Lecture 2. Electric charge, Coulomb's law and the electric field

Last time:

① Review of vector operations

let $\underline{a} = a_x \hat{e}_x + a_y \hat{e}_y + a_z \hat{e}_z$

$\underline{b} = b_x \hat{e}_x + b_y \hat{e}_y + b_z \hat{e}_z$

* addition

$$\underline{a} + \underline{b} = (a_x + b_x)\hat{e}_x + (a_y + b_y)\hat{e}_y + (a_z + b_z)\hat{e}_z$$

* dot product

$$\underline{a} \cdot \underline{b} = a_x b_x + a_y b_y + a_z b_z \leftarrow \text{scalar}$$

$$\underline{a} \cdot \underline{b} = ab \cos\theta$$

* Cross product

$$\underline{a} \times \underline{b} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = (a_y b_z - b_y a_z)\hat{e}_x + (-a_x b_z - a_z b_x)\hat{e}_y + (a_x b_y - a_y b_x)\hat{e}_z$$

$$\underline{a} \times \underline{b} = ab \sin\theta \, \hat{n}$$

in which direction does $\hat{n}$ point?

the right hand rule: Let your fingers point in the direction of the 1st vector and curl around the 2nd, then your thumb will point in the direction of $\hat{n}$.

② The "del" operator:

$$\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$ in cartesian coordinates.

Let $f(x,y,z) = f_x \hat{x} + f_y \hat{y} + f_z \hat{z}$

$$\nabla \cdot f = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}$$ Divergence (scalar)

$$\nabla f = \frac{\partial f_x}{\partial x}\hat{x} + \frac{\partial f_y}{\partial y}\hat{y} + \frac{\partial f_z}{\partial z}\hat{z}$$ Gradient (vector)

$$\nabla \times f = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} = \left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right)\hat{e}_x + \left(-\frac{\partial f_z}{\partial x} + \frac{\partial f_x}{\partial z}\right)\hat{e}_y + \left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right)\hat{e}_z$$

The curl (vector)

Electric force

Demo 416. Electric charge

Magnetic force

Demo 568. Magnet

Electromagnetic force

Demo 568. EM induction (Moving charges)

Electrostatics

From the demos we did, we can see there is a $\vec{F}$ that is "fighting" gravity. There must be a property of matter that makes it "feel" this $\vec{F}$. In the case of gravity we have mass. In the case of the electric force we have charge. Electrostatics is the study of the physics of stationary charges.

Electric charge: Fundamental facts

* Electric charge exists in two varieties: → positive $(+e)$, → negative $(-e)$

"All members of one class repel each other while attracting members of the other class"

Two charged bodies

(A) attracts (B)

if (A) attracts (C) $\Rightarrow$ (B) repels (C)

This is in contrast with gravitation where all bodies with mass attract each other.

NOTE: There is nothing inherently positive or negative about the charges. It is simply a convention. We could've called them left or right or anything really.

* Electric charge is conserved: The total electric charge in an isolated system, that is the algebraic sum of positive and negative charge present at any time, never changes.

Non-conservation of charge would be incompatible with the structure of our present electromagnetic theory. Thus, we state this as a postulate of theory and as an empirical law supported without exception by all observations so far.

* Electric charge is quantized: All charges (both positive and negative) have the same magnitude.

The magnitude of an electric charge is equal to the amount of charge carried by a single electron.

We denote the magnitude of the charge by e.

Equivalent to this statement is that the magnitude of the charge carried by any object can be written as:

