Solution Sheet 11

Discussion 26.11.2025

Solution 1 - Induction in a loop

a) We take a coordinate system where x points towards the right and y towards the bottom. The flux of magnetic field $\varphi_A(B)$ through the rectangle area A is:

$$\varphi_A(B) = \iint_A \vec{B}\, d\vec{A} = \int_d^{d+a} \int_0^b \frac{\mu_o I}{2\pi y} dx dy = \frac{\mu_o I b \ln\left(\frac{d+a}{d}\right)}{2\pi} \tag{1}$$

b) The current induced in the loop is:

$$I_{ind} = \frac{\Phi_{ind}}{R} = \frac{-\frac{\partial \varphi_A(B)}{\partial t}}{R} = -\frac{\mu_o b \ln\left(\frac{d+a}{d}\right)}{2\pi R} \frac{\partial I(t)}{\partial t} \tag{2}$$

Let's set $\tau = \frac{\mu_o b \ln\left(\frac{d+a}{d}\right)}{2\pi R}$, so that $I_{ind}(t) = -\tau \frac{\partial I(t)}{\partial t}$. By looking at the plot of $I(t)$ we can write its expression, and thus evaluate and plot $I_{ind}(t)$. The results are shown in Fig. 1.

$$I(t) = \begin{cases} \frac{I_1}{t_1^2} t^2 & \text{, for } t < t_1 \\ I_1 & \text{, for } t_1 < t < t_2 \\ \frac{-I_1}{t_3 - t_2} t + \frac{I_1 t_3}{t_3 - t_2} & \text{, for } t_2 < t < t_3 \\ 0 & \text{, for } t > t_3 \end{cases}$$

$$I_{ind}(t) = -\tau \frac{\partial I(t)}{\partial t} = \begin{cases} -2\tau \frac{I_1}{t_1^2} t & \text{, for } t < t_1 \\ 0 & \text{, for } t_1 < t < t_2 \\ \tau \frac{I_1}{t_3 - t_2} & \text{, for } t_2 < t < t_3 \\ 0 & \text{, for } t > t_3 \end{cases}$$

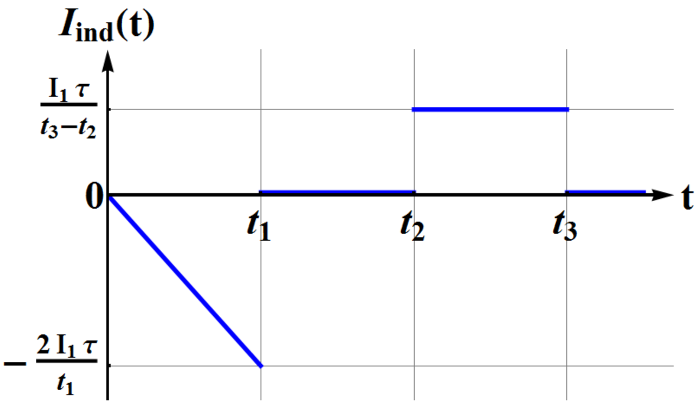

Figure 1:

Solution 2 - Rail gun

a) Let $\hat{x}$ be the coordinate towards the right. Because of the interaction between the current I and the magnetic field B, the bar experiences a force $\vec{F} = I\vec{d} \times \vec{B} = IdB\hat{x}$, so it starts to move towards the right. Since the area A of the circuit will vary with time, there will be an induced voltage:

$$\Phi_{ind} = -\frac{\partial \varphi_A(B)}{\partial t} = -B\frac{\partial A}{\partial t} = -Bd\frac{\partial x}{\partial t} = -Bdv(t) \tag{3}$$

where $v(t)$ is the speed of the bar. Therefore the current flowing in the bar is:

$$I = \frac{\Phi + \Phi_{ind}}{R} = \frac{\Phi - Bdv}{R} \tag{4}$$

b) The acceleration of the bar is:

$$\vec{a}(t) = \frac{\vec{F}}{m} = \frac{(\Phi - Bdv(t))\, Bd}{mR}\hat{x} \tag{5}$$

c) The speed $v(t)$ will increase while $a(t)$ decreases until $\Phi = Bdv$: here $a = 0$ and therefore v does not vary anymore. Thus $v_{max} = \frac{V}{Bd}$. Since this corresponds to $\Phi_{ind} = -\Phi$, $I = 0$. The projectile bar will move at uniform speed v_{max}.

