Solution Sheet 8

Discussion 05.11.2025

Solution 1 - Image charges

(a) The diagram of the image charges is shown on figure 1.

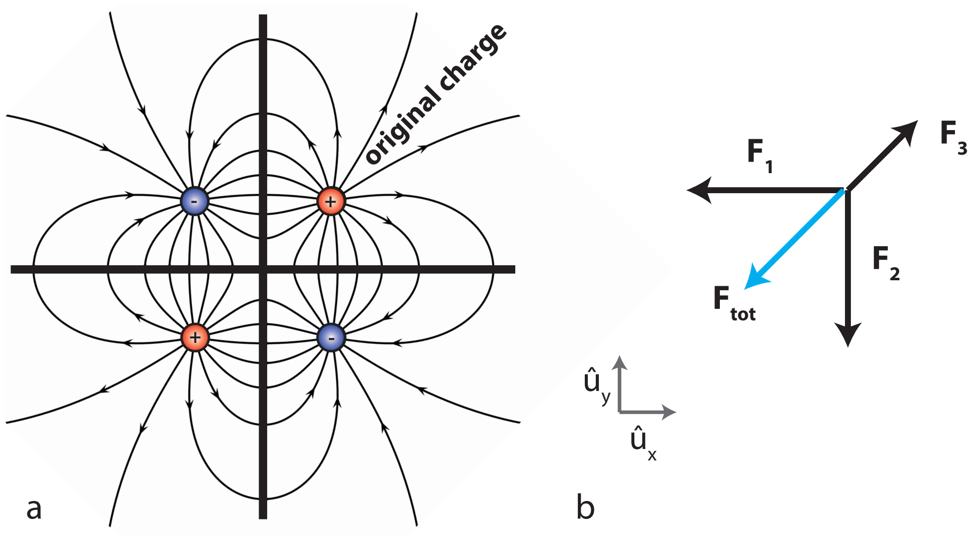

Figure 1: a) We look at the problem in two dimensions. The situation is like for a quadrupole. Note that the field lines are always perpendicular to a conducting object (here the two planes). b) Forces acting on the placed charge.

(b) The image charges create forces on the original charge, as in figure 1. For a coordinate system as defined in the figure, we can write:

$$\vec{F}_1 = \frac{k\ Q^2}{4d^2}\,(-\hat{u}_x) \tag{1}$$

$$\vec{F}_2 = \frac{k\ Q^2}{4d^2}\,(-\hat{u}_y) \tag{2}$$

$$\vec{F}_3 = \frac{k\ Q^2}{8d^2}\left(\frac{\hat{u}_x + \hat{u}_y}{\sqrt{2}}\right) \tag{3}$$

Where $k = 1/(4\pi\epsilon_0)$. Thus the total force is

$$\vec{F}_{tot} = \frac{k\ Q^2}{4d^2}\left(\frac{1}{2\sqrt{2}} - 1\right)(\hat{u}_x + \hat{u}_y) \tag{4}$$

So the force is pointing towards the intersection of the two planes, with magnitude :

$$\|\vec{F}_{tot}\| = \frac{k\ Q^2}{8d^2}\left(2\sqrt{2} - 1\right) \tag{5}$$

(c) From the lectures we know that the electric field, in cylindrical coordinates, exerted on a point at distance r from the dipole $\vec{p}$ and angle θ from the dipole axis is:

$$E_r = \frac{2\ k\ p \cos\theta}{r^3} \tag{6}$$
$$E_\theta = \frac{k\ p \sin\theta}{r^3} \tag{7}$$

We define our axis $\hat{u}_r$ in the direction of the dipole, with the origin at the dipole (see figure 2). In our case we are looking for the electric field along the dipole axis therefore we write for $r > 0$,

$$\theta = 0,\ \vec{E} = \vec{E}_r = \frac{2\ k\ p}{r^3}\hat{u}_r \tag{8}$$

and for $r < 0$,

$$\theta = 180°,\ \vec{E} = \vec{E}_r = -\frac{2\ k\ p}{r^3}\hat{u}_r \tag{9}$$

Or we can generally write

$$\vec{E}_r = \frac{2\ k\ p}{|r|^3}\hat{u}_r \tag{10}$$

The electric field of this system consists of a contribution from the dipole itself $\vec{E_p}$ and the field originating from the charges on the conducting surface that are represented by the image dipole $\vec{E_{p'}}$.