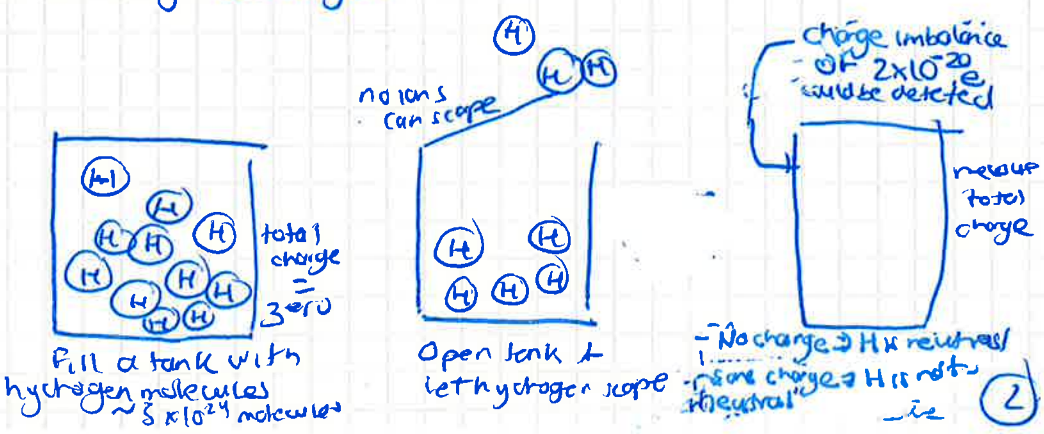
$$Ne \quad \text{, where } N \text{ is an integer}$$

$$e = 1.602176634 \times 10^{-19}\ C \quad (SI)$$

J. G King experiment → Experiment performed to prove that the magnitude of a positive charge is really the same as a negative charge.

Take hydrogen

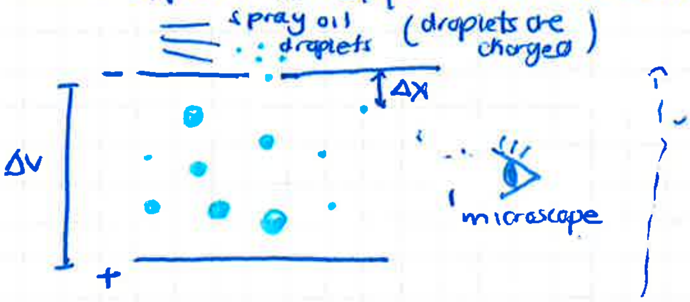
(H) each molecule has two e^- and two p^+



→ If each molecule was neutral, the charge measured after the whole gas escaped would be zero.
→ If H is not neutral → each molecule would leave behind a "residual charge" in the container since charge needs to be conserved. Since we have a lot of molecules, the imbalance in each one would add up and would be measurable (up to 2×10^{-20} e/hydrogen molecule)

At the end, charge measured was zero ⇒ H is neutral (to ~1 part in 10^{20}) ⇒ ~~the~~ the magnitude of charge of a proton is the same as the electron.

Millikan experiment: Experiment that measured the charge of $\bar{e}$ and showed charge is quantized.



forces acting on each droplet:

$F_{\bar{e}}$ electric $= qE$ ← electric field

q

F_g gravitational $= mg$

As we will soon learn the electric field between 2 // plates is given by $E = \frac{\Delta V}{\Delta x}$

$\Rightarrow F_{\bar{e}} = q\frac{\Delta V}{\Delta x}$

For droplets that are hovering at the center of the plates: $F_e = F_g \Rightarrow q\frac{\Delta V}{\Delta x} = mg$

$\Rightarrow q = \underbrace{\frac{\Delta x\, mg}{\Delta V}}_{\text{are all known experimental quantities}}$

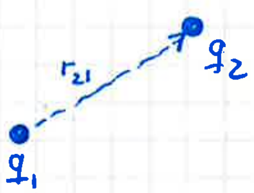
Δx, ΔV are experimentally observed
m is taken from $\rho = \frac{m}{V} \Rightarrow m = \frac{4}{3}\pi r^3 \rho_{oil}$ (r ← radius of droplet)
g is known

This was a clever way to measure q the elementary charge $e \equiv 1.6\times10^{-19}$ C
All measurements obtained were multiples of this qty ⇒ charge is quantized.

Now that we have established the basic facts about electric charge we are ready to talk about the force between charged objects.

