Solution Sheet 10

Discussion 19.11.2025

Solution 1 - Semicircle Wire

The magnetic field is $d\vec{B} = \frac{\mu_0 I}{4\pi r^2} d\vec{l} \times \hat{r}$. Since the two straight wires pass through the centre, $\vec{dl} \mid\mid \vec{r}$, they do not contribute to the magnetic field. Therefore the magnetic field at the centre is only due to the two semicircle parts.
The magnetic field at the centre of a circular wire is $B(a) = \mu_0 I/2a$ as given in the lecture. Thus the magnetic field at the centre of the two semicircles is

$$\begin{aligned} \vec{B} &= \vec{B_1} + \vec{B_2} \\ &= \frac{1}{2}\frac{\mu_0 I}{2 \cdot 2R}\widehat{z} - \frac{1}{2}\frac{\mu_0 I}{2 \cdot R}\widehat{z} \\ &= -\frac{\mu_0 I}{8R}\widehat{z} \end{aligned} \tag{1}$$

with $\vec{z}$ is the unit vector in the direction pointing out of the plane.

Solution 2 - Moving Charge

a) From the lecture we know that the magnetic field at the centre of a circular wire of radius a with current I is $B = \mu_0 I/2a$. The rotation leads to the current as

$$I = \frac{dQ}{dt} = Q\omega/2\pi \tag{2}$$

The magnetic field is thus

$$B = \frac{\mu_0 Q\omega}{4\pi a} \tag{3}$$

.

b) The surface charge density of the disk is

$$\sigma = \frac{Q}{\pi a^2} \tag{4}$$

We consider a thin circular wire on the disk of radius r, width dr and total charge $dQ = 2\pi r\sigma dr$. Using the result from the previous question, the magnetic field generated by this wire at the disk center is

$$\begin{aligned} dB &= \frac{\mu_0 \omega dQ}{4\pi r} \\ &= \frac{\mu_0 \sigma\omega dr}{2} \end{aligned} \tag{5}$$

Thus the total magnetic field is

$$\begin{aligned} B &= \int_0^a \frac{\mu_0 \sigma \omega \, dr}{2} \\ &= \frac{\mu_0 \sigma \omega \, a}{2} \\ &= \frac{\mu_0 \omega \, Q}{2\pi \, a} \end{aligned} \tag{6}$$

Solution 3 - Helmholtz coils

a) Both rings have the same radius R, and the same current I flows through them. Ring 1 is at $x = -h$ and ring 2 at $x = h$. By the principle of superposition, the magnetic field on the axis of the rings is $B_x = B_{1x} + B_{2x}$. The magnetic field created by each ring on its axis is:

$$B_{1x} = \frac{\mu_0 I R^2}{2\left(R^2 + (x+h)^2\right)^{3/2}} \quad \text{and} \quad B_{2x} = \frac{\mu_0 I R^2}{2\left(R^2 + (x-h)^2\right)^{3/2}}$$

The general expression thus becomes:

$$B_x = \frac{\mu_0 I R^2}{2}\left[\frac{1}{\left(R^2 + (x+h)^2\right)^{3/2}} + \frac{1}{\left(R^2 + (x-h)^2\right)^{3/2}}\right]$$

b) The two fields are in the same direction, and we can further simplify the expression by placing the origin of x half way between the coils such that x becomes the distance between the origin and coil. Now the field around the centre cane be written as:

$$B_x = \frac{\mu_0 I R^2}{\left(R^2 + x^2\right)^{3/2}}$$

To evaluate the uniformity of the B-field around $x = 0$ we calculate the second derivative of B with respect to x, evaluate it in $x = 0$ and find the value of h for which $\dfrac{\partial^2 B_x}{\partial x^2} = 0$ (minimum curvature $\Longrightarrow$ maximum curvature radius, i.e. the B field is constant over a wider region).

This yields:

$$\frac{d^2 B_x}{dx^2} = \frac{3R^2\left(R^2 - 4x^2\right)}{\left(x^2 + R^2\right)^{7/2}} = 0$$

This condition can only be fulfilled if

$$R^2 - 4x^2 = 0$$

Which gives $x = (h) = \frac{R}{2}$. Thus for a distance $2h = R$ between the coils the B-field at the centre of the system can be considered uniform.

Just like the parallel plate capacitor is the standard to produce uniform electric fields, Helmholtz coils in this configuration are the standard to produce uniform magnetic fields. Another application of Helmholtz coils is to use slightly different opposite currents to cancel the earth magnetic field and to create a field free region between the coils.

