Solution Sheet 12

Discussion 3.12.2025

Solution 1 - Impedance

a) To add impedances in series $Z_{eq} = Z_1 + Z_2$. For impedances in parallel $1/Z_{eq} = 1/Z_1 + 1/Z_2$ and $Z_{eq} = Z_1 Z_2 / Z_1 + Z_2$. The impedances of a resistor, capacitor and inductor are, respectively: $Z_R = R,\ Z_C = -i/\omega C,\ Z_L = i\omega L$

Then in our case we have 2 elements in parallel, which in turn, are comprised of 2 elements in series each:

$$Z_{tot} = \frac{Z_{coil} Z_{capacitor}}{Z_{coil} + Z_{capacitor}};\ Z_{coil} = R + i\omega L,\ Z_{capacitor} = R - i/\omega C$$

Substituting:

$$Z_{tot} = \frac{(R + i\omega L)(R - i/\omega C)}{R + i\omega L + R - i/\omega C} = \frac{(R^2 + L/C) + iR(\omega L - 1/\omega C)}{2R + i(\omega L - 1/\omega C)} \cdot \frac{2R - i(\omega L - 1/\omega C)}{2R - i(\omega L - 1/\omega C)} =$$

$$= ... = \frac{\frac{2R(R^2C + L)}{C} + R\left(\frac{\omega^2 LC - 1}{\omega C}\right)^2 + i\frac{\omega^2 LC - 1}{\omega C}(R^2 - L/C)}{\left[4R^2 + \left(\frac{\omega^2 LC - 1}{\omega C}\right)^2\right]}$$

b) We want to make the imaginary part zero by changing the value of R. Thus

$$\left(\frac{\omega^2 LC - 1}{\omega C}\right)(R^2 - \frac{L}{C}) = 0 \rightarrow R^2 - \frac{L}{C} = 0$$

and

$$R = \sqrt{\frac{L}{C}}$$

c) similar as above but now we want to change ω instead of R

$$\left(\frac{\omega^2 LC - 1}{\omega C}\right)(R^2 - \frac{L}{C}) = 0 \rightarrow \omega^2 LC - 1 = 0$$

and

$$\omega = \frac{1}{\sqrt{LC}}$$

Solution 2 - Resonance

In the lecture the current resonance was derived, which occurs at $\omega_0 = 1/\sqrt{LC}$. For this frequency, the current in the system peaks and the voltage and current are in phase, also the impedance becomes purely resistive (real). Therefore, this is also known as the phase resonance. For a mechanical system this corresponds to the velocity resonance.
The equivalence to the amplitude resonance in a mechanical system is the charge resonance in a LCR-circuit. The charge circulating in the circuit is given by $\frac{I}{\omega}$ and we can use the same expression as in the lecture as a starting point, but now divided by ω.

$$Q = \frac{I}{\omega} = \frac{\Phi}{\omega\sqrt{R^2 + (\omega L - 1/\omega C)^2}} = \frac{\Phi}{\sqrt{\omega^2 R^2 + (\omega^2 L - 1/C)^2}}$$

The maximum can be found by derivation and setting to zero, which yields

$$\omega^2 = \frac{1}{LC} - \frac{R^2}{2L^2}$$

and thus the charge resonance occurs at

$$\omega_Q = \sqrt{\frac{1}{LC} - \frac{R^2}{2L^2}}$$

. This is often written with respect to the current resonance frequency as

$$\omega_Q^2 = \omega_0^2 - \frac{R^2}{2L^2}$$

.
Note that the found frequency is different from the natural frequency with which an LCR circuit oscillates in the transient solution

$$\omega_N = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

. These three frequencies approach each other for the case that the resistance goes to zero.

Solution 3 - Mu-Metal

From the lecture we know that the component of the B-field orthogonal to the surface is conserved while the parallel component is changed according to the ratio of the magnetic permeability.

$$B_{in\perp} = B_{out\perp} = B/\sqrt{2} \tag{1}$$

$$B_{in||} = \frac{\mu_{mu-metal}}{\mu_{Air}} \cdot B_{out||} = \frac{10^5}{1} \cdot \frac{B}{\sqrt{2}} \tag{2}$$

