Lecture 15. The magnetic vector potential

Last time:

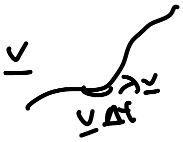
*Biot-Savart's law:

$$\underline{B} = \frac{\mu_0 I}{4\pi}\int \frac{d\underline{s}\times\hat{r}}{r^2}$$ line of current

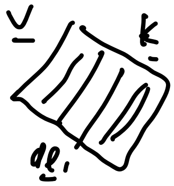
$$\underline{B} = \frac{\mu_0}{4\pi}\int \frac{\underline{K}(\underline{r})\times\hat{r}}{r^2}\,dA$$ surface current

$$\underline{B} = \frac{\mu_0}{4\pi}\int \frac{\underline{J}(\underline{r})\times\hat{r}}{r^2}\,d\tau$$ Volume current.

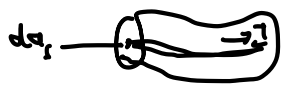
$\underline{I} = \frac{dq}{dt}$; $\underline{I} = \lambda \underline{v}$



$\underline{K} = \frac{dI}{dl_\perp}$; $\underline{K} = \sigma\underline{v}$



$\underline{J} = \frac{dI}{da_\perp}$; $\underline{J} = \rho\underline{v}$



*Divergence of $\underline{B}$:

$$\nabla\cdot\underline{B} = 0$$ Non ∃ of magnetic monopoles

*Curl of $\underline{B}$

$$\nabla\times\underline{B} = \mu_0\underline{J}$$ Ampère's law in differential form

$$\Updownarrow$$

$$\oint \underline{B}\cdot d\underline{l} = \mu_0 I_{enc}$$ Ampère's law in $\int$ form

Summary of $\underline{E}$ and $\underline{B}$

Electrostatics

$$\nabla\cdot\underline{E} = \frac{1}{\varepsilon_0}\rho$$ Gauss's law

$$\rightarrow \nabla\times\underline{E} = 0$$

Magnetostatics

$$\nabla\cdot\underline{B} = 0$$

$$\nabla\times\underline{B} = \mu_0\underline{J}$$ Ampère's law

$\nabla \cdot \underline{E} = \frac{1}{\varepsilon_0}\rho$ Relate $\underline{E}$ with its source → charge density

$\nabla \times \underline{E} = 0 \Leftrightarrow \oint \underline{E} \cdot d\underline{s} = 0 \Rightarrow \underline{E}$ is a conservative field

These eq $\Leftrightarrow$ coulomb's law + superposition principle

For magnetostatics:

$\nabla \cdot \underline{B} = 0$ ∄ of magnetic monopoles
Field lines form closed loops
There are no sources or sinks → source of $\underline{B}$ is charge in motion

$\nabla \times \underline{B} = \mu_0 \underline{J}$ Relates $\underline{B}$ with its source → current density

These eq $\Leftrightarrow$ Biot-Savart's law + superposition principle

If we add Lorentz's force law

$\underline{F} = Q(\underline{E} + \underline{v} \times \underline{B})$ Force experienced by a moving charged particle in the presence of $\underline{E}$ & $\underline{B}$

→ Typically electric $\bar{F}$'s >> magnetic $\bar{F}$'s
comes from the values of ε_0 & μ_0

→ We notice magnetic fields when we have electrically neutral objects (e.g. wire) with charges in motion

Magnetic vector potential

To simplify electrostatics we defined a scalar potential φ

$\underline{E} = -\nabla\varphi$, we could do this $\nabla \times \underline{E} = 0 \Rightarrow \nabla \times (\nabla\varphi) = 0$

Can we do the same for $\underline{B}$?

$\nabla \cdot \underline{B} = 0$

$\nabla \cdot (\nabla \times f) = 0$ for any f, so we define $\underline{A}$:

$\boxed{\underline{B} = \nabla \times \underline{A}}$ Magnetic vector potential

$\underline{A}$ is NOT connected to work or energy. $\Big\}$ in contrast with φ
$\underline{A}$ is NOT uniquely defined.

