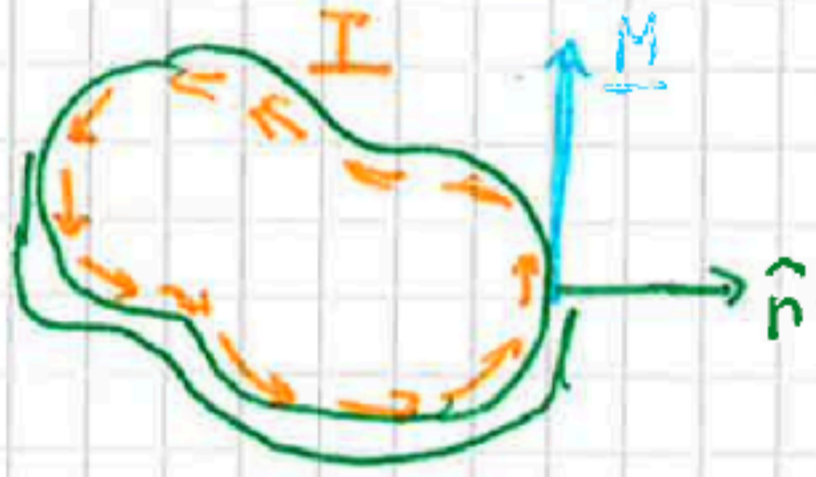


If we imagine this material full of tiny loops that each produce a dipole moment:



current of adjacent loops will cancel, but the boundary won't cancel

so we will get some effective surface current that looks like:



What is the current I in terms of M ?

Let's assume that each one of the tiny loops has an area a and thickness t.

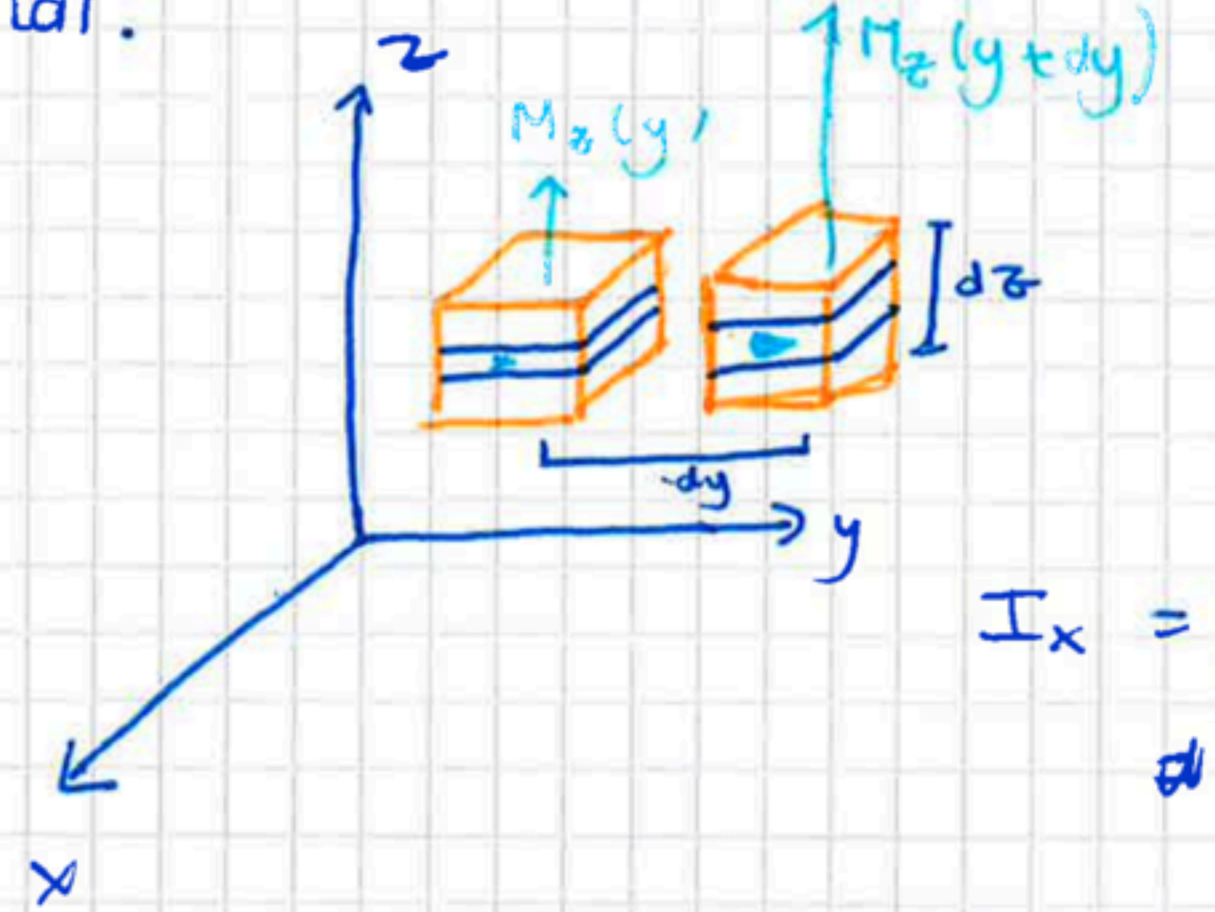
$\underline{K}_b = \underline{M} \times \hat{n}$

this expression also implies that there is no current on the top or bottom of the slab because there $\underline{M} \parallel \hat{n}$ so the cross product vanishes.

What are bound currents?

Now what happens if the magnetization is non-uniform?

In this case the internal currents of adjacent loops will no longer cancel, so there will be a net current between adjacent chunks of magnetized material.



If we have a larger magnetization at y + dy compared to y we will get a net current in the x direction where the 2 slabs join given by:

$$I_x = \underbrace{[M_z(y+dy) - M_z(y)]}_{\text{this is just the difference in magnetization at the points } y \text{ and } dy} dz$$

$$= \frac{\partial M_z}{\partial y}\, dz\, dy$$

$$\Rightarrow \underbrace{\frac{I_x}{dz\,dy}}_{\text{Volume current} \equiv J_b} = \frac{\partial M_z}{\partial y}$$

$$\Rightarrow (J_b)_x = \frac{\partial M_z}{\partial y}$$

Difference between multipole expansion and induced dipoles

1. Multipole expansion: how a charge distribution produces fields

What it is:

Multipole expansion is a **mathematical description** of the electric potential (or field) generated by a *fixed* charge distribution, especially when viewed from far away.

Core idea:

Any localized charge distribution can be represented as a sum of increasingly detailed moments:

$$V(\mathbf{r}) = V_{\text{monopole}} + V_{\text{dipole}} + V_{\text{quadrupole}} + V_{\text{octupole}} + \cdots$$

Terms mean:

- **Monopole:** total charge
- **Dipole:** separation of positive and negative charge
- **Quadrupole:** shape of charge distribution
- Higher moments: finer spatial detail

Key features:

- Describes the **source** of the electric field
- Assumes the charge distribution is **static**
- Accuracy improves by including higher-order moments
- Valid when observation distance $\gg$ size of charge distribution

Difference between multipole expansion and induced dipoles

2. Induced dipoles: how matter responds to fields

What it is:

An induced dipole arises when an **external electric field distorts** the electron cloud of an atom or molecule.

Core idea:

The field shifts positive and negative charges slightly apart:

$$\mathbf{p}_{\text{ind}} = \alpha \mathbf{E}$$

where:

- $\mathbf{p}_{\text{ind}}$ = induced dipole moment
- α = polarizability
- $\mathbf{E}$ = local electric field

Key features:

- Describes the **response** of matter
- Dipole exists **only while the field is present**
- Depends on material properties (polarizability)
- Can lead to macroscopic polarization in dielectrics

Example:

A neutral atom placed near a charged object develops an induced dipole, causing attraction.

