Exercise Sheet 2

Discussion 17.09.2025

Exercise 1 - Jet d'eau

We call A be the basis of the water jet and B its top point.

a) We can solve the problem with the Bernoulli equation :

$$\text{in A: } \rho g \cdot 0 + p_{atm} + \rho\frac{v^2}{2} = p_{atm} + \rho\frac{v^2}{2};$$

$$\text{in B: } \rho g \cdot h + p_{atm} + \rho\frac{0^2}{2} = p_{atm} + \rho g h.$$

So,

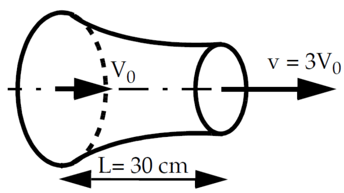
$$p_{atm} + \rho\frac{v^2}{2} = p_{atm} + \rho g h \ ;$$

$$\Rightarrow \quad v = \sqrt{2gh} \quad \Rightarrow \quad v = 52\ m/s\ .$$

b) Right at the exit of the nozzle, the pressure applied on the fluid is no other than the atmospheric pressure. Because the pressures balance out, there is only the atmospheric pressure in the water jet. To be convinced of that, if the water pressure was higher that p_{atm}, the fluid should laterally expand and make an "explosion" !

Discussion 1 - Steady flow and acceleration

One can take as an example a one-dimensional flow inside a pipe as shown in the figure. Let's suppose the velocity varies linearly from V_0 to $3\,V_0$. The velocity can be written as:



$$v(x,t) = V_0(1 + \frac{2x}{L}) \quad (1)$$

The acceleration of a fluid particle in one-dimension is:

$$a = \frac{\mathrm{d}v(x,t)}{\mathrm{d}t} = \frac{\partial}{\partial t}v(x,t) + v(x,t)\frac{\partial}{\partial x}v(x,t) = 0 + V_0(1+\frac{2x}{L})\frac{2\,V_0}{L} = \frac{2\,V_0^2}{L}(1+\frac{2x}{L}) \quad (2)$$

The acceleration is not zero here because of the change in section through the pipe leading to a convective acceleration $v(x,t)\frac{\partial v(x,t)}{\partial x} \neq 0$. A steady flow is defined as $\frac{\partial v(x,t)}{\partial t} = 0$ which means that the local acceleration of particles that pass by a given point x is nil: every particles has the same acceleration in x but the individual acceleration of these particles can also not be zero as here $a \neq 0$.

Exercise 2 - Tank with a hole

a) The hole being small ($r \ll R$) the flow can be considered as quasi-steady and Bernoulli's theorem can be applied along a streamline going from the fluid surface (height z) to the hole ($z = 0$):

$$p_{\text{atm}} + \rho g z + \frac{1}{2}\rho v_z^2 = p_{\text{atm}} + 0 + \frac{1}{2}\rho v^2 \tag{3}$$

with v_z the velocity at the fluid surface and v the velocity at the hole level.

Conservation of mass flow rate:

$$\rho v_z \pi R^2 = \rho v \pi r^2 \tag{4}$$

Assuming that $R \gg r$, one has $v_z \ll v$ which means that v_z can be neglected in equation 3. The fluid velocity at the hole is then:

$$v = \sqrt{2gz}. \tag{5}$$

The velocity v is constant over the hole section πr^2 because the fluid is ideal. The flow rate coming out from the hole is therefore $Q = v\pi r^2$.
This rate is related to the decrease in height of the fluid:

$$-\pi R^2 \frac{\mathrm{d}z}{\mathrm{d}t} = \pi r^2 \sqrt{2gz}. \tag{6}$$

Integrating this equation leads to the time evolution of the fluid height $z(t)$:

$$\frac{\mathrm{d}z}{\sqrt{z}} = -\left(\frac{r}{R}\right)^2 \sqrt{2g}\mathrm{d}t \implies 2\sqrt{z} + C = -\left(\frac{r}{R}\right)^2 \sqrt{2g}\, t. \tag{7}$$

The constant of integration C is determined by the initial condition $z(t = 0) = h$ and equals $C = -2\sqrt{h}$.

