Solution Sheet 4

Discussion 01.10.2025

Solution 1 - Magnus effect

Pressure forces on the cylinder:

$$\vec{F}_p = -\int_S p\, d\vec{S} \tag{1}$$

where S is the surface of the cylinder and $d\vec{S} = (\mathrm{d}S\,\cos\phi)\,\hat{e}_x + (dS\,\sin\phi)\,\hat{e}_y$ with $dS = R\,d\phi\,dz$.
The lift force is the y-component of this force:

$$F_{p,y} = -\int_0^L \int_0^{2\pi} pR\sin\phi\ d\phi\ \ dz \tag{2}$$

To find the expression of p as a function of R, ϕ, we use Bernoulli's theorem along a streamline from ∞ to R, ϕ, neglecting the height difference between the top and bottom of the cylinder:

$$p_\infty + \frac{1}{2}\rho v_\infty^2 = p(R,\phi) + \frac{1}{2}\rho v^2(R,\phi) \tag{3}$$

Using the given expression for the velocity field we obtain for $r = R$ (surface of the cylinder):

$$v(R,\phi) = -\left(2v_\infty \sin\phi + \omega R\right) \tag{4}$$

This expression shows that the velocities add up in the upper part of the cylinder ($0 < \phi < \pi$) and subtract in the lower part ($\pi < \phi < 2\pi$). We find the expression for the pressure $p(R,\phi)$:

$$p(R,\phi) = p_\infty + \frac{1}{2}\rho v_\infty^2 - \frac{1}{2}\rho\omega^2 R^2 - 2\rho v_\infty^2(\sin\phi)^2 - 2\rho v_\infty\,\omega R\,\sin\phi \tag{5}$$

Only the 4th $\sin^2$ term gives a non-zero contribution in the calculation of $F_{p,y}$:

$$F_{p,y} = -LR\int_0^{2\pi} -2\rho v_\infty(\sin\phi)^2\omega R\, d\phi = 2\rho\omega R^2 L v_\infty \int_0^{2\pi}(\sin\phi)^2 d\phi = 2\pi\omega R^2\rho L v_\infty \tag{6}$$

Note that the line path of the velocity around a circular loop at the surface of the cylinder (circulation) is

$$\Gamma = \left|\oint \vec{v}\cdot d\vec{s}\right| = \left|\int_0^{2\pi}\left[-\left(2v_\infty\sin\phi + \omega R\right)\right] R\, d\phi\right| = 2\pi\omega R^2 \tag{7}$$

such that the lift force can be expressed as

$$F_{p,y} = \Gamma\rho L v_\infty \tag{8}$$

Solution 2 - Reynolds number

a) The Reynolds number $\mathcal{R} = \dfrac{\rho}{\eta} vL$ must be the same.

For the 1:4 model in air $v = 120$ m/s.
For the 1:4 model in water $v = 12$ m/s.

b) The Reynolds number is the ratio of inertial forces to viscous forces within a fluid which is subjected to relative internal movement due to different fluid velocities. The Reynolds number can be used to characterize the stability of flows (laminar or turbulent regime) and to study flows on models (similarity). In the latter case, it allows to determine the mechanical effects on the model obstacle and the characteristics of the flow. For example, one can use a wind-tunnel to determine the flow around and past a model airplane. For a given model size, the use of a fluid with lower specific viscosity η/ρ allows to reduce the velocity to obtain the same Reynolds number.
Note also that, depending on the systems, it is not possible to find a fluid that satisfies the requirements on specific viscosity, and consequently the Reynolds number cannot be matched.
The limitations result from the main hypothesis under which the Reynolds number has been derived, i.e. the fluid incompressibility: at high velocities the compressibility cannot be neglected anymore. For instance, air compressibility cannot be neglected when dealing with airplane models.

Discussion 1 - Drag and boundary layer

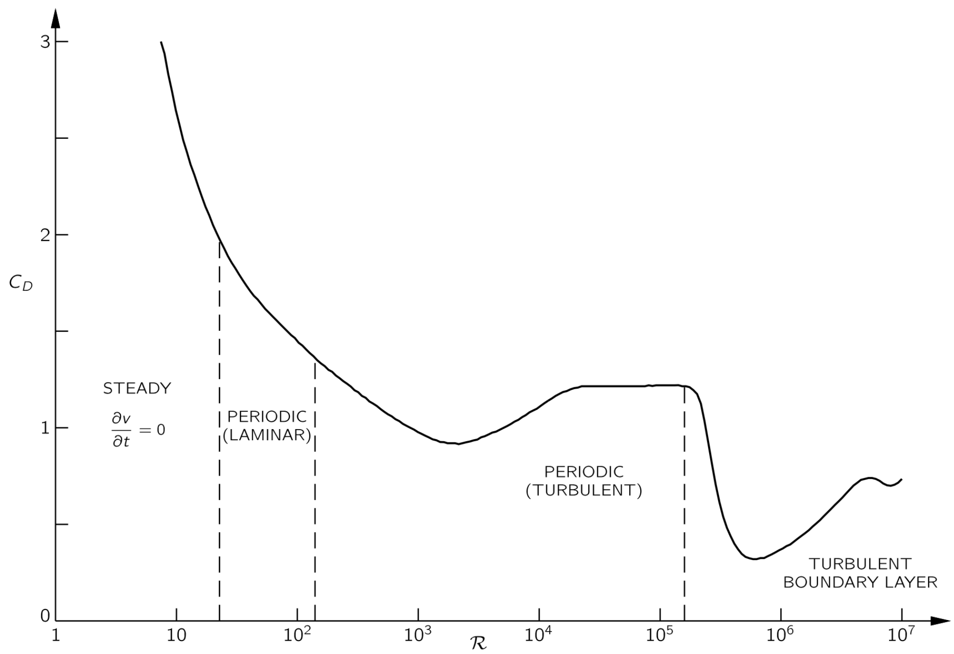


Fig. 41–4. The drag coefficient C_D of a circular cylinder as a function of the Reynolds number.

