Solution Sheet 5

Discussion 8.10.2025

Solution 1 - Longitudinal Poiseuille flow in a ring gap

Symmetry + continuity equation $\rightarrow \vec{v} = (0, 0, v_x(r))$ and $p(r, x)$.

Simplifying the Navier-Stokes equation (steady flow, no gravity, continuity equation + incompressible fluid) we find

$$-\nabla p + \eta \nabla^2 \vec{v} = 0$$

r component: $-\frac{1}{r}\partial_r p = 0$

φ component: $0 = 0$

x component $-\partial_x p + \frac{\eta}{r}\partial_r(r\partial_r v_x) = 0$

The pressure does not depend on r. Therefore each term in the x component equation depends on a different variable, such that we can write:

$$\partial_x p(x) = -K = \frac{\eta}{r}\partial_r(r\partial_r v_x)$$

As expected, the expression for the pressure is :

$$p(x) = -Kx + p_0$$

The other equation gives

$$-K = \frac{\eta}{r}\partial_r(r\partial_r v_x)$$

$$-\frac{K}{\eta}r = \partial_r(r\partial_r v_x)$$

$$-\frac{K}{2\eta}r^2 + C_1 = r\partial_r v_x$$

$$\partial_r v_x = -\frac{K}{2\eta}r + \frac{C_1}{r}$$

$$v_x(r) = -\frac{K}{4\eta}r^2 + C_1 \ln(r/r_0) + C_2$$

where we have used a reference length r_0 to express the ln.
With the no-slip conditions: $v_x(R_1) = v_x(R_2) = 0$:

$$0 = -\frac{K}{4\eta}R_1^2 + C_1 \ln(R_1/r_0) + C_2$$

$$0 = -\frac{K}{4\eta} R_2^2 + C_1 \ln(R_2/r_0) + C_2$$

Taking the difference between the two expressions we find

$$C_1 = -\frac{K}{4\eta}(R_2^2 - R_1^2)\frac{1}{\ln(R_1/R_2)}$$

Re-injecting C_1 in the expression for $v(R_2) = 0$

$$C_2 = \frac{K}{4\eta} R_2^2 + \frac{K}{4\eta}(R_2^2 - R_1^2)\frac{\ln(R_2/r_0)}{\ln(R_1/R_2)}$$

Replacing for C_1 and C_2 we find

$$v_x(r) = \frac{K}{4\eta}\left[R_2^2 - r^2 - (R_2^2 - R_1^2)\frac{\ln(r/R_2)}{\ln(R_1/R_2)}\right]$$

One can easily verify that this expression corresponds to the usual Poiseuille flow in a pipe of radius R_2 for $R_1 \to 0$.

Solution 2 - Charge Density

a) The total charge on the surface of the sheet is given by

$$Q = \int_0^{L_y} \int_0^{L_x} (x^2 y^3 + ln(x))\ dx\ dy$$

We first solve the inner integral,

$$\int_0^{L_x} (x^2 y^3 + ln(x))\ dx = y^3 \int_0^{L_x} x^2\ dx\ +\ \int_0^{L_x} ln(x)\ dx$$

for the left side:

$$y^3 \int_0^{L_x} x^2\ dx = y^3 \cdot \frac{1}{3} x^3 \Big|_0^{L_x}$$

for the right side we use integration by parts with $u(x) = ln(x)$, $dv = dx$ and we evaluate $du = 1/x\ dx$ and $v(x) = x$:

$$\int ln(x)\ dx = \int u\ dv = uv - \int v\ du$$

we substitute $u = ln(x)$, $v = x$ and $du = 1/x\ dx$:

$$\begin{aligned}\int ln(x)\ dx &= uv - \int v\ du \\ &= ln(x)\ x - \int x \cdot \frac{1}{x}\ dx \\ &= ln(x)\ x - \int\ dx \\ &= ln(x)\ x - x + C\end{aligned}$$

With boarders:

$$\int_0^{L_x} ln(x)\ dx = ln(x)\ x\Big|_0^{L_x} - x\Big|_0^{L_x}$$

So the inner integral writes as

$$\int_0^{L_x} (x^2y^3 + ln(x))\ dx = y^3 \cdot \frac{1}{3}x^3\Big|_0^{L_x} + ln(x)\ x\Big|_0^{L_x} - x\Big|_0^{L_x}$$