The displacement field D to understand polarization

1. Why the displacement field is needed

In matter, an applied electric field causes **bound charges** (from polarization) to appear inside materials and on their surfaces. These bound charges make Gauss's law for the electric field messy:

$$\nabla \cdot \mathbf{E} = \frac{\rho_{\text{free}} + \rho_{\text{bound}}}{\varepsilon_0}$$

The displacement field is defined so that **only free charge appears** in Gauss's law.

---

2. Definition of the displacement field

$$\boxed{\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}}$$

where:

- $\mathbf{E}$ = electric field
- $\mathbf{P}$ = polarization (dipole moment per unit volume)
- $\varepsilon_0 \mathbf{E}$ = vacuum response
- $\mathbf{P}$ = material response

The displacement field D to understand polarization

3. Gauss's law in terms of $\mathbf{D}$

With this definition:

$$\boxed{\nabla \cdot \mathbf{D} = \rho_{\text{free}}}$$

or in integral form:

$$\oint \mathbf{D} \cdot d\mathbf{A} = Q_{\text{free, enclosed}}$$

Key idea:

$\mathbf{D}$ "absorbs" the bound charges into $\mathbf{P}$, leaving only free charges as sources.

---

4. Physical interpretation

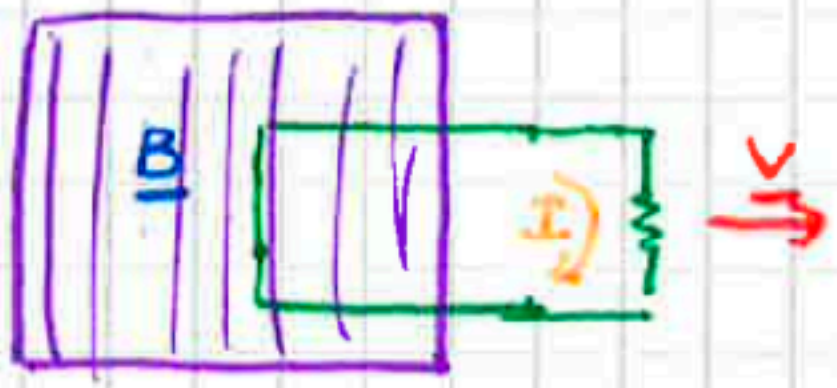
- $\mathbf{E}$: total electric field that actually acts on charges
- $\mathbf{P}$: how much the material is polarized
- $\mathbf{D}$: a **bookkeeping field** that tracks how free charges create fields *through* matter

Faraday's law

Faraday's law:

Recall Faraday's experiments (simplified). We have a magnet & a loop.

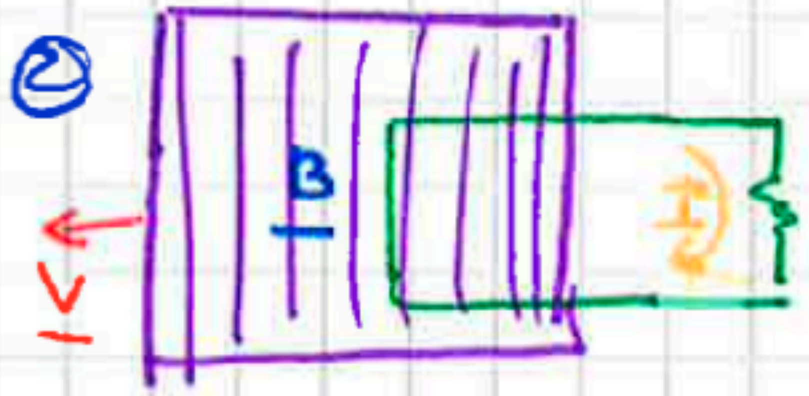
①



Pull loop to the right through $\underline{B}$ → induced current

Experiment 1: Motional emf drives the current. $\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int \underline{B}\cdot d\underline{a} = -\int \frac{\partial \underline{B}}{\partial t}\cdot d\underline{a}$ ⊛

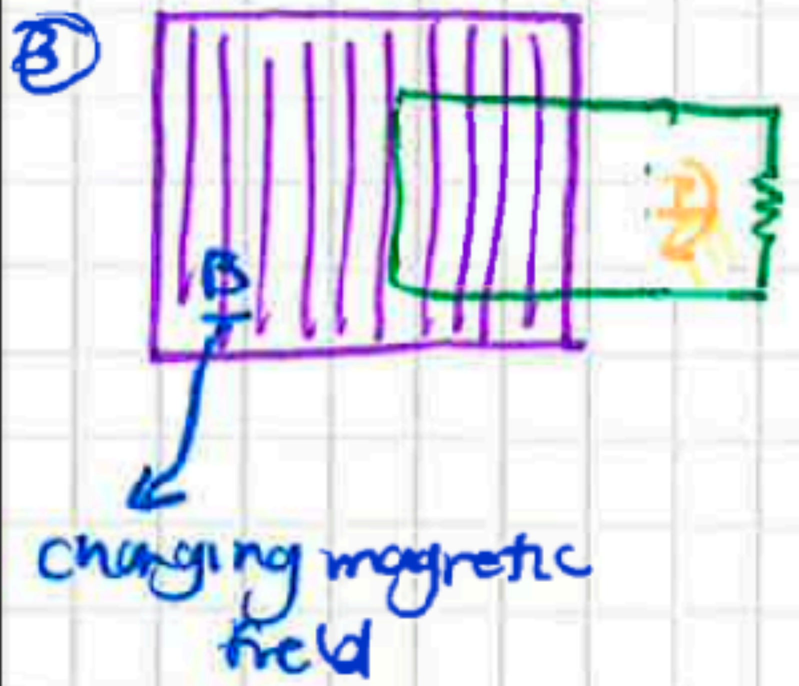
②



Pull magnet to the left. Don't move loop → induced current

Experiment 2: The same emf as in experiment 1 arises, since it's the "relative" motion of the magnet & the wire that matters. This makes sense in the light of special relativity. But Faraday knew nothing about it so this reciprocity of this point was a coincidence.

③



Don't move anything but have $\underline{B}$ that changes in time → induced current.

Motional emf drives the current. $\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int \underline{B}\cdot d\underline{a} = -\int \frac{\partial \underline{B}}{\partial t}\cdot d\underline{a}$ ⊛

A changing magnetic field induces an electric field

Faraday's law

A changing magnetic field induces an electric field

Equating the expressions for $\mathcal{E}$ (*) and (**):

$$\oint \underline{E} \cdot d\underline{l} = -\int \frac{\partial \underline{B}}{\partial t} \cdot d\underline{a}$$ Faraday's law in integral form

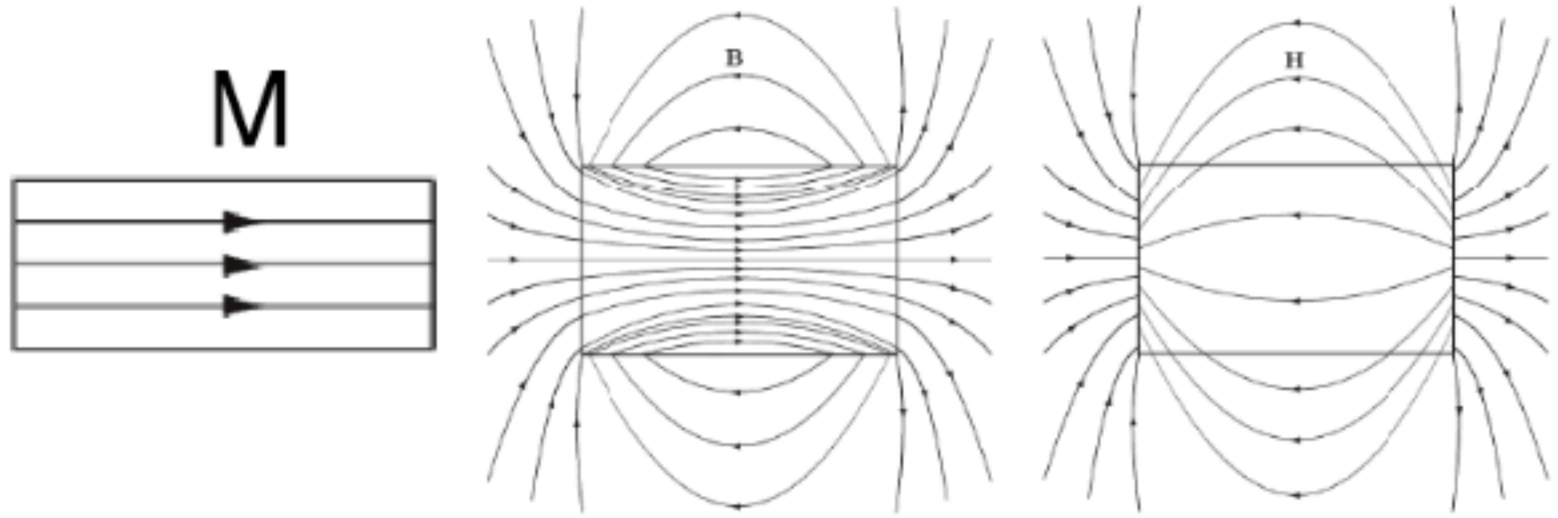
Which we can convert into differential form by applying Stokes's theorem:

$$\nabla \times \underline{E} = \frac{-\partial \underline{B}}{\partial t}$$ Faraday's law

Magnetic field lines

Problem 1. For a bar magnet (a short circular cylinder of radius a and length L with a "frozen-in" uniform magnetization M parallel to its axis), make careful sketches of $\mathbf{M}, \mathbf{B}$, and $\mathbf{H}$, assuming L is about $2a$. Compare with your results in Problem 3 from Exercise sheet 7.

Solution: $\mathbf{B}$ is the same as the field of a short solenoid; $\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}$. □



Bonus slides from last year’s student questions

How do we solve circuits?

Direct current circuits

- **Kirchhoff's rules**:

  (1) The sum of the currents flowing into a junction is equal to the sum of the currents flowing out of the junction:

  $$\sum I_{\text{in}} = \sum I_{\text{out}}$$

  (2) The algebraic sum of the changes in electric potential in a closed-circuit loop is zero.

  $$\sum_{\text{closed loop}} \Delta V = 0$$

How do we solve circuits?
Direct current circuits

(1) Draw a circuit diagram, and label all the quantities, both known and unknown. The number of unknown quantities is equal to the number of linearly independent equations we must look for.

(2) Assign a direction to the current in each branch of the circuit. (If the actual direction is opposite to what you have assumed, your result at the end will be a negative number.)

(3) Apply the junction rule to all but one of the junctions. (Applying the junction rule to the last junction will not yield any independent relationship among the currents.)

(4) Apply the loop rule to the loops until the number of independent equations obtained is the same as the number of unknowns. For example, if there are three unknowns, then we must write down three linearly independent equations in order to have a unique solution.

How do we solve circuits?

Direct current circuits

Traverse the loops using the convention below for ΔV :

resistor	travel direction; I; higher V, lower V; a, b; $\Delta V = V_b - V_a = -IR$	travel direction; I; lower V, higher V; a, b; $\Delta V = V_b - V_a = +IR$
emf source	travel direction; ε; lower V, higher V; $-$ $+$; a, b; $\Delta V = V_b - V_a = +\varepsilon$	travel direction; ε; higher V, lower V; $+$ $-$; a, b; $\Delta V = V_b - V_a = -\varepsilon$
capacitor	travel direction; lower V, higher V; $-q$ $+q$; a, b; $\Delta V = V_b - V_a = +q/C$	travel direction; higher V, lower V; $+q$ $-q$; a, b; $\Delta V = V_b - V_a = -q/C$

The same equation is obtained whether the closed loop is traversed clockwise or counterclockwise. (The expressions actually differ by an overall negative sign. However, using the loop rule, we are led to $0 = -0$, and hence the same equation.)

(5) Solve the simultaneous equations to obtain the solutions for the unknowns.

How do we solve circuits?

Alternating current circuits/ Review on phasors

1. Keep in mind the phase relationships for simple circuits

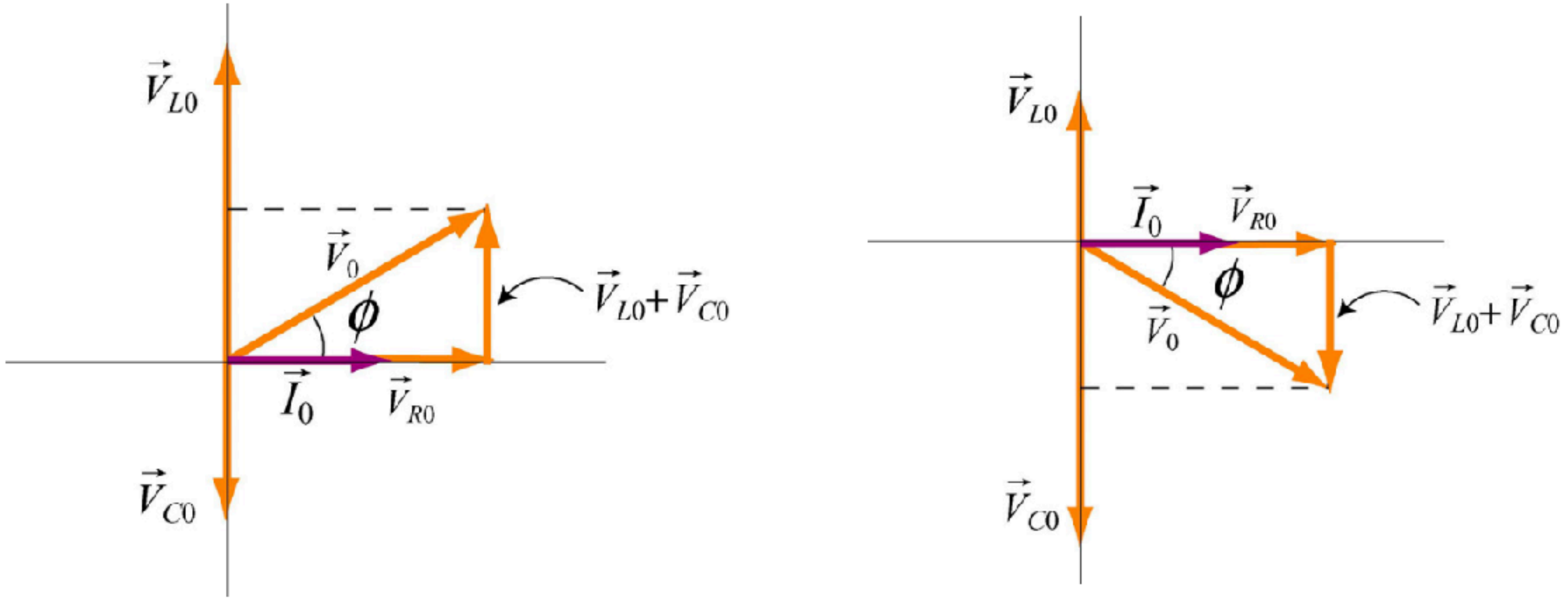
   (1) For a resistor, the voltage and the phase are always in phase.
   (2) For an inductor, the current lags the voltage by $90°$.
   (3) For a capacitor, the current leads to voltage by $90°$.

2. When circuit elements are connected in *series*, the instantaneous current is the same for all elements, and the instantaneous voltages across the elements are out of phase. On the other hand, when circuit elements are connected in *parallel*, the instantaneous voltage is the same for all elements, and the instantaneous currents across the elements are out of phase.

3. For series connection, draw a phasor diagram for the voltages. The amplitudes of the voltage drop across all the circuit elements involved should be represented with phasors. In Figure 12.8.1 the phasor diagram for a series *RLC* circuit is shown for both the inductive case $X_L > X_C$ and the capacitive case $X_L < X_C$.