As shown in the figure above, the drag force is lower in the turbulent boundary layer regime.

The separation takes place farther back in this regime, resulting in a narrower wake behind the object.
Golf balls are deliberately dimpled to induce a thin turbulent boundary layer at the surface of the ball. This lowers the drag and the ball can fly farther compared to a smooth ball.

Solution 3 - Planar flows between two plates

From the hypothesis, valid for both cases:
- incompressible fluid $\longrightarrow \rho = \text{cte}$;
- continuity equation: $\dfrac{\partial \rho}{\partial t} + \rho \nabla \cdot \vec{v} = 0 \longrightarrow \nabla \cdot \vec{v} = 0$;
- steady flow: $\dfrac{\partial \vec{v}}{\partial t} = 0$.

Considerations on geometry, symmetry and invariance allow us to determine the non-zero components of the velocity and their dependencies:
- y direction does not enter into the problem, so the y component of the velocity is zero, $v_y = 0$, and there will be no dependence on y for the physical quantities;
- in steady state, the streamlines are parallel to x, so $v_z = 0$;
- there is translational invariance in the x direction, so v_x does not depend on x.

In summary:

$$\vec{v} = (v_x(z), 0, 0) = v_x(z)\hat{e}_x \tag{9}$$

The Navier-Stokes equation

$$\rho \left(\frac{\partial \vec{v}}{\partial t} + \vec{\Omega} \times \vec{v} + \frac{1}{2} \nabla v^2 \right) = -\nabla p - \rho \nabla \phi + \eta \nabla^2 \vec{v} + (\eta + \eta^*) \nabla (\nabla \cdot \vec{v}) \tag{10}$$

Simplifying (steady, no gravity, incompressible)

$$\rho \left(\vec{\Omega} \times \vec{v} + \frac{1}{2} \nabla v^2 \right) = -\nabla p + \eta \nabla^2 \vec{v} \tag{11}$$

For the celocity components and dependency derived above, the term on the left is $= 0$ (verify that $\vec{\Omega} \times \vec{v} = -\frac{1}{2} \nabla v^2$, or equivalently that $(\vec{v} \cdot \nabla)\vec{v} = 0$).

Therefore, in this case the Navier-Stokes equation can be expressed as:

$$\nabla p = \frac{\eta}{\rho} \nabla^2 \vec{v} \tag{12}$$

a) $\vec{v} = v_x(z)\hat{e}_x$ and $\partial p / \partial x = 0$

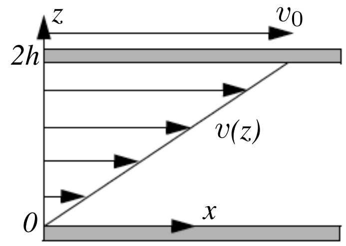
Note that only the projection along x is non-trivial. In this case we find:

$$\eta \frac{\partial^2 v_x}{\partial z^2} = 0 \tag{13}$$

Boundary conditions (no-slip): $v_x(z = 0) = 0$ and $v_x(z = 2h) = v_0$

$$\vec{v}(z) = v_x(z)\hat{e}_x = \frac{v_0}{2h} z\, \hat{e}_x \tag{14}$$

This is the Couette planar flow.



For this geometry it is intuitive to see that $v_{avg} = v_0/2$.

More formally, one can calculate v_{avg} considering the flow rate over a cross section. Since in this case the systems extends indefinitely along y, we consider the flow rate Q across an arbitrary section of area $A = a \cdot 2h$:

$$v_{avg} = \frac{Q}{A} = \frac{1}{A} \iint_A v_x(z)\, dA = \frac{1}{a\, 2h} \int_0^a dx \int_0^{2h} v_x(z)\, dz = \frac{1}{2h} \int_0^{2h} \frac{v_0}{2h} z\, dz = \frac{v_0}{(2h)^2} \frac{1}{2} z^2 \Big|_0^{2h} = \frac{v_0}{2} \tag{15}$$

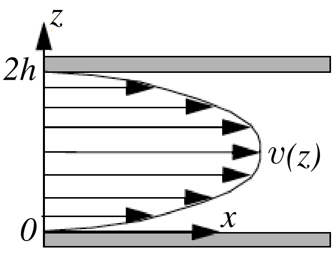
b) $\vec{v} = v_x(z)\hat{e}_x$ et $\partial p / \partial x = -K$

Again, only the projection along x is non trivial, and we obtain:

$$\eta \frac{\partial^2 v_x}{\partial z^2} = -K \tag{16}$$

Boundary conditions (no-slip): $v_x(z = 0) = 0$ and $v_x(z = 2h) = 0$

$$\vec{v}(z) = \frac{K}{2\eta}(2h - z) z \vec{e}_x \tag{17}$$



This is the Poiseuille planar flow.

Calculation of v_{avg} :

$$v_{avg} = \frac{Q}{A} = \frac{1}{2h} \int_0^{2h} v_x(z)\, dz = \frac{1}{2h} \int_0^{2h} \frac{K}{2\eta}(2h - z) z\, dz = \frac{K}{2\eta} h^2 \frac{2}{3} = \frac{2}{3} v_{max} \tag{18}$$

where in the last step we have used the fact that the velocity is maximum at mid-height (for $z = h$).