The time T_0 necessary to empty the tank is therefore:

$$T_0 = \sqrt{\frac{2h}{g}} \frac{R^2}{r^2}. \tag{8}$$

b) Now if there is no condition on r ($r \ll R$ not verified), v_z is not neglected and combining Bernoulli's theorem with the equation $v_z = v\, r^2/R^2$ leads to a corrective term:

$$v = \sqrt{\frac{2gz}{1 - (r^2/R^2)^2}}; \qquad 2(\sqrt{z} - \sqrt{h}) = -\frac{r^2}{R^2}\sqrt{\frac{2gz}{1 - (r^2/R^2)^2}}\, t \tag{9}$$

For T_0 we find :

$$T_0 = \sqrt{\frac{2h}{g}} \frac{R^2}{r^2} \sqrt{1 - \left(\frac{r^2}{R^2}\right)^2} \tag{10}$$

Exercise 3 - Egyptian water clock

Conditions are fulfilled to apply Bernoulli's theorem between $z = 0$ and z, one can write:

$$p_{\text{atm}} + \frac{1}{2}\rho v_z^2 + \rho g z = p_{\text{atm}} + \frac{1}{2}\rho v_0^2 \tag{11}$$

The equation of continuity is given by:

$$S_z v_z = S_0 v_0 \tag{12}$$

where S_z and S_0 are the revolution sections of the vessel in z and at the hole level respectively.

By combining these equations, the level position as a function of the velocity and the geometry is:

$$z = \frac{v_z^2}{2g}\left(\frac{S_z^2}{S_0^2} - 1\right) \tag{13}$$

As the fluid level decreases at equal time intervals with respect to equidistant graduations on the axis, one must have $v_z = v = cste$.

Since the vessel has a revolution symmetry around z, i.e. $S_z = \pi(r(z))^2$, the equation describing the vessel geometry is, with $S_0 = \pi r_0^2$:

$$r(z) = r_0\left(\frac{2gz}{v^2} + 1\right)^{\frac{1}{4}} \tag{14}$$

Exercise 4 - Vorticity vector

a) $\vec{v} = (v_x(z), 0, 0)$, with $v_x(z) = Cz$, in cartesian coordinates (x, y, z)

$$\nabla \times \vec{v} = (\partial_y v_z - \partial_z v_y, \partial_z v_x - \partial_x v_z, \partial_x v_y - \partial_y v_x) = (0, \partial_z v_x, 0) = (0, C, 0)$$

$\vec{\Omega} = \nabla \times \vec{v} = (0, C, 0)$. The flow is rotational. There is not global rotation of the fluid but there is a local rotation of the particles. The vorticity vector is in the y direction and it corresponds to the rotation axis of particles "rolling" in the flow direction x.

b) $\vec{v} = (0, v_\phi(r), 0)$, with $v_\phi(r) = \omega r$, in cylindrical coordinates in the basis $(\hat{e}_r, \hat{e}_\phi, \hat{e}_z)$

$$\nabla \times \vec{v} = (\frac{1}{r}\partial_\phi v_z - \partial_z v_\phi, \partial_z v_r - \partial_r v_z, \frac{1}{r}\partial_r(r v_\phi) - \frac{1}{r}\partial_\phi v_r) = (0, 0, \frac{1}{r}\partial_r(r v_\phi)) = (0, 0, \frac{1}{r}\partial_r(\omega r^2)) = (0, 0, 2\omega)$$

$\vec{\Omega} = (0, 0, 2\omega)$. This flow is rotational. There is a global rotation of the fluid and a local rotation of the particles around their own axis. The vorticity vector is in the z direction, it is the particles rotation axis. This flow corresponds to the rotation of a solid with angular frequency ω.

c) $\vec{v} = (0, v_\phi(r), 0)$, with $v_\phi(r) = C/r$, in cylindrical coordinates in the basis $(\hat{e}_r, \hat{e}_\phi, \hat{e}_z)$

$$\nabla\times\vec{v} = (\frac{1}{r}\partial_\phi v_z - \partial_z v_\phi, \partial_z v_r - \partial_r v_z, \frac{1}{r}\partial_r(rv_\phi) - \frac{1}{r}\partial_\phi v_r) = (0, 0, \frac{1}{r}\partial_r(rv_\phi)) = (0, 0, \frac{1}{r}\partial_r(r\frac{C}{r})) = (0, 0, 0)$$

Here, $\vec{\Omega} = \vec{0}$. The flow is irrotational. There is a global rotation of the fluid but no local rotation of the particles.

Below: representation of the velocity fields; for c), vectors are not shown close to the origin (divergence). Squares represent fluid particles and their motion in the flow.