Coulomb's law

Two stationary electric charges repel or attract each other with a force proportional to the product of the magnitude of the charges + inversely proportional to the square of the distance between them.



$$\underbrace{F_2}_{\text{Force acting on charge 2}} = \underbrace{\frac{1}{4\pi\varepsilon_0}}_{} \frac{q_1 q_2}{r_{21}^2}\hat{r}_{21}$$

$q_1 q_2$ → scalars giving the magnitude + sign of the respective charges

$\hat{r}_{21}$: unit vector going from q_1 to q_2

r_{21}^2: distance between charges squared

ε_0: permittivity of free space (fundamental constant)

what are the physics ~~underlying~~ in this mathematical statement?

- $F \propto \frac{1}{r^2}$ ⟶ electric force is much stronger than gravitational force
- Equal sign charges will repel, opposite sign charges will attract
- Newton's third law applies : $\underline{F}_{12} = -\underline{F}_{21}$

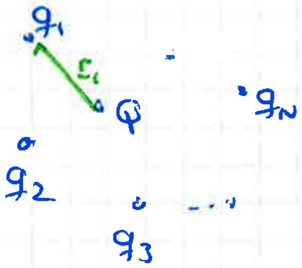
From Coulomb's law you can see that the only way to detect and measure electric charges is by observing the interactions between charged bodies.

Principle of superposition

"The interaction between two charges is completely unaffected by the presence of other charges"

Consider N discrete charges in the presence of a "test charge" Q :

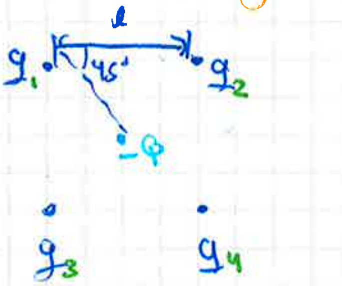
The force on Q due to all other charges is equal to the vector sum of the forces created by individual charges.



$$\underline{F}_Q = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1 Q \hat{r}_1}{|r_1|^2} + \frac{q_2 Q \hat{r}_2}{|r_2|^2} + \dots + \frac{q_N Q \hat{r}_N}{|r_N|^2}\right]$$

$$= \boxed{\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i Q}{|r_i|^2}\hat{r}_i}$$

EXAMPLE : Calculate the value of Q such that the force at the corners of the following charge distribution is zero.



- Charges form a square of side ℓ
- $-Q$ is located at the center of the square.

The force on one corner (let's say the top left) will be :

$$\underline{F}_{q_1} = \underline{F}_{q_1q_2} + \underline{F}_{q_1q_3} + \underline{F}_{q_1q_4} - \underline{F}_{q_1Q}$$ ⟵ this force will be directed along the diagonal of the square

$$|F_{q_1}| = \frac{1}{4\pi\varepsilon_0}\left[\frac{q^2}{\ell^2}(\cos 45) + \frac{q^2}{\ell^2}(\cos 45) + \frac{q^2}{2\ell^2} - \frac{qQ}{\ell^2/2}\right] = 0$$

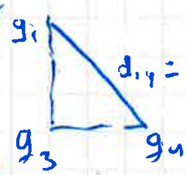
we want this equal to zero as required by the problem

Now from this expression we just need to get Q :

$$\frac{2q^{\cancel{2}}}{\cancel{\ell^2}}(\cos 45) + \frac{q^{\cancel{2}}}{2\cancel{\ell^2}} = \frac{\cancel{2}\cancel{q}Q}{\cancel{\ell^2}}$$

$$q(\cos 45) + \frac{q}{4} = Q$$

$$\Rightarrow Q = q\left[\cos 45 + 0.25\right]$$



d_{14}: $\ell^2 + \ell^2 = d_{14}^2$ $\Rightarrow d_{14} = \sqrt{2}\,\ell$

We can extend the principle of superposition from discrete to continuous charge distributions.

