Solution Sheet 9

Discussion 12.11.2025

Solution 1 - Coaxial capacitor

a) From Gauss's law we can evaluate the E field between the two cylinders as:

$$E = \frac{\lambda}{2\pi\epsilon_0\epsilon_{r_1} r} \tag{1}$$

By integrating along the radial direction between the two cylinders we get the electric potential:

$$\Phi = -\int_{R_1}^{R_2} E dr = \frac{-\lambda}{2\pi\epsilon_0\epsilon_{r_1}} \ln\left(\frac{R_2}{R_1}\right) \tag{2}$$

Therefore the capacitance, which is a positive value, for a coaxial cable of length L is:

$$C = \left|\frac{Q}{\Phi}\right| = \left|\frac{\lambda L}{\Phi}\right| = \frac{2\pi\epsilon_0\epsilon_{r_1} L}{\ln\left(\frac{R_2}{R_1}\right)} \tag{3}$$

b) This situation corresponds to the parallel of two coaxial capacitors of length $L/2$ and outer and inner radii R_2 and R_1. The first one has relative permittivity ϵ_{r_1} and the second one ϵ_{r_2}, thus we obtain:

$$C_1 = \frac{\pi\epsilon_0\epsilon_{r_1} L}{\ln\left(\frac{R_2}{R_1}\right)} \tag{4}$$

$$C_2 = \frac{\pi\epsilon_0\epsilon_{r_2} L}{\ln\left(\frac{R_2}{R_1}\right)} \tag{5}$$

$$C = C_1 \parallel C_2 = C_1 + C_2 = \frac{\pi\epsilon_0 L\left(\epsilon_{r_1} + \epsilon_{r_2}\right)}{\ln\left(\frac{R_2}{R_1}\right)} \tag{6}$$

c) This situation corresponds to the series of two coaxial capacitors of length L. The first one has outer and inner radii $(R_1 + R_2)/2$ and R_1 and relative permittivity ϵ_{r_1}, the second one has outer and inner radii R_2 and $(R_1 + R_2)/2$ and relative permittivity ϵ_{r_2}. Thus we obtain:

$$C_1 = \frac{2\pi\epsilon_0\epsilon_{r_1} L}{\ln\left(\frac{R_1+R_2}{2R_1}\right)} \tag{7}$$

$$C_2 = \frac{2\pi\epsilon_0\epsilon_{r_2} L}{\ln\left(\frac{2R_2}{R_1+R_2}\right)} \tag{8}$$

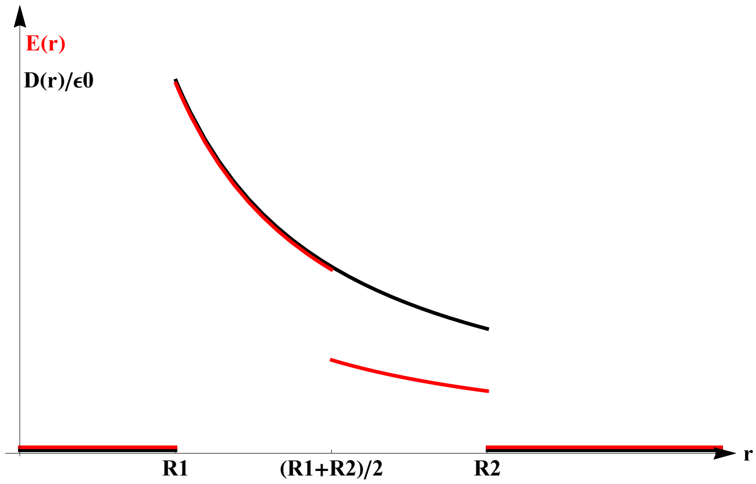
$$C = C_1 \oplus C_2 = \frac{C_1 C_2}{C_1 + C_2} = \frac{2\pi\epsilon_0\epsilon_{r_1}\epsilon_{r_2} L}{\epsilon_{r_1} \ln\left(\frac{2R_2}{R_1+R_2}\right) + \epsilon_{r_2} \ln\left(\frac{R_1+R_2}{2R_1}\right)} \tag{9}$$

d) By applying Gauss's law we obtain:

$$\vec{E} = \begin{cases} 0 & r < R_1 \\ \frac{\lambda}{2\pi\epsilon_0\epsilon_{r_1} r}\hat{u}_r & R_1 < r < \frac{R_1+R_2}{2} \\ \frac{\lambda}{2\pi\epsilon_0\epsilon_{r_2} r}\hat{u}_r & \frac{R_1+R_2}{2} < r < R_2 \\ 0 & r > R_2 \end{cases} \tag{10}$$

$$\vec{D} = \begin{cases} 0 & r < R_1 \\ \frac{\lambda}{2\pi r}\hat{u}_r & R_1 < r < R_2 \\ 0 & r > R_2 \end{cases} \tag{11}$$

In the figure the plots of $E(r)$ and $D(r)/\epsilon_0$ are shown for the case $\epsilon_{r_1} = 1$ and $\epsilon_{r_2} = 2$.