How do we solve circuits?

Alternating current circuits



A phasor is a rotating vector w/ the following properties:
(i) length → amplitude
(ii) angular speed ω → freq of current (angular)
(iii) Projection along vertical is the value of alternating quantity at t (5)

Figure 12.8.1 Phasor diagram for the series *RLC* circuit for (a) $X_L > X_C$ and (b) $X_L < X_C$.

From Figure 12.8.1(a), we see that $V_{L0} > V_{C0}$ in the inductive case and $\vec{V}_0$ leads $\vec{I}_0$ by a phase ϕ. On the other hand, in the capacitive case shown in Figure 12.8.1(b), $V_{C0} > V_{L0}$ and $\vec{I}_0$ leads $\vec{V}_0$ by a phase ϕ.

How do we solve circuits?

Alternating current circuits

5. For parallel connection, draw a phasor diagram for the currents. The amplitudes of the currents across all the circuit elements involved should be represented with phasors. In Figure 12.8.2 the phasor diagram for a parallel *RLC* circuit is shown for both the inductive case $X_L > X_C$ and the capacitive case $X_L < X_C$.

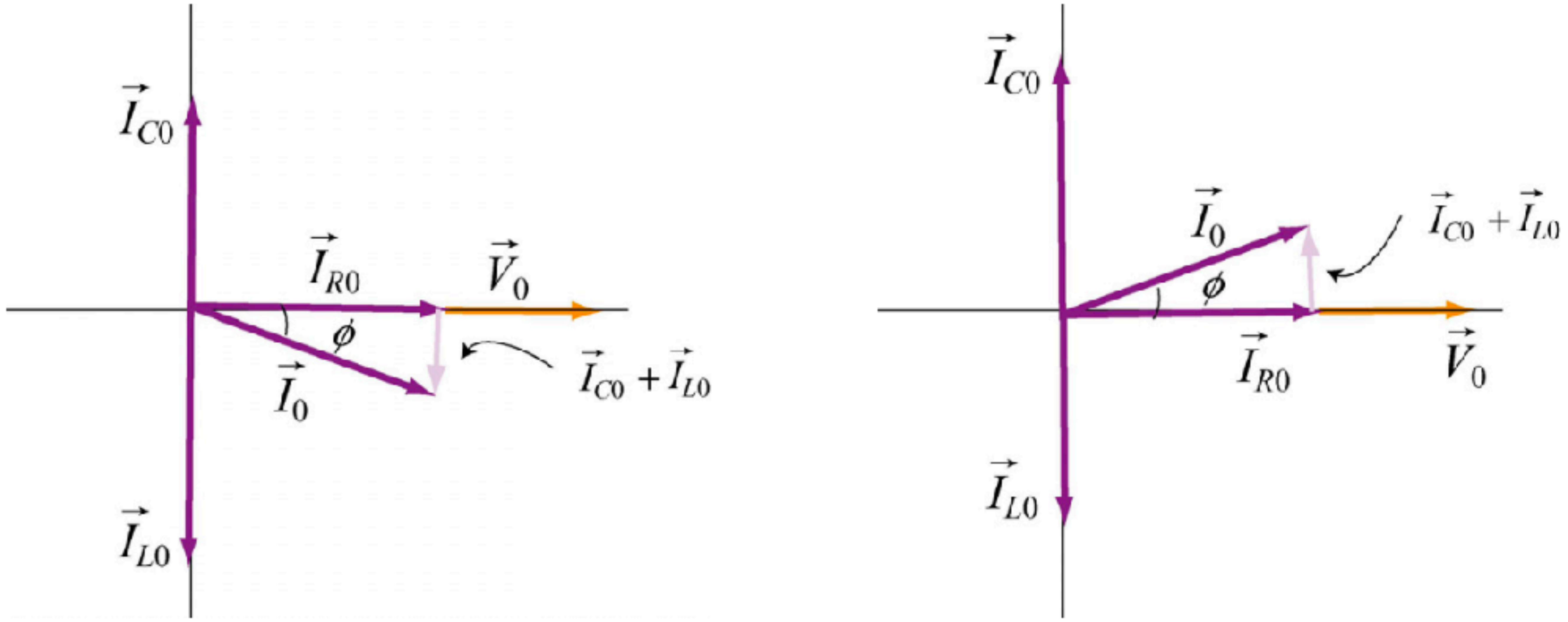


Figure 12.8.2 Phasor diagram for the parallel *RLC* circuit for (a) $X_L > X_C$ and (b) $X_L < X_C$.

From Figure 12.8.2(a), we see that $I_{L0} > I_{C0}$ in the inductive case and $\vec{V}_0$ leads $\vec{I}_0$ by a phase ϕ. On the other hand, in the capacitive case shown in Figure 12.8.2(b), $I_{C0} > I_{L0}$ and $\vec{I}_0$ leads $\vec{V}_0$ by a phase ϕ.

Electromagnetic waves why use sine or cosine?

The choice between using **sine** or **cosine** for representing electromagnetic waves depends on the **initial conditions**, **convention**, and **phase of the wave**. Both are mathematically equivalent, as one can always be expressed in terms of the other with a phase shift. Here's a breakdown of when to use each:

1. Initial Conditions of the Wave

- If the wave starts at a **maximum or minimum amplitude** (e.g., at $t = 0$ or $x = 0$), the **cosine form** is typically used:

$$\psi(x,t) = A\cos(kx - \omega t)$$

- If the wave starts at **zero amplitude** and is increasing or decreasing at $t = 0$, the **sine form** is used:

$$\psi(x,t) = A\sin(kx - \omega t)$$

This choice aligns with the fact that cosine starts at 1 (maximum amplitude at $\theta = 0$), while sine starts at 0.

2. Phase of the Wave

- If the wave has a **phase shift**, you can use either sine or cosine, but include the appropriate phase factor ϕ:
  - Cosine form with phase:
  $$\psi(x,t) = A\cos(kx - \omega t + \phi)$$
  - Sine form with phase:
  $$\psi(x,t) = A\sin(kx - \omega t + \phi)$$

The phase ϕ determines the starting point of the wave. For example: - $\phi = 0$: A cosine wave starts at maximum. - $\phi = \pi/2$: A sine wave and a cosine wave are equivalent.

3. Physical or Practical Context

- **Cosine** is often the default in textbooks and analysis because it is mathematically convenient to start at a maximum amplitude when describing standing waves or harmonic oscillators.
- **Sine** may be more convenient in contexts where initial conditions or symmetry suggest a starting point of zero amplitude (e.g., Fourier series often uses sine terms for odd symmetry).

If you need help getting started with your formula sheet check out appendix D.4 of Purcell

D.4 The formulas

Chapter 1

Coulomb's law (1.4):	$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2 \hat{\mathbf{r}}}{r^2}$	$\mathbf{F} = \frac{q_1 q_2 \hat{\mathbf{r}}}{r^2}$
potential energy (1.9):	$U = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r}$	$U = \frac{q_1 q_2}{r}$
electric field (1.20):	$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q\hat{\mathbf{r}}}{r^2}$	$\mathbf{E} = \frac{q\hat{\mathbf{r}}}{r^2}$
force and field (1.21):	$\mathbf{F} = q\mathbf{E}$	(same)
flux (1.26):	$\Phi = \int \mathbf{E} \cdot d\mathbf{a}$	(same)
Gauss's law (1.31):	$\int \mathbf{E} \cdot d\mathbf{a} = \frac{q}{\epsilon_0}$	$\int \mathbf{E} \cdot d\mathbf{a} = 4\pi q$
field due to line (1.39):	$E_r = \frac{\lambda}{2\pi\epsilon_0 r}$	$E_r = \frac{2\lambda}{r}$
field due to sheet (1.40):	$E = \frac{\sigma}{2\epsilon_0}$	$E = 2\pi\sigma$
$\Delta\mathbf{E}$ across sheet (1.41):	$\Delta\mathbf{E} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}$	$\Delta\mathbf{E} = 4\pi\sigma\hat{\mathbf{n}}$
field near shell (1.42):	$E_r = \frac{\sigma}{\epsilon_0}$	$E_r = 4\pi\sigma$
$F/$(area) on sheet (1.49):	$\frac{F}{A} = \frac{1}{2}(E_1 + E_2)\sigma$	(same)
energy in E field (1.53):	$U = \frac{\epsilon_0}{2}\int E^2\, dv$	$U = \frac{1}{8\pi}\int E^2\, dv$

