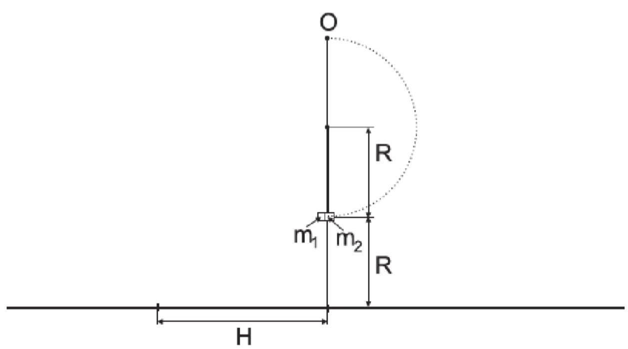
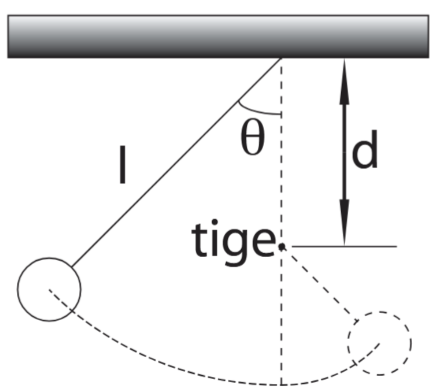
Exercices

Exercice 1 Consider a body composed of two point masses m_1 and m_2 connected to each other by an explosive (of zero mass). The body is suspended from a flexible string of length R (see diagram), attached to mass m_2. The system is initially at rest, and mass m_1 is ejected horizontally at velocity v_1 by the explosion, landing at a distance H from the starting point.
The mass m_2 is pulled by the string with a velocity v_2 immediately after the explosion along a circular trajectory. When m_2 reaches point O, the string breaks and m_2 hits the ground at the same place as m_1. We consider that the mass m_2 has just enough velocity to reach O.



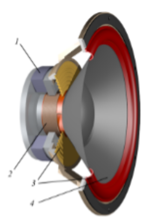
Calculate the ratios $\frac{v_1}{v_2}$ and $\frac{m_1}{m_2}$, as well as the value of H.

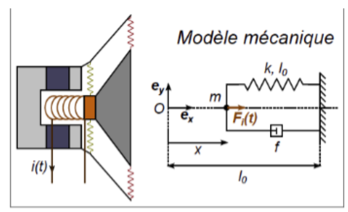
Exercice 2 A simple pendulum consists of a small sphere of mass $m = 3$ kg attached to an inextensible string of length $l = 3$ m. The pendulum is released without initial velocity at an angle $\theta_0 = -8°$ to the vertical. At $t = 0$, when the string is vertical, it strikes a horizontal rod located at a distance $d = 1.30$ m below the suspension point. The sphere continues its motion along a circular path with radius $r = l - d$. Use the approximation of small-amplitude oscillations.



1. Calculate the speed v of the sphere and the tension $T_<$ in the string just before the collision.
2. Calculate the tension $T_>$ in the string immediately after the collision.
3. Calculate the angle $\theta_M > 0$ of maximum deviation that the wire makes with respect to the vertical.
4. Calculate the period P of the sphere's motion.

Exercice 3 This exercise was given in the 2013 General Physics I exam.
We propose to study the mechanical operation of an electrodynamic loudspeaker. The loudspeaker shown in the figure below consists of a magnet (1), a coil (2), suspensions (3), and a diaphragm or cone (4). When an electric current flows through the coil, a Laplace force is applied axially to it, moving the diaphragm with it. The suspensions keep the diaphragm moving in line with the axis and provide a restoring force.

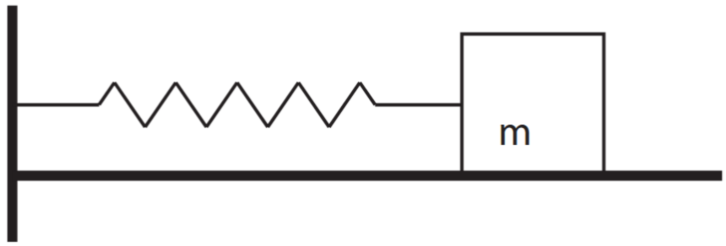




The mechanical part of the loudspeaker is modeled using a point mass m, moving horizontally without friction along the axis $(O, \vec{e}_x)$. This mass is connected to a spring with an unstretched length l_0 and stiffness k, as well as to a fluid damper with constant f. It is subjected to a force $\vec{F}_i(t) = Ki(t)\vec{e}_x$ imposed by the sinusoidal current $i(t) = I\cos(\omega_e t)$ entering the loudspeaker. We are working in the Galilean terrestrial reference frame $R_g(O, \vec{e}_x, \vec{e}_y)$.
Data : $m = 0.02$ kg, $k = 20\,000$ N/m, $K = 200$ N/A, $I = 1$ A, $Q = 2$.