Solution 2 - Perturbing capacitors

a) The capacitance of a parallel-plate capacitor is $C = \epsilon_0 S/d$, and the electric potential energy stored in it is $U = \frac{1}{2}C\Phi^2$. In the initial case we have:

$$U_i = \frac{1}{2}C_i\Phi^2 = \frac{1}{2}\frac{\epsilon_0 S}{d}\Phi^2 \tag{12}$$

whereas when the separation of the plates has been doubled we have:

$$U_f = \frac{1}{2}C_f\Phi^2 = \frac{1}{2}\frac{\epsilon_0 S}{2d}\Phi^2 \tag{13}$$

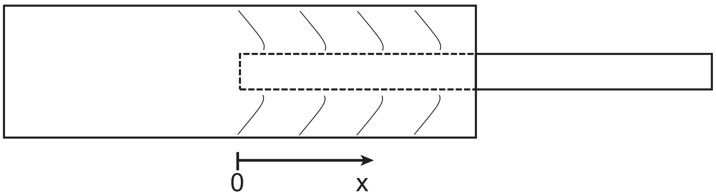
The work done in the process correspond to the opposite of the variation of potential energy, thus:

$$W = -\Delta U = -(U_f - U_i) = \frac{\epsilon_0 S}{4d}\Phi^2 \tag{14}$$

b) In this case it would be very difficult to evaluate the $\vec{E}$ field (qualitative sketch in Fig. 2), which thus is not useful to calculate the force acting on the inner cylinder. However, since $\vec{F} = (\nabla U)_\Phi$ and using the derivation from the lecture, if we call x the axis direction as shown in the figure we can write:

$$\vec{F}_x = \left(\frac{dU}{dx}\right)_\Phi = \frac{1}{2}\Phi^2\frac{dC}{dx}\hat{u}_x = \frac{\pi\epsilon_0\Phi^2}{\ln\left(\frac{R_2}{R_1}\right)}\frac{d}{dx}\left(\frac{L}{2} - x\right)\hat{u}_x = -\frac{\pi\epsilon_0\Phi^2}{\ln\left(\frac{R_2}{R_1}\right)}\hat{u}_x \tag{15}$$

where we have considered the varying length of the inner cylinder inside the outer cylinder as $(\frac{L}{2} - x)$ and we have used the expression for the capacitance of a coaxial capacitor of length ℓ: $C = \frac{2\pi\epsilon_0\ell}{\ln(R_2/R_1)}$. The inner cylinder is thus pulled inside. Here, we have not considered the (self-)capacitance of the outside part of the inner cylinder.



Solution 3 - Coalescing charges

a) The capacitance of a sphere of radius R is $C = 4\pi\epsilon_0 R$. The total amount of charge in Coulombs stored in all the droplets is:

$$Q_{tot} = \sum_k C_k\Phi_k = 4\pi\epsilon_0 R(4+5) = 36\pi\epsilon_0 R \tag{16}$$

The total volume of all the droplets is:

$$\tau_{tot} = 27\frac{4}{3}\pi R^3 = \frac{4}{3}\pi(3R)^3 \tag{17}$$

that can be seen as the volume of one single sphere of radius $3R$. By assuming that the charge and the volume are maintained during the process, the electric potential in Volts of the final big droplet will be:

$$\Phi_f = \frac{36\pi\epsilon_0 R}{4\pi\epsilon_0 3R} = 3 \tag{18}$$

b) In the initial case the electric potential energy in Joules of the system is:

$$U_i = \frac{1}{2}4\pi\epsilon_0 R\left(4^2 + 5^2\right) = 82\pi\epsilon_0 R \tag{19}$$

whereas in the final case we have:

$$U_f = \frac{1}{2}4\pi\epsilon_0 3R\left(3^2\right) = 54\pi\epsilon_0 R \tag{20}$$

Therefore the variation of internal energy in Joules of the system is:

$$\Delta U = U_f - U_i = -28\pi\epsilon_0 R \tag{21}$$

Discussion 1 - Conductivity

If the distance between atoms is different for different direction, or especially if their bonding varies as a function of direction, then the electrons can be expected to move easier in one direction as in another. A clear example of this is graphite, which is formed by stacked single atom thick layers of carbon. In the layers the bonding is strong and the conductivity high, but between the layers the bonding is weak and the conductivity lower. This weak bonding between graphite layers is used when writing with a pencil.
In the Drude model this anisotropy can be in the effective mass and in the Fermi velocity. Why these depend on the direction will be explained in electronic band theory in condensed matter physics. Beyond the Drude model there are other effects that can cause the conductivity to be anisotropic. For example quantum confinement effects or the interference of an electron with itself. All this goes well beyond the scope of this lecture and we will consider the conductivity and resistivity as a scalar.

Solution 4 - Spherical resistor

The electric field and the electric potential between two concentric charged spheres of radii $R_1 < R_2$ are:

$$E = \frac{Q}{4\pi\epsilon_0 r^2} \tag{22}$$

$$|\Phi| = \frac{Q\left(R_2 - R_1\right)}{4\pi\epsilon_0 R_1 R_2} \tag{23}$$

as evaluated in the lectures. By using the continuum form of the Ohm's law $E = \rho j$, where ρ is the resistivity and j is the current density defined by $j = I/A$, we can evaluate the current flowing between the two spheres:

$$I = j4\pi r^2 = \frac{E}{\rho}4\pi r^2 = \frac{Q}{\rho\epsilon_0} \tag{24}$$

Now we can evaluate the resistance between the two spheres:

$$R = \frac{\Phi}{I} = \frac{\rho\left(R_2 - R_1\right)}{4\pi R_1 R_2} \tag{25}$$