Chapter 11

dipole moment (11.9):	$\mathbf{m} = I\mathbf{a}$	$\mathbf{m} = \frac{I\mathbf{a}}{c}$
vector potential (11.10):	$\mathbf{A} = \frac{\mu_0}{4\pi}\frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2}$	$\mathbf{A} = \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2}$
dipole (B_r, B_θ) (11.15):	$\frac{\mu_0 m}{4\pi r^3}(2\cos\theta, \sin\theta)$	$\frac{m}{r^3}(2\cos\theta, \sin\theta)$
force on dipole (11.23):	$\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})$	(same)
orbital $\mathbf{m}$ for e (11.29):	$\mathbf{m} = \frac{-e}{2m_e}\mathbf{L}$	$\mathbf{m} = \frac{-e}{2m_e c}\mathbf{L}$
polarizability (11.41):	$\frac{\Delta m}{B} = -\frac{e^2 r^2}{4m_e}$	$\frac{\Delta m}{B} = -\frac{e^2 r^2}{4m_e c^2}$
torque on dipole (11.47):	$\mathbf{N} = \mathbf{m} \times \mathbf{B}$	(same)
polarization density (11.51):	$\mathbf{M} = \frac{\mathbf{m}}{\text{volume}}$	(same)
susceptibility χ_m (11.52):	$\mathbf{M} = \chi_m \frac{\mathbf{B}}{\mu_0}$	$\mathbf{M} = \chi_m \mathbf{B}$
χ_{pm} for weak B (11.53):	$\chi_{pm} \approx \frac{\mu_0 N m^2}{kT}$	$\chi_{pm} \approx \frac{N m^2}{kT}$
surface density $\mathcal{J}$ (11.55):	$\mathcal{J} = M$	$\mathcal{J} = Mc$
volume density $\mathbf{J}$ (11.56):	$\mathbf{J} = \text{curl}\,\mathbf{M}$	$\mathbf{J} = c\,\text{curl}\,\mathbf{M}$
$\mathbf{H}$ field (11.68):	$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}$	$\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}$
curl of $\mathbf{H}$ (11.69):	$\text{curl}\,\mathbf{H} = \mathbf{J}_{\text{free}}$	$\text{curl}\,\mathbf{H} = \frac{4\pi}{c}\mathbf{J}_{\text{free}}$
(integrated form) (11.70):	$\int \mathbf{H} \cdot d\mathbf{l} = I_{\text{free}}$	$\int \mathbf{H} \cdot d\mathbf{l} = \frac{4\pi}{c}I_{\text{free}}$
χ_m (accepted def.) (11.72):	$\mathbf{M} = \chi_m \mathbf{H}$	(same)
permeability (11.74):	$\mu = \mu_0(1 + \chi_m)$	$\mu = 1 + 4\pi\chi_m$
$\mathbf{B}$ and $\mathbf{H}$ (11.74):	$\mathbf{B} = \mu\mathbf{H}$	(same)

Don't forget to include your fundamental constants in your formula sheet

Fundamental constants

speed of light	c	$2.998 \cdot 10^{8}$ m/s
elementary charge	e	$1.602 \cdot 10^{-19}$ C
		$4.803 \cdot 10^{-10}$ esu
electron mass	m_{e}	$9.109 \cdot 10^{-31}$ kg
proton mass	m_{p}	$1.673 \cdot 10^{-27}$ kg
Avogadro's number	N_{A}	$6.022 \cdot 10^{-23}$ mole^{-1}
Boltzmann constant	k	$1.381 \cdot 10^{-23}$ J/K
Planck constant	h	$6.626 \cdot 10^{-34}$ J s
gravitational constant	G	$6.674 \cdot 10^{-11}$ m^3/(kg s^2)
electron magnetic moment	μ_{e}	$9.285 \cdot 10^{-24}$ J/T
proton magnetic moment	μ_{p}	$1.411 \cdot 10^{-26}$ J/T
permittivity of free space	ϵ_0	$8.854 \cdot 10^{-12}$ C^2 s^2/(kg m^3)
permeability of free space	μ_0	$1.257 \cdot 10^{-6}$ kg m/C^2

- To get started you can look at the list on Purcell page 825 appendix K.1

Problems

Biot-Savart

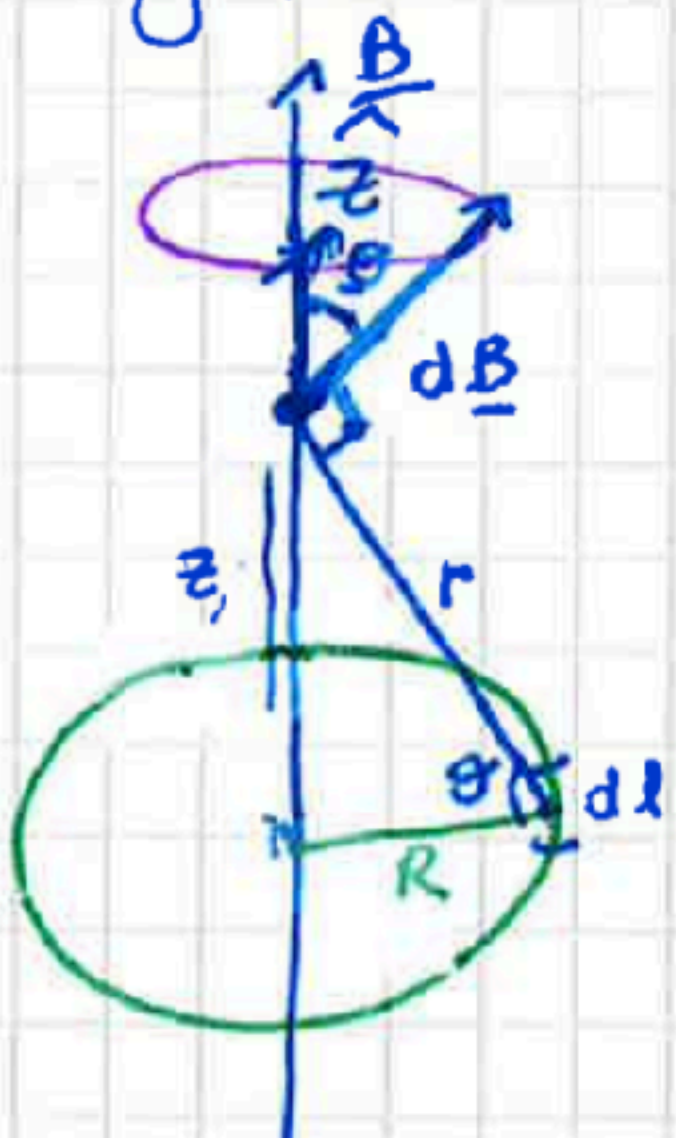
EXAMPLE: Find the magnetic field a distance z above the center of a circular loop of radius R, which carries a steady current I.

steady current → we can use Biot-Savart's law

$$B(\underline{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\underline{l} \times \hat{\underline{r}}}{r^2}$$

(here Don't be confused by the change in notation, here $d\underline{l} = ds$ from before)

Always make a drawing first:



Observations:

* $\underline{B}$ should point in the direction $d\underline{l} \times \hat{\underline{r}}$, i.e. it will be ⊥ to the plane these vectors make.
* As we integrate around the circle, the vector $d\underline{B}$ will "sweep out" a cone.