Solution 3 -Current loops

a) We know the magnetic field generated by the current I_1 in a round loop of radius b at distance d over its axis: $B_1 = \frac{\mu_0 I_1 b^2}{2(b^2+d^2)^{3/2}}$. Therefore the flux of B_1 through the loop 2 is $\varphi_2 = B_1 \pi a^2$ (we consider B constant through the small area of loop 2).

The mutual inductance of loop 2 with respect to loop 1 is defined as $M_{21} = \varphi_2 / I_1$, we obtain:

$$M_{21} = \frac{\mu_0 \pi b^2 a^2}{2\left(b^2 + d^2\right)^{3/2}} \tag{6}$$

b) The current in loop 1 is $I_1(t) = I_1^0 \cos(2\pi f \cdot t)$, with $I_1^0 = 0.5\ A$ and $f = 50\ Hz$. The voltage induced in loop 2 can be evaluated with Faraday's law as:

$$\Phi_{ind} = -\frac{\partial \varphi_2}{\partial t} = -M_{21}\frac{\partial I_1}{\partial t} = M_{21} I_1^0 2\pi f \sin(2\pi f \cdot t) = \Phi_{ind}^{max} \sin(2\pi f \cdot t) \tag{7}$$

We can therefore evaluate $\Phi_{ind}^{max} \approx 9.8\ nV$.

c) In this case the change in flux through the small coil is due to it's motion and the changing B-field magnitude.

$$\Phi_{ind}(z) = -\frac{\mathrm{d}\varphi(z)}{\mathrm{d}t} = -\frac{\mathrm{d}\varphi(z)}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{3\mu_0 I_1 b^2 \pi a^2 z}{2\left(b^2 + z^2\right)^{5/2}} v_z \tag{8}$$

The magnetic dipole moment is dependent on the direction of the current, which can be determined from the sign of Φ_{ind} or by realising that the induced current will be such as to counter the effect of a change in flux. For negative z the flux is increasing and and the current will be clockwise to counter this. This means that $\vec{m}$ points along $-\hat{z}$. For positive z the flux is decreasing and the current will run counterclockwise, with $\vec{m}$ thus along $+\hat{z}$.

d) The magnetic dipole moment will precess around the magnetic field lines. The frequency of this precession is given by $\nu_L = \frac{\gamma}{2\pi}B$ with B given above. Here $\gamma = \frac{e}{2m_e}$ is the gyromagnetic ratio, which does not depend on the magnitude of the current that is induced. However, this current is reduced by the inclination of the loop and would go to zero for $90°$.

When looking from positive z at the approaching small coil, the precession will be counterclockwise for negative z, go to zero when also the induced current goes to zero at $z = 0$ and become clockwise for positive z.

In this we have ignored the mass of the wire of the small loop.

Solution 4 - LR circuit

The solution to this problem is very similar to the RC circuit treated earlier in this course. However, now the variable is the current and not the charge. As a reminder, in the case of the RC circuit the time constant is $\tau = RC$.

a) We identify the voltage of the different resistive elements of the system:

$$\Phi_L = L\frac{dI}{dt}; \quad \Phi_R = RI; \quad \Phi_0 = RI_0$$

Kirchhoff's second law gives $\Phi_0 - L\frac{dI}{dt} - RI = 0$. With $\Phi_0 = I_0 R$ this can be reorganised and solved by integration. The boundary conditions are that $I = 0$ for $t = 0$. This yields

$$\int_0^t \frac{R}{L}dt = \int_0^I \frac{dI}{I_0 - I} = -\int_0^I \frac{dI}{I - I_0}$$

$$\frac{R}{L}t = ln\left(\frac{I_0}{I - I_0}\right)$$

$$I(t) = I_0\left(1 - e^{-\frac{Rt}{L}}\right) = I_0\left(1 - e^{-\frac{t}{\tau}}\right)$$

with $\tau = L/R$.

b) Now K2 gives $L\frac{dI}{dt} + RI = 0$ because there is no source. Now the boundary condition is $I = I_0$ for $t = 0$. This can be easily solved to yield

$$I(t) = I_0 e^{-\frac{t}{\tau}}$$

again with $\tau = L/R$.