$$E_{total} = E_p + E_{p'} \tag{11}$$
$$E_{total} = 2k\left(\frac{p}{|r|^3} + \frac{p'}{|2d+r|^3}\right)\hat{u}_r \tag{12}$$
$$E_{total} = 2kp\left(\frac{1}{|r|^3} + \frac{1}{|2d+r|^3}\right)\hat{u}_r \tag{13}$$

Using the relation $p = p'$. This expression for the electric field is correct for $r > -d$ or in words for points above the surface.

(d) The general expression for the potential energy of a dipole $\vec{p}$ in an electric field $\vec{E}$ is:

$$U = -\vec{p}\cdot\vec{E} \tag{14}$$

In this particular exercise, we have the E-field generated by the image dipole, acting on the "real" dipole. We write the energy as:

$$U = -\vec{p}_{\text{"real"}}\cdot\vec{E}_{\text{"image"}} = -p\ \frac{2\ k\ p'}{|r|^3} \tag{15}$$

Knowing the distance between the two dipoles $|r| = 2d$ and $p = p'$

$$U = -\frac{2\ k\ p^2}{(2d)^3} = -\frac{k\ p^2}{4\ d^3} \tag{16}$$

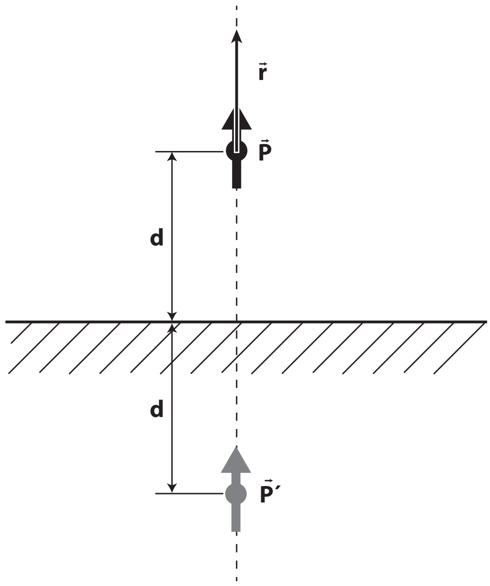

Figure 2: Sketch of the dipole above the surface and its image dipole

Discussion 1 - Dipole moment

For the perpendicular orientation, the image dipole moment is oriented in the same direction as the original dipole. In the case that it is oriented parallel to the plane the image dipole moment will be opposite. This can be easily realised by reversing the sign of the charges when making the image. These two results indicate that the electric dipole moment is a pseudovector; i.e. its direction is reversed when performing a mirror operation. A typical pseudovector is the angular momentum, but, in contrast to the magnetic dipole moment we will discuss later, for the electric dipole moment there is no angular rotation involved. What breaks the symmetry and makes it a pseudovector is the opposite charges that compose the dipole moment. Thus the dipole moment is composed of two ingredients; the normal vector describing the position of the charges and the relative position of the positive and negative charges.

Solution 2 - Capacitance of wires

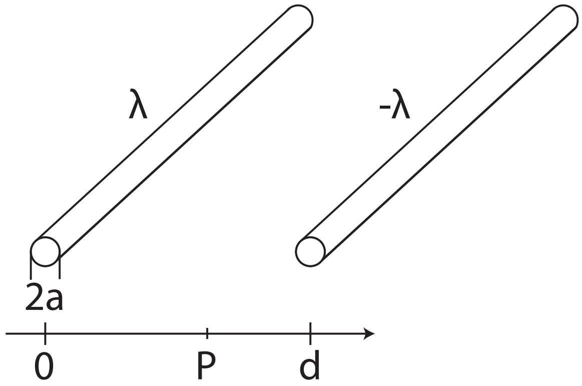

Figure 3: Situation sketch of the two wires.

