Solution Sheet 1

Discussion 10.09.2025

Solution 1 - Vector calculus with ∇

a) $\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$

b) $\nabla \times \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z}$

c) Because ∇ is an operator you can't change the order

$$(\vec{A} \cdot \nabla)\vec{A} = \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \vec{A} = (A_x \frac{\partial}{\partial x} + A_y \frac{\partial}{\partial y} + A_z \frac{\partial}{\partial z})\vec{A} = \begin{bmatrix} A_x \frac{\partial A_x}{\partial x} + A_y \frac{\partial A_x}{\partial y} + A_z \frac{\partial A_x}{\partial z} \\ A_x \frac{\partial A_y}{\partial x} + A_y \frac{\partial A_y}{\partial y} + A_z \frac{\partial A_y}{\partial z} \\ A_x \frac{\partial A_z}{\partial x} + A_y \frac{\partial A_z}{\partial y} + A_z \frac{\partial A_z}{\partial z} \end{bmatrix}$$

d) zero (0); the curl of a gradient of a scalar field is always zero, but is a good exercise to write it out.

$$\nabla \times (\nabla B) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial B}{\partial x} & \frac{\partial B}{\partial y} & \frac{\partial B}{\partial z} \end{vmatrix} = \begin{bmatrix} \frac{\partial^2 B}{\partial z \partial y} - \frac{\partial^2 B}{\partial y \partial z} \\ \frac{\partial^2 B}{\partial x \partial z} - \frac{\partial^2 B}{\partial z \partial x} \\ \frac{\partial^2 B}{\partial y \partial x} - \frac{\partial^2 B}{\partial x \partial y} \end{bmatrix} = 0$$

These, and similar, expressions will be encountered repeatedly during the course and it will be important to develop a feeling for their use.

Solution 2 - Divergence and curl of fields

The expressions $\nabla \cdot \vec{A} = \oiint \vec{A} \cdot d\vec{S}$ and $\nabla \times \vec{A} = \oint \vec{A} \cdot d\vec{l}$ are valid for *any* closed surface or loop. This means we are free to choose the surface or loop that is most convenient and that best reflects the symmetry of the field. This smart choice of how we calculate surface and line integrals and the consideration of the symmetry of the system is something that will come back throughout this course. It will make our lives much easier and shows that brute force calculations are often not the way to solve problems.
The chosen closed surfaces and loops are indicated in the figure. It is very important to reflect on why these were chosen. Afterwards the considerations can become trivial, especially if the symmetry is taken into account.

a) $\nabla \cdot \vec{A} = 0$ and $\nabla \times \vec{A} = 0$
Both the divergence and curl are differential operators. Their action on a *constant* field will thus yield zero. It can also be seen that just as much field enters the closed surface as that leaves it. For the loop the parts that are perpendicular to the field don't contribute ($\vec{A} \cdot d\vec{l} = 0$) and the other parts parallel to the field have equal length and field contribution.

b) $\nabla \cdot \vec{A} = 0$ and $\nabla \times \vec{A} \neq 0$
Just as much field enters the faces of the bent bar as that leaves on the other side, and the other faces don't contribute because $\vec{A} \cdot d\vec{S} = 0$. Because the two circle sections with components parallel to the field have different length and the field strength is constant, the loop integral will be nonzero.

c) $\nabla \cdot \vec{A} \neq 0$ and $\nabla \times \vec{A} = 0$
Taking a cylinder around the origin it is easy to see that there are only field contributions entering. The divergence is thus nonzero. On the other hand there are no components parallel to the circle used for the loop integral, thus the curl is zero.

d) $\nabla \cdot \vec{A} = 0$ and $\nabla \times \vec{A} = 0$
This is similar to (b), but now the field amplitude varies with $1/r$. Considering both integrals yield:

$$\int_1 \vec{A} \cdot \mathrm{d}\vec{\ell} = + \int_1 ||A|| \cdot ||\mathrm{d}\ell|| \propto + \int_1 \frac{1}{r} r \mathrm{d}\theta = \theta$$

$$\int_3 \vec{A} \cdot \mathrm{d}\vec{\ell} = - \int_3 ||A|| \cdot ||\mathrm{d}\ell|| \propto - \int_3 \frac{1}{r + \mathrm{d}r}(r + \mathrm{d}r)\mathrm{d}\theta = -\theta$$

The contributions are equal but opposite and the curl is thus also zero.

e) $\nabla \cdot \vec{A} \neq 0$ and $\nabla \times \vec{A} \neq 0$
One can look at the surface and loop and realise that the different components can't cancel each other. However, with the field given it is also possible to explicitly calculate the divergence:

$$\nabla \cdot \vec{A} = \partial_1 A_1 + \partial_2 A_2 + \partial_3 A_3 = -1 - 1 + 0 \neq 0$$

and the curl:

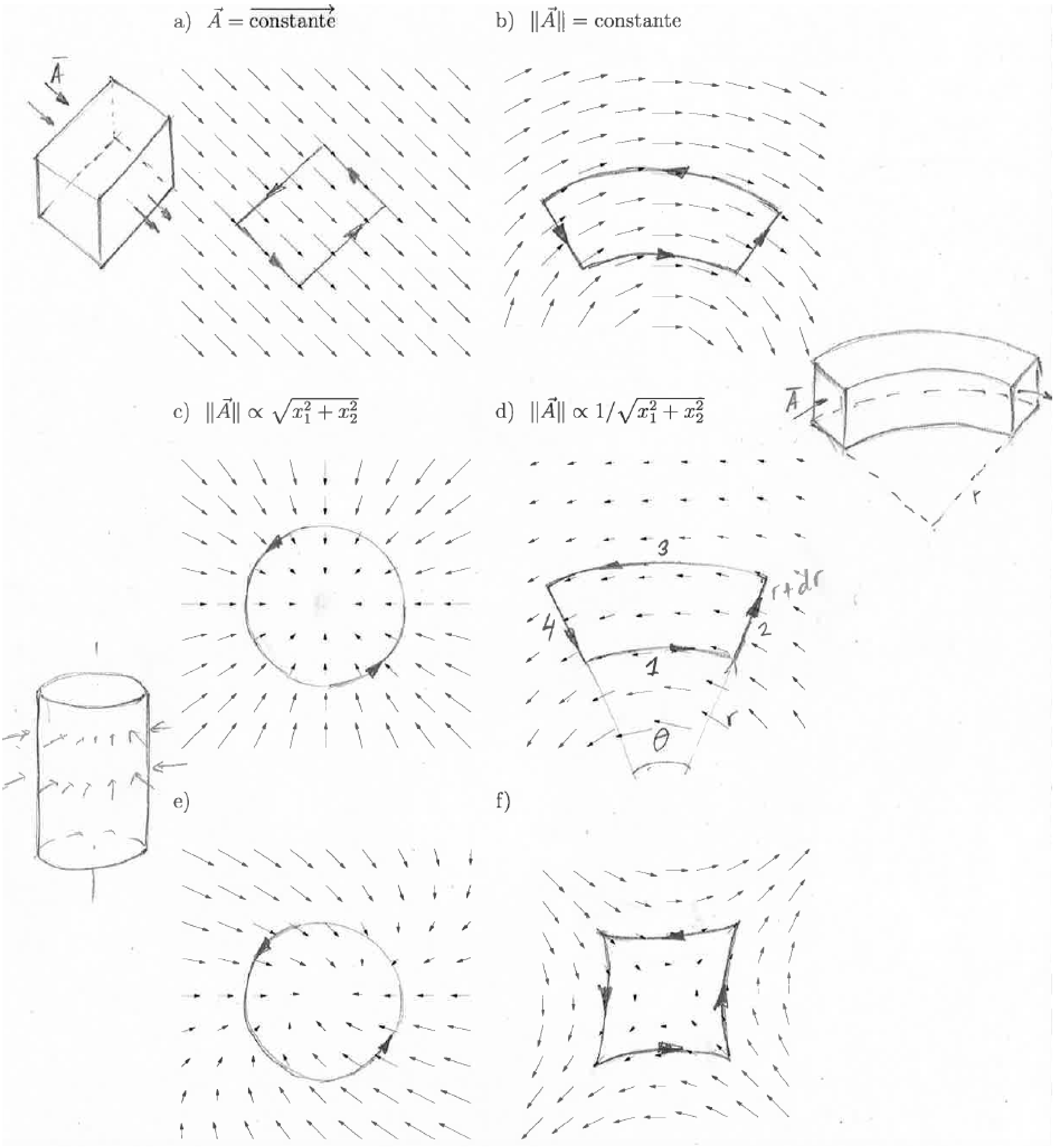
$$\nabla \times \vec{A} = (\partial_2 A_3 - \partial_3 A_2, \partial_3 A_1 - \partial_1 A_3, \partial_1 A_2 - \partial_2 A_1) = (0, 0, -1) \neq 0$$

f) $\nabla \cdot \vec{A} = 0$ and $\nabla \times \vec{A} = 0$
For the divergence all the components are parallel, except at the corners. Here two enter and two exit and they will thus compensate. For the loop integral all portions have the same length and the field is symmetric, so again all contributions will cancel. Also here one can explicitly calculate the divergence:

$$\nabla \cdot \vec{A} = \partial_1 A_1 + \partial_2 A_2 + \partial_3 A_3 = 0 + 0 + 0 = 0$$

and the curl:

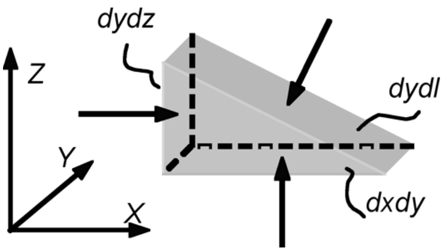
$$\nabla \times \vec{A} = (\partial_2 A_3 - \partial_3 A_2, \partial_3 A_1 - \partial_1 A_3, \partial_1 A_2 - \partial_2 A_1) = (0, 0, 1 - 1) = 0$$


a) $\vec{A} = \overrightarrow{\text{constante}}$
b) $\|\vec{A}\| = \text{constante}$
c) $\|\vec{A}\| \propto \sqrt{x_1^2 + x_2^2}$
d) $\|\vec{A}\| \propto 1/\sqrt{x_1^2 + x_2^2}$
e)
f)
$\bar{A}$
3
r+dr
4
2
1
r
θ

Solution 3 - Pressure in hydrostatics

As a reminder the definitions:
Isotropic: Having the same property in all directions
Homogeneous: Having the same property in one region as in every other region.
The absence of sheer stress means that there is no force component parallel to a surface. Let's now consider a sphere as it is the simplest and most symmetric surface to place around an arbitrary point. The absence of a parallel component means that the force always has to be perpendicular to the surface and that there can be no change in magnitude.
Another way of considering the issue is that in the absence of shear stress the force per area becomes a scalar instead of a vector (or tensor to be exact). A scalar is always isotropic, in contrast to a vector which is anisotropic by definition (except the null vector). Thus a vector field can be homogeneous but anisotropic and a scalar field can be isotropic but inhomogeneous.

More formally one can also consider the following corner or fluid and again the fact that the absence of sheer stress means that there is no force component parallel to a surface.. Let P_x, P_z and P_l be the pressures associated to the three forces of the figure, θ the angle



between the normal of the tilted face and the z axis, and ρ the fluid density. The conditions of equilibrium are :

$$x-\text{axis}: \quad P_x\, dy\, dz = P_l \sin\theta\; dy\, dl = P_l\, dy\, dz \rightarrow P_x = P_l\ ;$$

$$z-\text{axis}: \quad P_z\, dx\, dy = P_l \cos\theta\; dy\, dl + \frac{1}{2}\rho g\; dx\, dy\, dz = P_l\, dx\, dy + \frac{1}{2}\rho g\; dx\, dy\, dz\ ;$$

$$P_z = P_l \quad \text{for } dz \rightarrow 0\ .$$

For a different orientation of the corner, we obtain also : $P_y = P_l$. The angle θ being arbitrary, $P_i = P = const$ for a given axis i.

Solution 3 - Hydrostatic pressure in a bottle

a) First one needs to calculate the volume of the liquid in the bottle to determine the filling in the glass. It is composed of two cylinders

$$\begin{aligned} V &= \pi \left(r_1^2 h_1 + r_2^2 h_2 \right) \\ &\approx 334.6 \text{ cm}^3. \end{aligned} \tag{1}$$

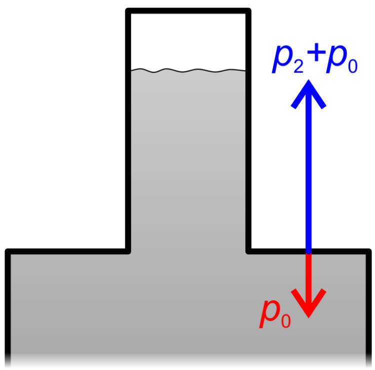
This gives the height in the glass

$$\begin{aligned} h_{\text{Gl}} &= \frac{V}{\pi r_1^2} = h_1 + h_2 \frac{r_2^2}{r_1^2} \\ &= \frac{71}{6} \text{ cm} \approx 11.8 \text{ cm} \end{aligned} \tag{2}$$

and the different forces due to the different hydrostatic pressures:

$$\begin{aligned} \Delta F &= \rho g A \left(h_1 + h_2 - h_{\text{Gl}} \right) \\ &= \pi \rho g h_2 \left(r_1^2 - r_2^2 \right) \\ &\approx 0.74 \text{ N}. \end{aligned} \tag{3}$$

Even though the hydrostatic pressure exerts a larger force on the bottom of the bottle, a scale will indicate the same weight because in the bottle there will also be an upward force on the area around the bottleneck due to the hydrostatic pressure there, as indicated in the figure.



b) This can be shown mathematically by considering the pressure around the bottleneck:

$$p_2 = p_0 + \rho g h_2.$$

The external air pressure p_0 will be compensated because it also presses from outside to the bottle. The effective force along the z-direction thus is:

$$F_z = \pi \rho g h_2 \left(r_1^2 - r_2^2 \right), \tag{4}$$

whereby $\pi \left(r_1^2 - r_2^2 \right)$ gives the area of the ring around the bottleneck. Eq. (4) is identical to Eq. (3), which shows that the total force on the scale for the bottle is the same as for the glass.