Example: Consider $\underline{B} = B_0 \hat{z}$. Find $\underline{A}$ that creates $\underline{B}$:

$$\underline{B} = \nabla \times \underline{A}$$

$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = 0$$

$$B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = 0$$

$$B_z = \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} = B_0$$

Here are a few possible solutions:

$$\underline{A} = -y B_0 \hat{x}$$

$$\underline{A} = x B_0 \hat{y}$$

$$\underline{A} = \frac{B_0}{2}(-y\hat{x} + x\hat{y})$$

$$\dots \infty$$

For convenience we choose the solution that fulfills $\nabla \cdot \underline{A} = 0$

Poisson's equation for $\underline{A}$:

For electrostatics we obtained Poisson's eq:

$$\underline{E} \equiv \nabla \varphi \quad \text{and} \quad \nabla \cdot \underline{E} = \frac{\rho}{\varepsilon_0} \Rightarrow \boxed{\nabla^2 \varphi = -\frac{\rho}{\varepsilon_0}}$$

For magnetostatics we do something similar:

$$\underline{B} \equiv \nabla \times \underline{A} \quad \text{and} \quad \nabla \times \underline{B} = \mu_0 \underline{J}$$

$$\nabla \times (\nabla \times \underline{A}) \underset{\substack{\uparrow \\ \text{identity}}}{=} \nabla \underbrace{(\nabla \cdot \underline{A})}_{\text{if } \nabla \cdot \underline{A} = 0} - \nabla^2 A = \mu_0 \underline{J}$$

$$\Rightarrow \boxed{\nabla^2 A = -\mu_0 \underline{J}}$$

choosing $\nabla \cdot \underline{A} = 0$

Poisson's eq
(3 equations)

Remember: $\nabla^2 \varphi = \frac{-\rho}{\varepsilon_0}$ has a solution $\varphi(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho}{r}\, dV$

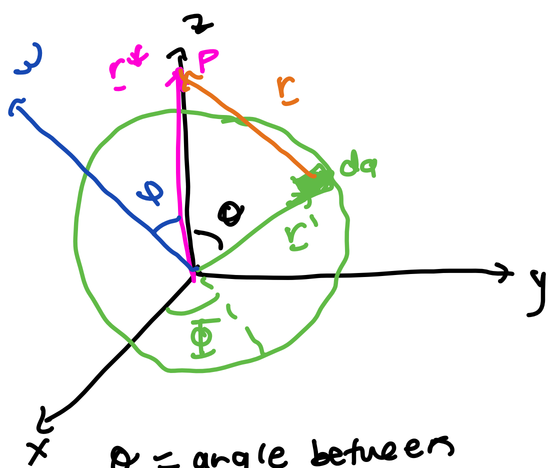
In this case the solution for $\underline{A}(r)$ is:

$$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi}\int \frac{\underline{J}(\underline{r})}{r}\, dV$$ For volume current densities

$$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi}\int \frac{\underline{I}}{r}\, dl = \frac{\mu_0}{4\pi}\underline{I}\int \frac{1}{r}\, dl$$ line current

$$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi}\int \frac{\underline{K}}{r}\, da$$ surface current.

Example: A spherical shell of radius R carrying a uniform surface charge density σ is spinning at angular velocity $\underline{\omega}$. Find the vector potential it produces at some point P in the $\hat{z}$ axis.



θ = angle between $\underline{r}$ and $\underline{r}'$

$\underline{r}^*$ vector from the origin to the point where we calculate $\underline{A}$

$\underline{r}'$ vector from the origin to the patch da on the surface of the sphere with charge $d\sigma$

$\underline{r}$ vector from source point to the point where we want to calculate $\underline{A}$

We can calculate $\underline{A}$ using the definition

$$\underline{A} = \frac{\mu_0}{4\pi}\int \frac{\underline{K}(\underline{r}')}{r}\, da \quad (*)$$

We will calculate each part in the $\int$:

① $K(\underline{r}) = \sigma \underline{v}$ where σ is the surface charge on sphere
$\underline{v}$ velocity of sphere

what is $\underline{v}$?
The velocity of a point $\underline{r}'$ in a rotating rigid body is $\underline{\omega} \times \underline{r}'$ in this case:

$$\underline{v} = \underline{\omega} \times \underline{r}' = \begin{array}{c} \\ \\ \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \omega \sin\psi & 0 & \omega\cos\psi \\ R\sin\theta\cos\Phi & R\sin\theta\sin\Phi & R\cos\theta \end{vmatrix} \end{array} \begin{array}{l} \\ \leftarrow \omega \\ \leftarrow \underline{r}' \end{array}$$

$$= R\omega\left[-(\cos\psi\sin\theta\sin\Phi)\hat{x} + (\cos\psi\sin\theta\cos\Phi - \underbrace{\sin\psi\cos\theta}_{\text{this term will survive.}})\hat{y} + (\sin\psi\sin\theta\sin\Phi)\hat{z}\right]$$

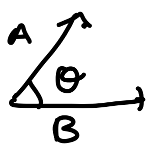
* will be zero upon integration

For terms involving $\sin\Phi$ or $\cos\Phi$:

$$\int_0^{2\pi} \sin\Phi \, d\Phi = \int_0^{2\pi} \cos\Phi \, d\Phi = 0$$

so we are left w/ only $\hat{y}$ term

② $\underline{r} = [R^2 + r^2 - 2Rr\cos\theta]^{1/2}$



$c^2 = A^2 + B^2 - 2AB\cos\theta$

③ $da = R^2 \sin\theta \, d\theta \, d\Phi$

so we can subs everything in the eq (*) for A:

$$\underline{A} = \frac{-\mu_0}{4\pi} \sigma R \omega \sin\psi \int_0^{2\pi}\int_0^{\pi} \underbrace{\frac{\cos\theta \overbrace{(R^2 \sin\theta\, d\theta\, d\Phi}^{da}}{[R^2+r^2-2Rr\cos\theta]^{1/2}}}_{r} \hat{y}$$

$$= \frac{\mu_0 \sigma}{4\pi} R\omega\sin\psi\,(2\pi) \int_0^{\pi} \frac{\cos\theta\sin\theta\, d\theta}{[R^2+r^2-2Rr\cos\theta]^{1/2}} \vec{y}$$

To solve the remaining $\int$ we set $u = \cos\theta$

$$= \frac{-1}{3R^2r^2}\left[(R^2+r^2+Rr)\underbrace{|R-r|}_{} - (R^2+r^2-Rr)(R+r) \right]$$

this will change depending if $r < R$ or $r > R$

For $R > r$ the expression reduces to $\frac{2r}{3R^2}$
inside sphere

For $R < r$ $\frac{2R}{3r^2}$
outside sphere

Then

$$A(\underline{r}) = \begin{cases} \dfrac{-\mu_0 R^3 \omega \sin\psi\,\sigma}{2}\left(\dfrac{2r}{3R^2}\right)\hat{y} & \text{for } r < R \\ \dfrac{-\mu_0 R^3 \omega\sin\psi\,\sigma}{2}\left(\dfrac{2R}{3r^2}\right)\hat{y} & \text{for } r > R \end{cases}$$

noting

$$\underline{\omega} \times \underline{r} = -\omega r \sin\psi \vec{y}$$

$$A(\underline{r},\theta,\Phi) = \begin{cases} \dfrac{\mu_0 R\sigma}{3} (\underline{\omega}\times\underline{r})\,\hat{\Phi} & \text{for } r < R \\ \dfrac{\mu_0 R^4 \sigma}{3r^3} (\underline{\omega}\times\underline{r})\,\hat{\Phi} & \text{for } r > R \end{cases}$$

How would you calculate $\underline{B}$ if you have $\underline{A}$?

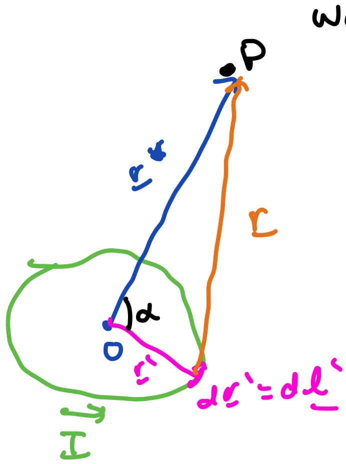
$$\underline{B} = \nabla \times \underline{A} = \frac{2\mu_0 R \omega \sigma}{3}(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})$$

$$= \frac{2}{3}\mu_0 \sigma R \omega \hat{z}$$

$$= \frac{2}{3}\mu_0 \sigma R \underline{\omega}$$ uniform inside the sphere.