Solution 4 - Mass Spectrometer

a) In accordance with kinetic energy theorem for a charged particle in electric field:

$$q\Phi = \frac{1}{2}mv^2 \Rightarrow v = \sqrt{2q\Phi/m}$$

so,

$$v_{U_{235}} = \sqrt{2q\Phi/m_{U_{235}}} = \sqrt{\frac{2 * 3.2 * 10^{-19} * 100 * 10^3}{3.903 * 10^{-25}}} \simeq 4.049 * 10^5 \mathrm{m/s}$$

$$v_{U_{238}} = \sqrt{2q\Phi/m_{U_{238}}} = \sqrt{\frac{2 * 3.2 * 10^{-19} * 100 * 10^3}{3.953 * 10^{-25}}} \simeq 4.024 * 10^5 \mathrm{m/s}$$

b) The Lorentz force provides the centrifugal force:

$$qBv = \frac{mv^2}{r} \Rightarrow r = \frac{mv}{qB} = \sqrt{\frac{2\Phi m}{q}}/B$$

so the spatial separation:

$$\begin{aligned} d = 2(r_{238} - r_{235}) &= 2\left(\sqrt{\frac{2\Phi m_{U_{238}}}{q}}/B - \sqrt{\frac{2\Phi m_{U_{235}}}{q}}/B\right) \\ &= \frac{2}{B}\sqrt{\frac{2\Phi}{q}}\left(\sqrt{m_{U_{238}}} - \sqrt{m_{U_{235}}}\right) \\ &\simeq 1.26 \mathrm{cm} \end{aligned}$$

c)

$$d = 2\left(\sqrt{\frac{2\Phi m_{U_{238}}}{q}}/B - \sqrt{\frac{2\Phi m_{U_{235}}}{q}}/B\right) \tag{7}$$

$$\Rightarrow \Phi = \frac{q}{2}\left(\frac{Bd}{2\left(\sqrt{m_{U_{238}}} - \sqrt{m_{U_{235}}}\right)}\right)^2 \simeq 251 \mathrm{kv} \tag{8}$$

Solution 5 - Electron beam

a) according to the solution of exercise 1 from sheet 7,

$$E = -\frac{n_e e}{2\epsilon_0} r \vec{e}_r$$

where e is unit positive charge.

b) With Ampere's law:

$$B * 2\pi r = \mu_0 I$$

where $I = \frac{Q}{t} = \frac{Q}{\frac{\Delta l}{v}} = \frac{en_e V}{\frac{\Delta l}{v}} = \frac{en_e \Delta l S}{\frac{\Delta l}{v}} = n_e evS = n_e ev\pi r^2$, where V is the volume and S is the surface of the considered cylinder.

$$\Rightarrow B = \frac{\mu_0 n_e ev}{2} r$$

c) The total force is

$$\begin{aligned}
\vec{F}_{total} &= \vec{F}_{electric} + \vec{F}_{lorentz} \\
&= -Ee\vec{e}_r - eBv\vec{e}_r \\
&= \frac{n_e er}{2\epsilon_0} e\vec{e}_r - e\frac{\mu_0 n_e ev}{2} rv\vec{e}_r \\
&= \frac{n_e e^2 r}{2} \left(\frac{1}{\epsilon_0} - \mu_0 v^2 \right) \vec{e}_r
\end{aligned}$$

$$F_{total} = 0 \Longrightarrow v^2 = \frac{1}{\epsilon_0 \mu_0}$$

where ϵ_0 is the vacuum permittivity ($\epsilon_0 = 8.854 \cdot 10^{-12} \frac{F}{m}$) and $\mu_0 = 4\pi \cdot 10^{-7} \frac{H}{m}$ is the vacuum permeability.

$$\begin{aligned}
v^2 &= \frac{1}{\epsilon_0 \mu_0} = \frac{1}{8.854 \cdot 10^{-12} \frac{F}{m} \cdot \mu_0 = 4\pi \cdot 10^{-7} \frac{H}{m}} \\
&= \frac{1}{111.216 \cdot 10^{-19} \frac{s^2}{m^2}} \\
&= 8.99 \cdot 10^{16} \frac{m^2}{s^2} \\
v &= 3 \cdot 10^8 \frac{m}{s}
\end{aligned}$$

Since this is equal to the speed of light in vacuum, a beam of electrons has a constant radius only if the velocity of the electrons corresponds to the speed of light. However, since the electron is not a massless particle, it cannot reach the speed of light