* Because of symmetry the horizontal components of $d\underline{B}$ cancel, leaving only the projection with the z axis given by $\cos\theta$.

Taking these observations into account we can see:

$$B(z) = \frac{\mu_0}{4\pi} I \int \frac{d\underline{l} \times \hat{r}}{r^2} = \frac{\mu_0}{4\pi} I \int \frac{dl}{r^2} \cos\theta = \frac{\mu_0}{4\pi} I \frac{\cos\theta}{r^2} \int_0^{2\pi} dl$$

$$= \frac{\mu_0}{4\pi} I \frac{\cos\theta}{r^2} 2\pi R = \frac{\mu_0 I R}{2r^2} \left(\frac{R}{r}\right)$$

this is $\cos\theta$ by trigonometry

$$= \frac{\mu_0 I R^2}{2r^3} = \frac{\mu_0 I R^2}{2(R^2+z^2)^{3/2}} \blacksquare$$

↑ because $r = (R^2+z^2)^{1/2}$ by pythagoras.

Observations:

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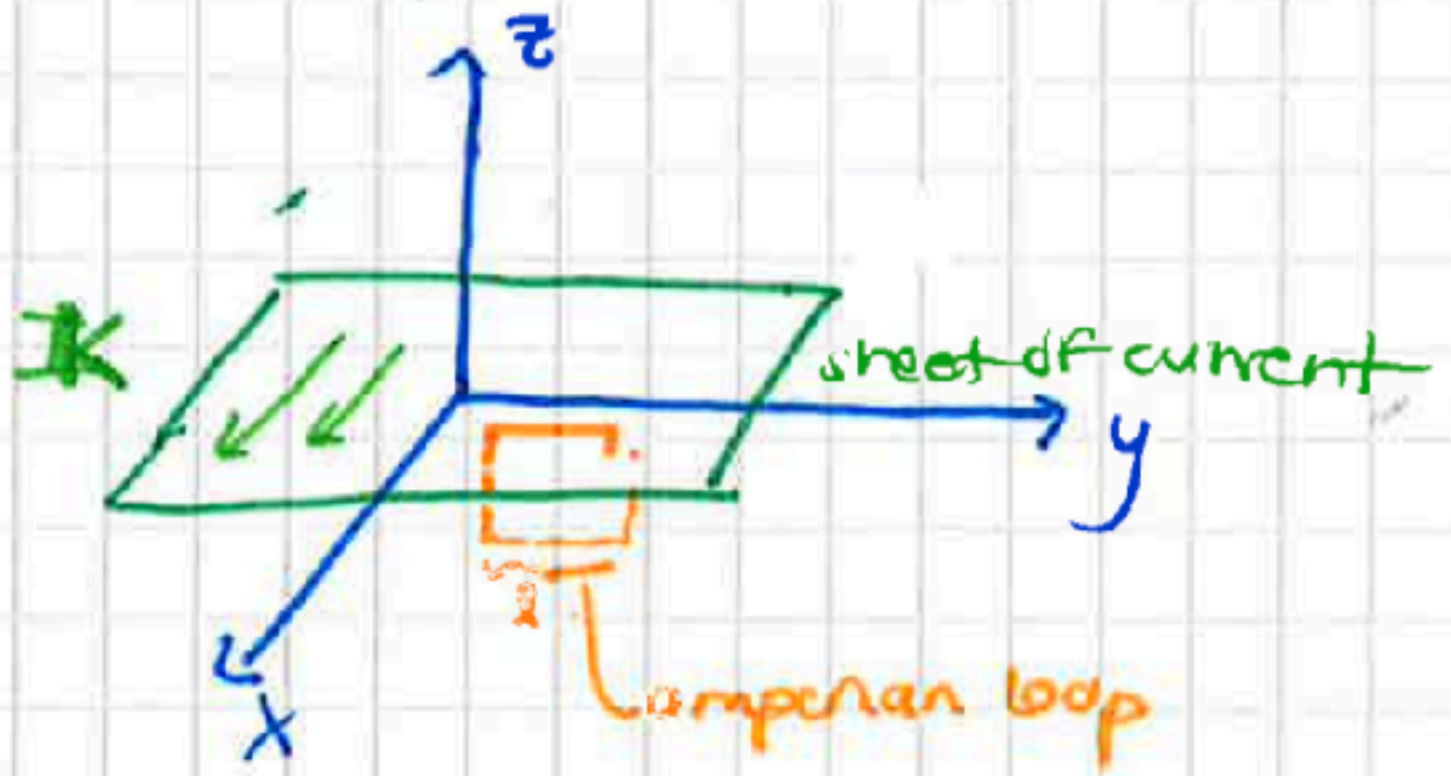
Using Ampere to calculate B

EXAMPLE: Calculate the magnetic field of an infinite uniform surface current

$\underline{K} = K\hat{x}$ flowing over the x, y plane.



K is flowing in the x direction

First of all we want to determine the direction of $\underline{B}$? We know that it can't have an x component, since the source is ∥ to x

$$B(\underline{r}) = \frac{\mu_0}{4\pi}\int \frac{\underline{K}(\underline{r}) \times \hat{r}}{r^2}\, d\underline{a}$$

then the x component term in the cross prod is zero

The field is ⊥ to K, so in this case it is ⊥ to x

Can the field have a z component? No, because of symmetry.

Any vertical contribution from a filament at +y will be cancelled by a filament at −y

Using Ampere to calculate B

Symmetry argument in other words:

1. Start from what magnetic fields "like to do"

A basic fact students usually know:

> **Magnetic fields form closed loops and wrap around currents.**

You can see this from:

- The right-hand rule for a straight wire
- Iron-filings pictures around current-carrying conductors

So before any math, we expect:

- The field should **circulate around the current**, not point straight along it.

2. Build the sheet from many parallel wires

This is the most physically intuitive argument.

Imagine the surface current as many wires

Picture the sheet as being made of:

- Infinitely many, very closely spaced
- Straight wires running in the $+x$ direction

Each tiny wire produces a magnetic field that:

- Circles around that wire
- Has no component along the wire itself

3. Add up the fields from all the wires

Now think about what happens when we superpose the fields.

(a) No x-component

Each individual wire:

- Produces a circular magnetic field
- That field is always **perpendicular to the wire**

So:

- No single wire produces a B_x
- Adding many wires cannot magically create one

$$\Rightarrow B_x = 0$$

(b) No z-component

Consider a point **above the sheet**.

- Wires to the left produce some upward $(+z)$ field
- Wires to the right produce an equal downward $(-z)$ field

Because the sheet is infinite and uniform:

- For every wire contributing $+B_z$, there is another contributing $-B_z$

They cancel **exactly**.

$$\Rightarrow B_z = 0$$

Using Ampere to calculate B

Symmetry argument in other words:

(c) The y-component survives

Now look at the sideways (y) direction.