continuous charge distributions

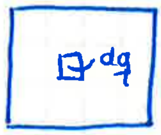
a line of charge q

$$\lambda \equiv \frac{dq}{dl}$$

total charge in this wire:

$$q = \int dq = \int_l \lambda\, dl$$

a sheet of charge q



$$\sigma = \frac{dq}{dA}$$

total charge in this sheet:

$$q = \int dq = \int_A \sigma\, dA$$

a volume of charge q

$$\rho = \frac{dq}{dV}$$

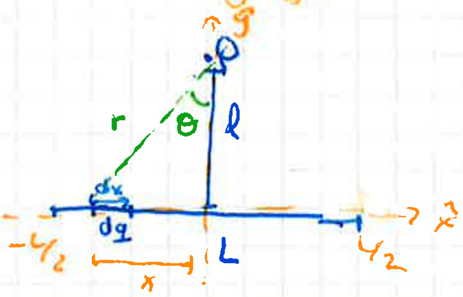
total charge in this volume:

$$q = \int dq = \int_V \rho\, dV$$

We can extend coulomb's law for continuous charge distributions, the force on a "test charge" Q due to a continuous distribution is:

$$\underline{F}_Q = \frac{Q}{4\pi\varepsilon_0} \int \frac{\lambda\, dl}{|r|^2}\,\hat{r} \qquad \underline{F}_Q = \frac{Q}{4\pi\varepsilon_0} \int_A \frac{\sigma\, dA}{|r|^2}\,\hat{r} \qquad F_Q = \frac{Q}{4\pi\varepsilon_0} \int_V \frac{\rho\, dV}{|r|^2}\,\hat{r}$$

EXAMPLE: A rod of length L has a charge q uniformly spread on it. A test charge Q is positioned at a distance l from the rod's midpoint. What is the force $\underline{F}$ that the rod exerts on charge q?



- because the rod is uniformly charged we can define a linear charge density:

$$\lambda \equiv \frac{dq}{dx} \quad (*)$$

- because the charge is located at the midpoint of the wire, by symmetry the force exerted on Q by the wire will be directed in the $\hat{y}$ direction (only $\hat{y}$ component, others cancel out)

- we can now calculate the force exerted by a piece of wire dx with charge dq on Q:

$$dF = dF_y = \frac{1}{4\pi\varepsilon_0} Q \frac{dq}{r^2} \underbrace{\cos\theta}_{\text{we project along y direction}}$$

now using $(*)$: $dF_y = \frac{1}{4\pi\varepsilon_0} Q \frac{\lambda\, dx}{r^2} \cos\theta \quad (**)$

We can look at the geometry of the problem to further simplify this expression; we would like to find an expression of dx in terms of θ so we integrate over that variable:

$$\tan\theta = \frac{x}{l} \qquad \frac{d}{d\theta}(\tan\theta) = \frac{dx}{l}$$

$$\Rightarrow \sec^2\theta\, d\theta = \frac{1}{\cos^2\theta}\, d\theta = \frac{dx}{l} \quad \Rightarrow dx = \frac{l}{\cos^2\theta}\, d\theta$$

. Subs this into $(**)$:

$$dF_y = \frac{1}{4\pi\varepsilon_0} \frac{Q\lambda}{r^2} \frac{l}{\cos^2\theta}\, d\theta \cos\theta$$

. Now we can also rewrite r^2 in terms of θ:

$$\cos\theta = \frac{l}{r} \Rightarrow r^2 = \frac{l^2}{\cos^2\theta}$$

so subs this into the last expression for dF_y:

$$dF_y = \frac{1}{4\pi\varepsilon_0} \frac{Q\lambda l}{\cos\theta} \frac{\cos^2\theta}{l^2}\, d\theta = \frac{Q\lambda}{4\pi\varepsilon_0 l} \cos\theta\, d\theta$$

so now we integrate to find the total force:

$$F = \int dF_y = \frac{Q\lambda}{4\pi\varepsilon_0 l}\int_{-L/2}^{L/2}\cos\theta \, d\theta = \frac{Q\lambda}{4\pi\varepsilon_0 l}\left[\sin\theta\right]_{-L/2}^{L/2} \quad (**)$$

Now we would like to go from θ to x to evaluate the integral, so we use trigonometry:

$\sin\theta = \frac{x}{r}$; but $r^2 = x^2 + l^2$ (pythagoras) $\Rightarrow \sin\theta = \frac{x}{(x^2+l^2)^{1/2}}$ so subs this into (**):

$$F = \frac{Q\lambda}{4\pi\varepsilon_0 l}\left[\frac{x}{(x^2+l^2)^{1/2}}\right]_{-L/2}^{L/2} = \frac{Q\lambda}{4\pi\varepsilon_0 l}\left[\frac{(L/2)}{\left(\frac{L^2}{4}+l^2\right)^{1/2}} + \frac{(L/2)}{\left(\frac{L^2}{4}+l^2\right)^{1/2}}\right]$$

$$= \frac{Q\lambda}{4\pi\varepsilon_0 l}\left[\frac{L}{\left(\frac{L^2}{4}+l^2\right)^{1/2}}\right]$$

and finally we subs $\lambda = \frac{dq}{dx} \Rightarrow \lambda\int dx = \int dq \Rightarrow \lambda L = q \Rightarrow \lambda = \frac{q}{L}$

$$= \frac{Qq}{4\pi\varepsilon_0 l L}\left[\frac{L}{\left(\frac{L^2}{4}+l^2\right)^{1/2}}\right] = \frac{Qq}{4\pi\varepsilon_0 l}\left(\frac{L^2}{4}+l^2\right)^{-1/2}$$

are we done? we still need to include the direction

$$\boxed{\underline{F} = \frac{Qq}{4\pi\varepsilon_0 l}\left(\frac{L^2}{4}+l^2\right)^{-1/2}\hat{y}}$$

What is a general strategy to solve these types of problem?

1. Look at the symmetry of the problem and choose an appropriate coordinate system.
   In this case $\rightarrow$ $F \parallel \hat{y}$, we chose origin of coord system such that we define the int limits $-L/2 : \frac{L}{2}$
2. Define a charge density (linear, area or volume as appropriate)
3. Consider the force exerted by an infinitesimal charge element dq or piece of wire dx
4. Use trigonometric relations to change the variables of integration
5. Integrate
6. Go back to the original variables of the problem
7. Don't forget to include the direction of the force in the final answer.

EXAMPLE: Consider now a uniformly charge rod of infinite length and a point charge Q located above it. What is the force that the rod exerts on the charge?

Do we need to solve the problem again from the beginning? Let's think first about what it means for the wire to be ∞

For all practical purposes, infinite means >> than any other distances in the problem.
In the case of our rod of length L, if it's now ∞, this means $L \gg l$.

Let's look at the solution of our last problem:

$$\underline{F} = \frac{Qq}{4\pi\varepsilon_0 l}\left(\frac{L^2}{4}+l^2\right)^{-1/2}\hat{y}$$

Let's rewrite this in terms of λ

$$= \frac{Q\lambda}{4\pi\varepsilon_0 l}\left[\frac{L}{\left(\frac{L^2}{4}+l^2\right)^{1/2}}\right]$$

now we factor out $\frac{L}{2}$ from the square root:

$$F = \frac{Q\lambda}{4\pi\varepsilon_0 l}\left[\frac{L}{\frac{L}{2}\left(1+\frac{4l^2}{L^2}\right)^{1/2}}\right]$$

Now we know that $L >> l$ so $\frac{4l^2}{L^2} << 1$ Let $\varepsilon = \frac{4l^2}{L^2}$

Given that $\varepsilon << 1$ we can rewrite the binomial using a Taylor expansion:

$$(1+\varepsilon)^{-1/2} = 1 - \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 + \ldots \quad \left| \text{Remember: } (1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} \ldots \right)$$

So:

$$F = \frac{Q\lambda}{2\pi\varepsilon_0 l}\left[1 - \frac{2l^2}{L^2} + \ldots\right) \simeq \frac{Q\lambda}{2\pi\varepsilon_0 l} - \frac{Q\lambda l^2}{\pi\varepsilon_0 l L^2} + \ldots$$ when $L \to \infty$ all higher order terms vanish

$$\therefore \boxed{\underline{F} = \frac{Q\lambda}{2\pi\varepsilon_0 l}\hat{y}}$$

we will see a simpler way to arrive at this core result in a few lectures.