For the angle α of the B-field with respect to the surface we can write

$$\tan(\alpha) = \frac{B_{in\perp}}{B_{in||}} = \frac{B}{\frac{10^5}{1} \cdot B} \cdot \frac{\sqrt{2}}{\sqrt{2}} = 10^{-5} \tag{3}$$

$$\alpha = 5.73 \cdot 10^{-4}\ {}^\circ \tag{4}$$

In order to distinguish the thickness of the mu-metal we assume the longest possible path from one corner to the other:

$$\tan(\alpha) = \frac{x}{L} = 10^{-5} \tag{5}$$

$$x = d \cdot \tan(\alpha) = d \cdot 10^{-5} = 20\mu m \tag{6}$$

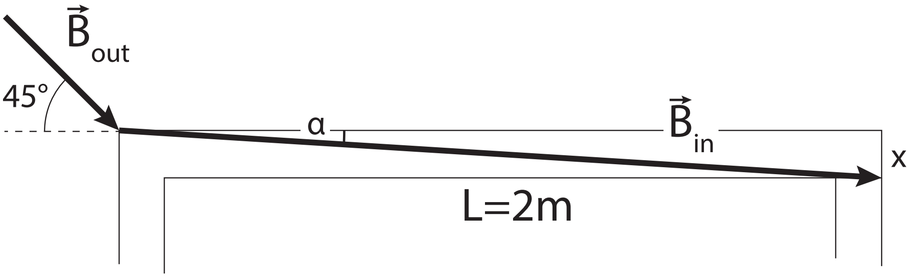

Figure 1: Zoom into edge of mu-metal box

Solution 4 - Hysteresis

a) At the beginning, we start magnetizing with h_0, and obtain a saturation value of the magnetisation (m_s). So all magnetic domains are oriented along the applied H-field. With decreasing H, M goes down until the residual value (m_r), which occurs at $H = 0$. This sometimes also called the remanent magnetisation or remanence. The material has a majority of its domains still oriented along one direction (ferromagnet).

For negative H, the magnetisation goes down until it reaches $-m_s$, the negative saturation at h_s. So all the magnetic domains are now oriented in the opposite direction.

When increasing H again, the curve follows the path through the negative residual value at $H = 0$, up to saturation.

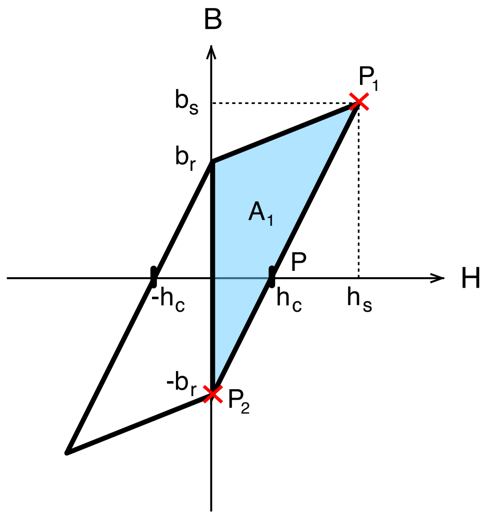

Figure 2: Diagram for exercises b and c.

b) At saturation (b_s) and remanence (b_r) we have (refer to Figure 2):

$$b_s = \mu_0 \left(h_s + m_s\right) \tag{7}$$
$$b_r = \mu_0 \, m_r \tag{8}$$

for: $\mu_0 = 4\pi \cdot 10^{-7}$, $m_r = 200 \, \frac{A}{m}$, $m_s = 300 \, \frac{A}{m}$, and $h_s = 1000 \, \frac{A}{m}$

$$b_s = 1.634 \; mT \tag{9}$$
$$b_r = 0.251 \; mT \tag{10}$$

To compute the coercive field or coercivity (h_c), we first find the following line equation using points $P_1(h_s, b_s)$ and $P_2(0, -b_r)$

$$b = \frac{b_s + b_r}{h_s} \, h - b_r \tag{11}$$

where $\frac{b_s+b_r}{h_s}$ is the slope of the line. Evaluating the equation in $P(h_c, 0)$, we obtain

$$h_c = b_r \, \frac{h_s}{b_s + b_r} = 133 \; \frac{A}{m}$$

c) The work done per unit volume in one cycle equals the area of the hysteresis curve in the B vs. H plot (refer to Figure 2):

$$\frac{W}{V} = \oint BdH = \oint HdB \tag{12}$$

$$\frac{W}{V} = 2 \cdot A_1 = 2\, b_r\, h_s \tag{13}$$

for: $b_r = 0.251\ mT$, and $h_s = 1000\ \frac{A}{m}$

$$\frac{W}{V} = 0.502\ \frac{J}{m^3} \tag{14}$$

The energy is required for rotating all the magnetic domains from one direction to the other. It is dissipated in the form of heat and sound (Barkhausen effect).