Note: Since $ln(0) = -\infty$ the solution for $x \cdot ln(x)$ is not defined for $x = 0$. However we can still solve the integral in the given boarders, since the limit $lim_{x\to 0}\,(x \cdot ln(x)) = 0$ is defined. To proof $lim_{x\to 0}\,(x \cdot ln(x)) = 0$ we use the rule of l'Hopital:

$$\begin{aligned}&\text{if } lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{0}{0} \text{ or if } lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{\pm\infty}{\pm\infty} \text{ then} \\ &lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = lim_{x\to a}\left(\frac{f'(x)}{g'(x)}\right)\end{aligned}$$

So we can apply this to our problem:

$$\begin{aligned}lim_{x\to 0}\,(x \cdot ln(x)) &= lim_{x\to 0}\left(\frac{ln(x)}{\frac{1}{x}}\right) \\ &= lim_{x\to 0}\left(\frac{\frac{1}{x}}{-\frac{1}{x^2}}\right) \\ &= lim_{x\to 0}\,(-x) \\ &= 0\end{aligned}$$

Now, we continue with the outer integral

$$\int_0^{L_y} \left(y^3 \cdot \frac{1}{3}x^3\Big|_0^{L_x} + ln(x)\ x\Big|_0^{L_x} - x\Big|_0^{L_x} \right) dy = \frac{1}{4}y^4\Big|_0^{L_y} \cdot \frac{1}{3}x^3\Big|_0^{L_x} + y\Big|_0^{L_y} \cdot ln(x)\ x\Big|_0^{L_x} - y\Big|_0^{L_y} \cdot x\Big|_0^{L_x}$$
$$= \frac{1}{4}L_y^4 \cdot \frac{1}{3}L_x^3 + L_y \cdot ln(L_x)\ L_x - L_y \cdot L_x$$
$$= L_x L_y \left(\frac{1}{12}L_x^2 L_y^3 + ln(L_x) - 1 \right)$$

substituting the dimensions of the sheet, $L_x = 0.6\ m$ and $L_y = 0.5\ m$, we have

$$Q = \int_0^{L_y} \int_0^{L_x} (x^2y^3 + ln(x))\ dx\ dy$$
$$= \left[0.6 \cdot 0.5 \left(\frac{1}{12}0.6^2 0.5^3 + ln(0.6) - 1 \right) \right]$$
$$= -0.4521\ C$$

b) For a disk of radius R, the mean charge density on its surface can be computed by

$$\overline{\sigma_{disk}} = \frac{\textit{Total charge on disk}}{\textit{Disk area}} = \frac{Q(R)}{\pi \cdot R^2}$$

The total charge on the disk, in the polar coordinate system, is given by the expression

$$Q = \int_0^{2\pi} \int_0^R r \cdot \sigma(r, \theta)\ dr\ d\theta$$

where σ is the charge distribution of the disk.

Then, Q needs to be computed in order to find $\overline{\sigma_{disk}}(R)$. The double integral to solve is the following

$$Q(R) = \int_0^{2\pi} \int_0^R r \cdot e^r\ dr\ d\theta$$

The inner most integral can be solved by parts, using the following substitutions: $u = r$,
$du = dr$, $v = e^r$ and $dv = e^r\ dr$. That is

$$\int u\, dv = u\, v - \int v\, du$$
$$\int r\, e^r\, dr = r\, e^r - \int e^r\, dr$$
$$= e^r\,(r-1)$$

Now, the outer integral can be solved

$$\int_0^{2\pi} \Big[\, e^r\,(r-1)\Big]_0^R\, d\theta = \int_0^{2\pi} (\, e^R\,(R-1)+1\,)\, d\theta$$
$$= (\, e^R\,(R-1)+1\,)\cdot \theta\Big|_0^{2\pi}$$
$$= 2\pi\,(\, e^R\,(R-1)+1\,)$$

The mean charge density is:
$\overline{\sigma_{disk}}(R) = \frac{2\pi\,(\,e^R\,(R-1)+1\,)}{\pi\, R^2} = \frac{2}{R^2}\Big(\, e^R\,(R-1)+1\Big)\ \frac{C}{m^2}$

Solution 3 - Point charges: triangle

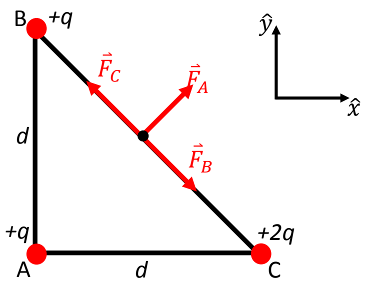

Figure 1: The forces only indicate the direction

The Coulomb force acting on a charge q_0 fixed in the midpoint of the hypotenuse will be given by the contributions of the three charges in A, B and C considerated separately because of the superposition principle. Therefore, we consider a coordinate system as shown in Fig. 1, and evaluate the three forces:

$$\vec{F}_A = \frac{1}{4\pi\epsilon_0}\frac{q q_0}{(d\sqrt{2}/2)^2}\frac{\hat{x}+\hat{y}}{\sqrt{2}} = \frac{\sqrt{2} q q_0}{4\pi\epsilon_0 d^2}(\hat{x}+\hat{y})$$
$$\vec{F}_B = \frac{1}{4\pi\epsilon_0}\frac{q q_0}{(d\sqrt{2}/2)^2}\frac{\hat{x}-\hat{y}}{\sqrt{2}} = \frac{\sqrt{2} q q_0}{4\pi\epsilon_0 d^2}(\hat{x}-\hat{y})$$
$$\vec{F}_C = \frac{1}{4\pi\epsilon_0}\frac{2 q q_0}{(d\sqrt{2}/2)^2}\frac{-\hat{x}+\hat{y}}{\sqrt{2}} = \frac{2\sqrt{2} q q_0}{4\pi\epsilon_0 d^2}(-\hat{x}+\hat{y})$$

Therefore we can evaluate the total force

$$\vec{F} = \vec{F}_A + \vec{F}_B + \vec{F}_C = \frac{2\sqrt{2}qq_0}{4\pi\epsilon_0 d^2}\hat{y}$$

It has to be noted that the calculation would be a bit easier by choosing a coordinate system rotated by 45°.

Discussion 1 - E-field Lines and Equipotential surfaces

The E-field lines are perpendicular to the equipotential lines. The E-field lines are from +15V object (high potential) to the +10V object (low potential), so between the objects, the +15V object is positively charged at the surface and +10V object is negatively charged at the surface. In addition, as both objects carry net positive charge, besides the E-field lines connecting these two objects, each of them should have other E-field lines which go to infinity.

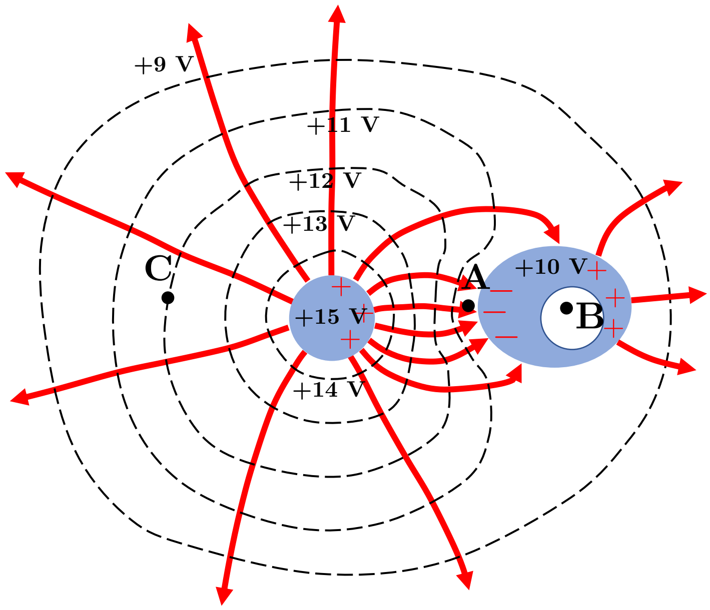

Figure 2: Equipotential maps and E-field lines around charged objects.

Solution 4 - Unstable Uranium Core

a) From the course, we know that the potential energy between two charges Q_1 and Q_2 separated by a distance d is given by :

$$U = \frac{Q_1 \cdot Q_2}{4\pi\epsilon_0 d} \quad (1)$$

In the case of an equal separation of the core, the two equal parts both have a charge equal to half the number of initial protons.

$$Q_1 = Q_2 = \frac{92e^+}{2} = 46e^+ \approx 7.4 \times 10^{-18} \tag{2}$$

The distance separating both equal parts can be estimated as the nucleus radius r, which can be approximated as $r \approx 10^{-14}$m. With the charges described above, we obtain :

$$U = 9 \cdot 10^9 \cdot \frac{54 \cdot 10^{-36}}{1 \cdot 10^{-14}} \approx 4.9 \times 10^{-11}\text{J/atom} \tag{3}$$

This is the energy for one single atom of uranium.

Using Avogadro's number, we find that 1 gram of Uranium-235 contains :

$\frac{N_A}{235} \approx 2.5 \times 10^{21}$atoms.

The total energy per gram is then $E = 1.2 \times 10^{11}$J

b) By converting the energy in J/gram instead of MJ/kg, we obtain that the energy produced from the nuclear reactor is 8×10^{10}J/gram. The two energies are of the same order of magnitude.