- For every wire:
  - The magnetic field above the wire points in the same sideways direction
- Fields from all wires add **constructively** in y

Nothing cancels this component.

$$\Rightarrow B_y \neq 0$$

4. Why the field flips direction below the sheet

Go below the sheet and repeat the argument:

- Each wire's magnetic field reverses direction
- All the sideways components flip together

So:

- Same magnitude
- Opposite direction

5. Why the field is uniform (constant magnitude)

Because:

- Every point at a given height sees the same infinite arrangement of wires
- There is no "edge" or "center" to the sheet

So:

- The magnetic field cannot get stronger or weaker as you move sideways
- Only the sign changes across the sheet

6. Final physical picture

You should picture:

- The current flowing uniformly in the x direction
- Magnetic field lines running straight in the y direction
- One direction above the sheet
- The opposite direction below the sheet

Like this (schematic):

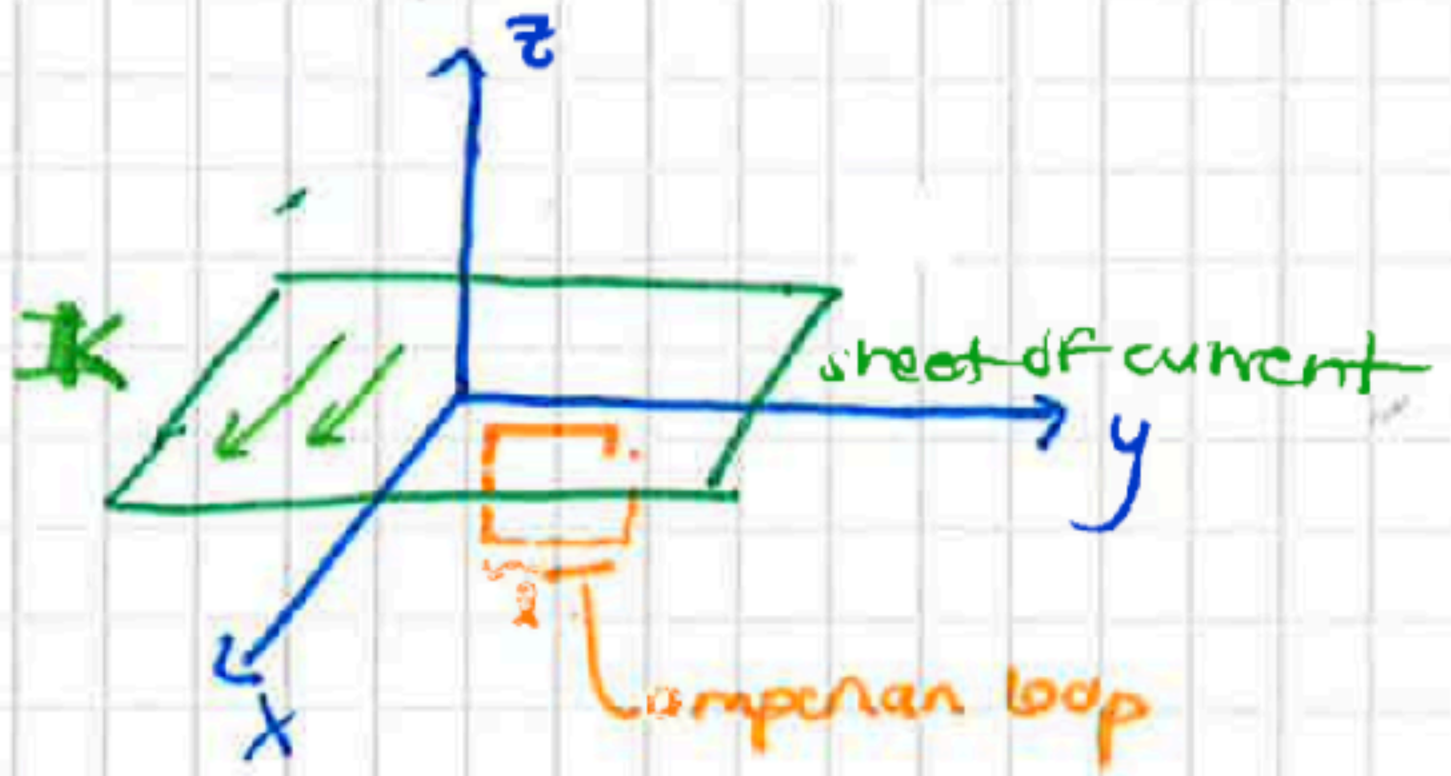
css Copy code

```css
  +y  ←  ←  ←  ←  ←   (B field above)
----------------------  current sheet (current → x)
  →  →  →  →  →   −y  (B field below)
```

Using Ampere to calculate B

EXAMPLE: Calculate the magnetic field of an infinite uniform surface current $\underline{K} = K\hat{x}$ flowing over the x, y plane.

K is flowing in the x direction



Now let's draw a loop as shown in the figure parallel to the x, z plane.
Applying Ampère's law:

$$\oint \underline{B} \cdot d\underline{l} = B(2\ell) = \mu_0 I_{enc} = \mu_0 K \ell$$

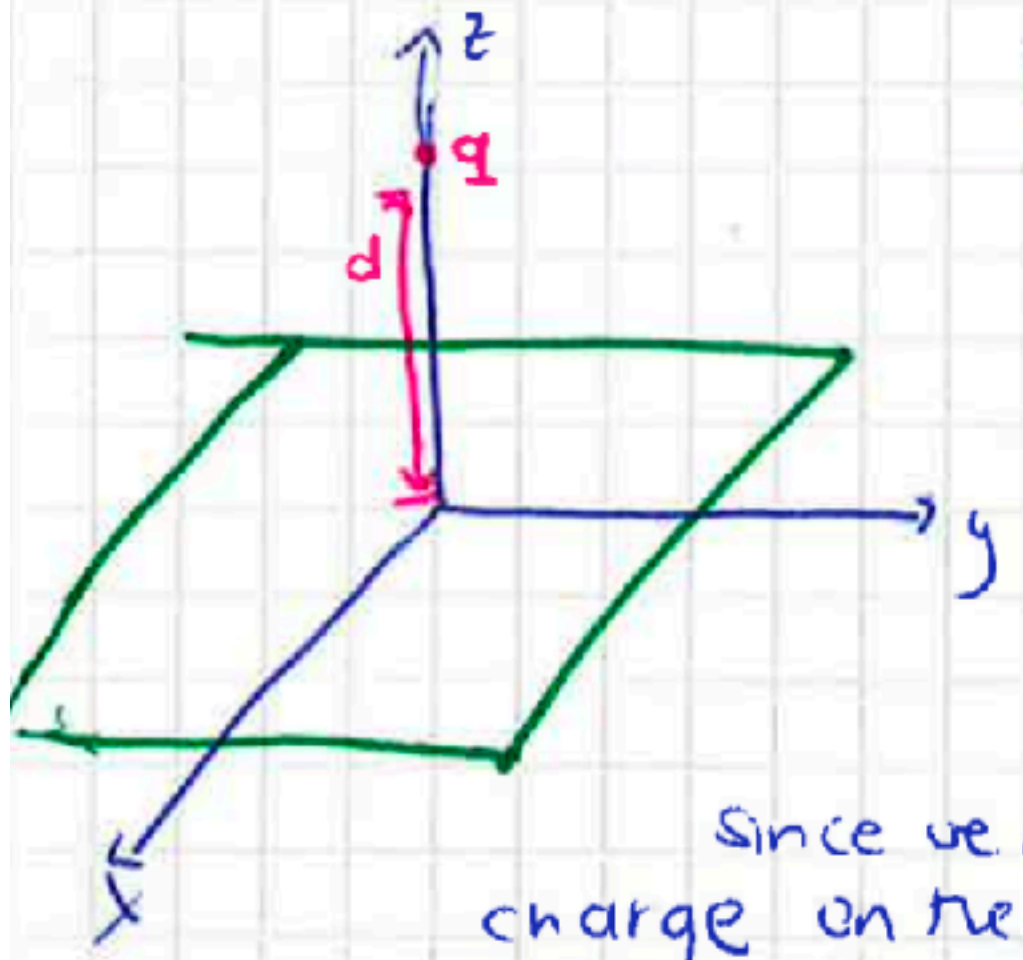
↑ by definition of K

$\Rightarrow B = \frac{\mu_0 K}{2}$ and more precisely $B = \begin{cases} \mu_0/2 \, K\hat{y}; & z<0 \\ -\mu_0/2 \, K\hat{y}; & z>0 \end{cases}$ (10)

Method of images

Applications of the uniqueness Theorems. Method of images

The classic image problem:



Suppose we have a point charge q above a grounded infinite conducting plane.

(grounded) means $\varphi = 0$ at the conductor's surface

what is the potential in the region above the plane?

We're tempted to say $\frac{1}{4\pi\varepsilon_0}\frac{q}{r}$ but this is WRONG, we need to consider the induced charge.