Multipole expansion of the magnetic vector potential

We want to approximate $\underline{A}$ due to a localized current distribution, at a point far away from it.
We can use a multipole expansion



$\underline{r}^*$ vector from origin to the point where we want to approximate $\underline{A}$
$r \gg r'$ far away

$\underline{r}'$ vector from origin to source of $\underline{A}$, in this case source $d\underline{l}$

$\underline{r}$ vector from source pt to the point P

α = angle between $\underline{r}'$ and $\underline{r}^*$

Our goal is to expand the expression for A as a series of $\frac{1}{r}$ terms

We need to expand the expression for A

$$A(\underline{r}) = \frac{\mu_0 I}{4\pi}\int \frac{1}{r}\,d\underline{l}'$$

As we found in Lecture 8, we can expand $\frac{1}{r}$ as:

$$\frac{1}{r} = \frac{1}{[r^2 + (r')^2 - 2rr'\cos\alpha]^{1/2}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n \underbrace{P_n(\cos\alpha)}_{\text{generating function for Legendre polynomials}}$$

More explicitly:

$$A(\underline{r}) = \frac{\mu_0 I}{4\pi}\left[\overbrace{\frac{1}{r}\oint d\underline{l}'}^{\text{monopole}} + \overbrace{\frac{1}{r^2}\oint r' \cos\alpha \, d\underline{l}'}^{\text{dipole}} + \underbrace{\frac{1}{r^3}\oint (r')^2 \left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right) d\underline{l}' + \dots}_{\text{quadrupole}} \right]$$

monopole

$$\oint d\underline{l}' = 0$$

↑ integral of a displacement around a closed loop

consistent $\nexists$ magnetic monopoles

$$\nabla \cdot \underline{B} = 0$$

dipole:

$$A_{dip}(\underline{r}) = \frac{\mu_0 I}{4\pi r^2}\oint r' \cos\alpha \, d\underline{l}' = \frac{\mu_0 I}{4\pi r^2}\oint (\hat{r}\cdot\underline{r}')\, d\underline{l}'$$

We can rewrite this term using the identity:

$$\oint (\underline{c}\cdot\underline{r})\, d\underline{l} = \underline{a} \times \underline{c}$$

; for the dipole if $\underline{c} = \hat{r}$

↑

$$\underline{a} \equiv \int_S d\underline{a}$$ area vector

then $\oint (\hat{\underline{r}}\cdot\underline{r}')\, d\underline{l}' = -\hat{r} \times \int d\underline{a}'$

with this the dipole contribution becomes:

$$\underline{A}_{dip}(\underline{r}) = \frac{\mu_0}{4\pi}\frac{I}{r^2}\left[-\hat{r} \times \int d\underline{a}'\right] = \frac{\mu_0}{4\pi}\underbrace{\frac{I}{r^2}\int d\underline{a}'}_{m} \times \hat{r}$$

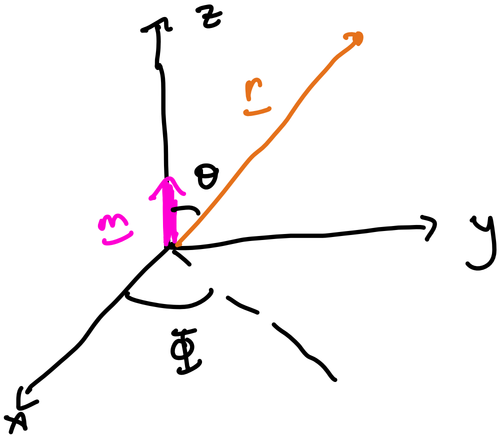
$$\therefore \quad \underline{A}_{dip}(\underline{r}) = \frac{\mu_0}{4\pi}\frac{\underline{m} \times \hat{r}}{r^2}$$ where $\underline{m} \equiv I\int d\underline{a} = I\underline{a}$