1. Using force analysis, write the differential equation verified by mass m as a function of natural frequency Ω_0, system quality factor Q, K, I, and m. Express Ω_0 as a function of the problem data and give its numerical value.
2. Express the coefficient of friction f as a function of Q, k, and m, and give its numerical value.
3. Schematically represent the evolution of the amplitude of the membrane vibrations when the loudspeaker is turned on at a frequency ω_e. What is the condition for the sound reproduction to be as faithful as possible to the frequency ω_e ?
4. The forced steady-state response is of the form $x(t) = A\sin(\omega_e t + \varphi)$. Give the amplitude $A(X)$ as a function of Q, K, I, k, and the reduced angular frequency $X = \frac{\omega_e^2}{\Omega_0^2}$.
5. What is the amplitude A^0 for $\omega_e = 0$? Express the maximum amplitude A^{max} and the corresponding angular frequency ω_e^{max} as functions of Q, Ω_0, K, I, and k. Plot the graph of $A(X)$. What happens when $f \to 0$?
6. The -3 dB bandwidth of the system corresponds to the frequency range for which $A(X) \geq \frac{A^{max}}{\sqrt{2}}$. Express the cutoff frequencies ω_c^1 and ω_c^2 as functions of Q and Ω_0 and plot them on the graph $A(X)$ from the previous question. We will not attempt to perform the numerical application.
7. Calculate the average power dissipation $\langle P^{diss}\rangle(X)$ and find the maximum power dissipation P^{max} as a function of Q, K, I, k, and m. Give the numerical value of P^{max}.

Exercice 4 Consider the following setup :



This time, unlike in the course where fluid friction was considered, we are considering dry friction. The coefficient of friction is $\mu_c = \mu_s$. We assume that the mass is displaced a distance x_0 to the right, so that the restoring force of the spring is equal to 4 times the frictional force. We release the mass with no initial velocity.
What will be the nature of the movement ? Plot x as a function of time.

Solutions

Solution 1 We define the coordinate system such that gravity is antiparallel to the y axis. After the explosion, mass m_1 follows the law of free fall, with a velocity according to $\vec{e}_x$ immediately after the explosion.
Thus :

— $\vec{e}_x : v = \text{const} = v_1 \Rightarrow x(t) = v_1 t$

— $\vec{e}_y : a = \text{const} = -g; \quad v(t=0) = 0 \Rightarrow y(t) = -\frac{1}{2}gt^2 + R$

At time t_1, the mass m_1 hits the ground (i.e. $y = 0$) :

$$y(t_1) = 0 = -\frac{1}{2}gt_1^2 + R \Rightarrow t_1 = \sqrt{\frac{2R}{g}}$$

Furthermore,

$$x(t_1) = H = v_1 t_1 \Rightarrow H = v_1\sqrt{\frac{2R}{g}}$$

During the explosion, conservation of momentum applies :

$$m_1 v_1 = m_2 v_2 \tag{1}$$

The mass m_2 does not experience any force from the string at point O. Therefore :

$$-m_2 g + \frac{m_2 \hat{v}_2^2}{R} = 0 \Rightarrow \hat{v}_2 = \sqrt{gR} \tag{2}$$

where $\hat{v}_2$ is the velocity of mass m_2 at point O. This velocity is related to that of mass m_2 immediately after the explosion by the conservation of energy :

$$\frac{1}{2}m_2 v_2^2 + m_2 gR = \frac{1}{2}m_2\hat{v}_2^2 + 3m_2 gR$$

And so, with (2), we obtain :

$$v_2^2 = \hat{v}_2^2 + 4gR = 5gR \Rightarrow v_2 = \sqrt{5gR} \tag{3}$$

After the string has come loose, the mass also describes a free-fall motion, with :

— $\vec{e}_x : v = \text{const} = \hat{v}_2 \Rightarrow x(t) = \hat{v}_2 t$

— $\vec{e}_y : a = \text{const} = -g; \quad y(t=0) = 3R \Rightarrow y(t) = -\frac{1}{2}gt_2^2 + 3R$

The time at which mass m_2 hits the ground is given by :

$$y(t_2) = 0 = -\frac{1}{2}gt_2^2 + 3R \Rightarrow t_2 = \sqrt{\frac{6R}{g}}$$

Since mass m_2 hits the ground at the same place as mass m_1, we have :

$$x(t_2) = H = \hat{v}_2 t_2 \Rightarrow H = \hat{v}_2\sqrt{\frac{6R}{g}}$$

Finally,

$$H = \sqrt{gR}\sqrt{\frac{6R}{g}} = \sqrt{6}R$$

and so

$$v_1 = H\sqrt{\frac{g}{2R}} = \sqrt{6}R\sqrt{\frac{g}{2R}} = \sqrt{3gR} \tag{4}$$

The velocity ratios are obtained through (3) and (4) :

$$\frac{v_1}{v_2} = \frac{\sqrt{3gR}}{\sqrt{5gR}} = \sqrt{\frac{3}{5}}$$

And consequently, that of the masses with (1) :

$$\frac{m_1}{m_2} = \frac{v_2}{v_1} = \sqrt{\frac{5}{3}}$$

Solution 2 The tension in the string is perpendicular to the displacement and therefore does not perform any work. The change in kinetic energy of the sphere is equal to the work done by the weight (conservative force).