Since we have a conductor q will induce a certain amount of negative charge on the nearby surface of the conductor

$\Rightarrow \varphi_{tot} = \varphi_q + \varphi_{q_{ind}}$

How can we calculate φ_{tot} if we don't know q_{ind}?

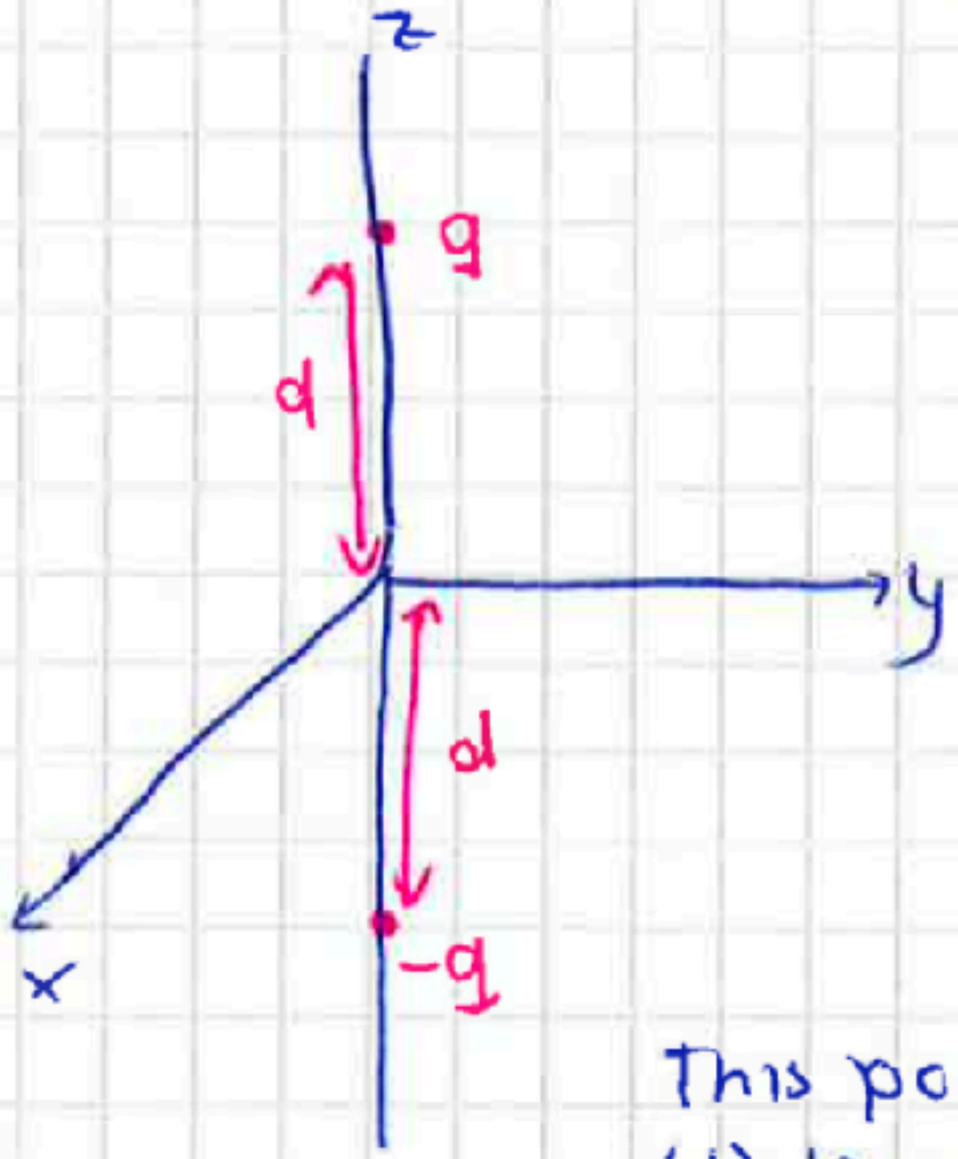
Mathematically our problem is to solve Poisson's eq in the region $z > 0$ with a single point charge q at $(0, 0, d)$, subject to the BC:

(1) $\varphi = 0$ at $z = 0$ (i.e the conductor is grounded)
(2) $\varphi \to 0$ away from q, i.e for points $(x^2+y^2+z^2)^{1/2} >> d$

By the 1st uniqueness theorem, if φ satisfies Poisson's eq & the BC $\Rightarrow \varphi$ is a unique solution to the problem.

⇒ We can substitute our problem for one that is much easier to solve. As long as we satisfy the BC the solution that we find will be the correct one!

we will substitute our problem by the following configuration of charges



We have 2 point charges: $+q$ at $(0,0,d)$ and $-q$ at $(0,0,-d)$
we got rid of the conducting plane

For this charge configuration we can easily write the potential:

$$\varphi(x,y,z) = \frac{1}{4\pi\varepsilon_0}\left\{\frac{q}{[x^2+y^2+(z-d)^2]^{1/2}} - \frac{q}{[x^2+y^2+(z+d)^2]^{1/2}}\right\}$$

by superposition we can just add the potential of 2 charges

denominator is just the distance of (x,y,z) to the charge $+q$

distance of a pt (x,y,z) to the charge $-q$

This potential satisfies the BC of the original problem, namely:

(1) $\varphi = 0$ at $z=0$

(2) $\varphi \to 0$ for $(x^2+y^2+z^2)^{1/2} \gg d$

And the only charge in the region $z>0$ is q.
So we have all the conditions of the original problem.

∴ This potential is the solution to our original problem
the uniqueness theorem guarantees this.

So now that we have the potential, we can actually calculate the induced surface charge on the conductor:

We know that at the surface of the conductor

$$\underline{E} = \frac{\sigma}{\varepsilon_0}\hat{n}$$ rewriting this in terms of φ:

$$\sigma = -\varepsilon_0 \frac{\partial \varphi}{\partial n}$$ where $\frac{\partial \varphi}{\partial n}$ is the normal derivative at the surface, in this case:

$$\sigma = -\varepsilon_0 \frac{\partial \varphi}{\partial z}\Big|_{z=0}$$

Now if we calculate $\frac{\partial \varphi}{\partial z}$ for the potential we found:

$$\frac{\partial \varphi}{\partial z} = \frac{1}{4\pi\varepsilon_0}\left\{ \frac{-q(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}} + \frac{q(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}} \right\}$$

$$\Rightarrow \boxed{\sigma(x,y) = \frac{-qd}{2\pi(x^2+y^2+d^2)^{3/2}}}$$

induced surface charge is negative ✓
it's greatest at $x=y=0$

Now if we compute the total induced charge:

$$Q = \int \sigma \, da$$ for convenience we compute this $\int$ in polar coordinates (r, φ) with $r^2 = x^2+y^2$ & $da = r\,dr\,d\varphi$

so $$\sigma(r) = \frac{-qd}{2\pi(r^2+d^2)^{3/2}}$$

Now computing the $\int$:

$$Q = \int_0^{2\pi}\int_0^{\infty} \frac{-qd}{2\pi(r^2+d^2)^{3/2}} r\,dr\,d\varphi = \frac{qd}{(r^2+d^2)^{1/2}}\Big|_0^{\infty} = -q$$ which makes sense.

Remember Purcell Chapter 12

- This Chapter has the solutions to the exercises in the textbook!

Final studying tips

- Make sure you understand the concepts behind all problems, rather than: "which formula to use"

- Read the class notes for conceptual explanations if you don't have time to read the textbook. However, our textbooks also have great explanations of the material.

- Here are some helpful notes form the MIT electromagnetism course for first year bachelor students:

  https://ocw.mit.edu/courses/8-02-physics-ii-electricity-and-magnetism-spring-2007/pages/readings/

- Study in groups, discussion with your peers is very important to understand concepts.

- We will answer questions intermittently in the ED forum until Thursday January 8th.

Good luck! You will do great in the exam if you have been following the course readings, lectures and exercise sheets.