direction of $\underline{m}$ is given by rhr

+ Magnetic Dipole moment is independent of the choice of coord. system

→ This dipole is a "good" approx whenever $r \gg \int dl$

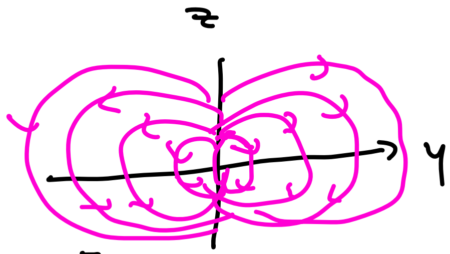
Field of a dipole



We just calculated that for a dipole:

$$\underline{A}_{dip}(\underline{r}) = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\Phi}$$

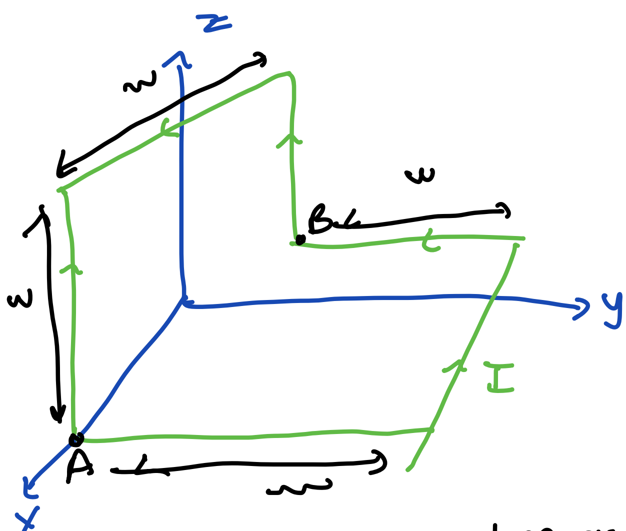
$$\Rightarrow \underline{B}(\underline{r}) = \nabla\times\underline{A} = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$



In coord free form:

$$\underline{B}_{dip}(\underline{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\underline{m}\cdot\hat{r})\hat{r} - \underline{m}\right]$$

Example: Find the magnetic dipole moment of the loop shown below ("bookend" shape loop)



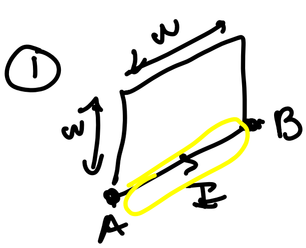
All the sides of the loop have length w. There is a current I going through the loop.

The goal is to calculate

$$\underline{m} = I\,\underline{a}$$

magnetic dipole moment

We consider this loop as a superposition of:



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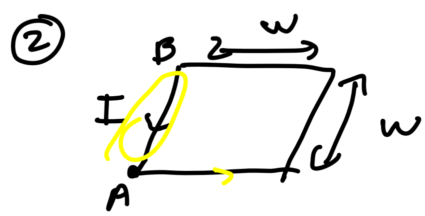

The magnetic dipole moment is

$$\mathbf{m} = \underbrace{I w^2 \hat{y}}_{\text{horizontal loop (2)}} + \underbrace{I w^2 \hat{z}}_{\text{vertical loop (1)}}$$

Its magnitude is $\sqrt{2}\, I w^2$ and points 45° line $z = y$

Now we have defined $\mathbf{A}$, we can summarize the relations between sources, fields and potential for $\mathbf{B}$

$\mathbf{J}$

$\mathbf{A}$

$\mathbf{B}$

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}}{r}\, dV$$

Solution to Poisson's eq.

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

Poisson's eq.

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J} \times \hat{r}}{r^2}\, dV$$

Biot-Savart's law

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \; ; \; \nabla \cdot \mathbf{B} = 0$$

Curl and divergence of $\mathbf{B}$

$$\mathbf{B} = \nabla \times \mathbf{A} \; ; \; \nabla \cdot \mathbf{A} = 0$$

Magnetic vector potential definition