(a) We already know (or distinguish with Gauss's law) that the E-field for a single wire is

$$\vec{E}(r) = \frac{2\ k\ \lambda}{r}\hat{u}_r \tag{17}$$

where λ is the charge line density. At a point r between between two wires separated by a distance d, with charge density λ and $-\lambda$, the electric field is by superposition:

$$\vec{E}(r) = \left(\frac{2\ k\ \lambda}{r} + \frac{2\ k\ \lambda}{d-r}\right)\hat{u}_r \tag{18}$$

Note that the negativ sign of $-\lambda$ cancels out due to $r_{-\lambda} = -\hat{u}_r$. We now distinguish the potential difference $\Delta\Phi$ over the gap between the two wires by integration.

$$\Delta\Phi = -\int_a^{d-a} E(r)dr = 2\ k\ \lambda \int_a^{d-a}\left(-\frac{1}{r} - \frac{1}{d-r}\right)dr = -4\ k\ \lambda \ln\frac{d-a}{a} \tag{19}$$

Now the capacitance per unit length is found:

$$\frac{C}{L} = \frac{Q}{L\,|\Delta\Phi|} = \frac{\lambda L}{L\,|\Delta\Phi|} = \frac{\pi\epsilon_0}{\ln\frac{d-a}{a}} \simeq \frac{\pi\epsilon_0}{\ln\frac{d}{a}} \tag{20}$$

where we have used that $a \ll d$.

(b) As in exercise 3 with image charges, we place an image wire at distance $-d$ from the ground. Therefore the potential difference between the real wire at $+d$ and the image wire at $-d$ is, using eq. 19 of part a),

$$|\Delta\Phi_{\pm}| \simeq 4\ k\ \lambda \ln\frac{2d}{a} \tag{21}$$

However what we want is the potential difference to the ground, not between the two wires. Given the symmetry of the problem the integral from one wire or the other to the ground is the same. Therefore:

$$|\Delta\Phi| = \frac{1}{2}\,|\Delta\Phi_{\pm}| \simeq 2\ k\ \lambda \ln\frac{2d}{a} \tag{22}$$

And finally the capacitance per unit length:

$$\frac{C}{L} \simeq \frac{2\pi\epsilon_0}{\ln\frac{2d}{a}} \tag{23}$$

Solution 3 - Barium titanate

(a) If we look at the unit cell, we see 6 O^{2-} ions. Each one of them is shared with the next unit cell, so only half is inside. This means we get a contribution of $6 \cdot 1/2 \cdot -2e$ charge from the Oxygen ions. The 8 Ba^{2+} ions at the corners are shared between 8 unit cells each. So their contribution is $8 \cdot 1/8 \cdot +2e$. So the total charge is given as:

$$Q_{virtual\ \ ion} = (6 \cdot 1/2 \cdot -2e) + (8 \cdot 1/8 \cdot +2e) = -4e \tag{24}$$

(b) This virtual charge forms together with the displaced Ti^{4+} ion a dipole. We know all the parameters to distinguish the dipole moment of the unit cell.

$$\vec{p} = q \cdot \vec{d} \tag{25}$$

$$|\vec{p}| = p = q \cdot d \tag{26}$$

$$p = 4e \cdot 0.012\ nm = 4 \cdot \left(1.6 \cdot 10^{-19}\right) C \cdot \left(0.012 \cdot 10^{-9}\right) m \tag{27}$$

$$p = 7.68 \cdot 10^{-30}\, Cm \tag{28}$$

(c) From the lecture we know:

$$\vec{P} = \frac{\vec{p}}{d\tau} \tag{29}$$

From the figure c) in the exercise we read the polarization of bariumtitanate at room temperature:

$$\left|\vec{P}\right| = 15 \cdot 10^{-2}\ \frac{C}{m^2} \tag{30}$$

And from the measures in figure a) we can distinguish the volume of a unit cell

$$d\tau = 0.403 \cdot 0.398 \cdot 0.398\ nm^3 = 6.3837 \cdot 10^{-29}\ m^3 \tag{31}$$

With this numbers we can distinguish the dipole moment of a unit cell to be:

$$|\vec{p}| = \left|\vec{P}\right| \cdot d\tau = 15 \cdot 10^{-2}\ \frac{C}{m^2} \cdot 6.3837 \cdot 10^{-29}\ m^3 = 9.57 \cdot 10^{-30}\ Cm \tag{32}$$

This result is very close to the dipole moment we calculated in part b). The difference can be explained by additional contribution from the electrons to the dipole moment of the unit cell. But this is beyond the scope of this course.