1. $$\frac{1}{2}mv^2 = mgl(1 - \cos\theta_0) \Rightarrow v^2 = 2gl(1 - \cos\theta_0)$$

   Vertically, the normal component of the resultant force is equal to the product of mass times centripetal acceleration :

   $$T_< - mg = \frac{mv^2}{l} \Rightarrow T_< = mg(3 - 2\cos\theta_0)$$

   N.A. : $v = 0.76$ m/s, $T_< = 30$ N.
2. Immediately after the collision with the rod, the radius of curvature becomes $r = l - d = 1.7$ m. Therefore :

   $$T_> = mg + \frac{mv^2}{r} = mg\left[1 + \frac{2l(1 - \cos\theta_0)}{l - d}\right]$$

   N.A. : $T_> = 30.4$ N.
3. The theorem of conservation of mechanical energy tells us that the two turning points of the sphere's motion are at the same height. Therefore :

   $$l(1 - \cos\theta_0) = r(1 - \cos\theta_M) \Rightarrow \cos\theta_M = 1 - \frac{l}{r}(1 - \cos\theta_0)$$

   N.A. : $\theta_M = 10.6°$.
4. In the small angle approximation, the period of oscillation of a pendulum of length L is given by $2\pi\sqrt{\frac{L}{g}}$. The period of the sphere's motion is given by half a period of oscillation with length l plus half a period of oscillation with length r :

   $$P = \pi\sqrt{\frac{l}{g}} + \pi\sqrt{\frac{l - d}{g}}$$

   N.A. : $P = 3.045$ s.

Solution 3 1. (1.5 pts) The mass is subject to the following forces :

$$\vec{F}_k = -kx\vec{e}_x, \quad \vec{F}_f = -f\dot{x}\vec{e}_x, \quad \vec{F}_i = KI\cos(\omega_e t)\vec{e}_x$$

Applying the fundamental relationship of dynamics gives us

$$\sum \vec{F} = m\vec{a} = m\ddot{x}\vec{e}_x$$

Projected onto the axis $(O, \vec{e}_x)$:

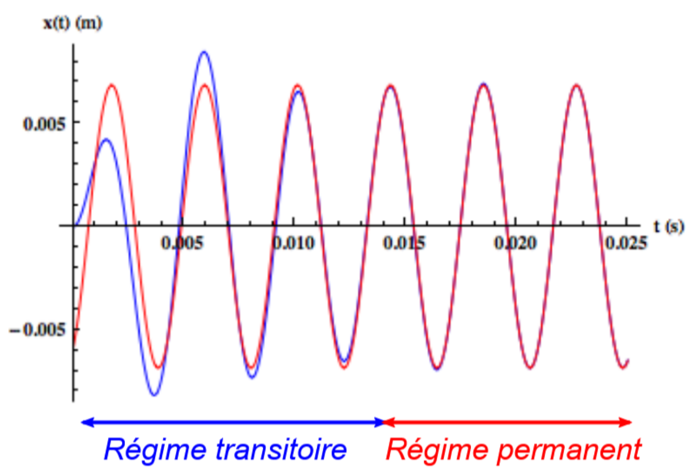
$$\ddot{x} + \frac{f}{m}\dot{x} + \frac{k}{m}x = \frac{KI}{m}\cos(\omega_e t)$$

The differential equation is written as a function of Ω_0 and Q :

$$\ddot{x} + \frac{\Omega_0}{Q}\dot{x} + \Omega_0^2 x = \frac{KI}{m}\cos(\omega_e t)$$

with $\Omega_0 = \sqrt{\frac{k}{m}}$ et $Q = \frac{m\Omega_0}{f}$. N.A. : $\Omega_0 = 1 \times 10^3$ rad/s.

2. (0.5 pts) $f = \frac{\sqrt{km}}{Q} = 10$ N s/m $= 10$ kg/s.
3. (1 pt) The solution is the superposition of a decreasing solution (transient regime) and the steady-state solution.



In order to achieve good sound quality, the steady state must be established quickly.

4. (1 pt) In steady-state, $x(t) = A\sin(\omega_e t + \varphi)$.

$$A(\omega_e) = \frac{(KI/m)}{\sqrt{(\omega_e^2 - \Omega_0^2)^2 + 4\gamma^2\omega_e^2}}$$

or

$$A(X) = \frac{KI}{k}\frac{Q}{\sqrt{Q^2(1-X)^2 + X}}$$

Note : The solution uses $X = \omega_e^2/\Omega_0^2$.

5. (2.5 pts) For $X = 0$, $A^0 = \frac{KI}{k}$. The maximum amplitude A^{max} is found when $g(X) = Q^2(1-X)^2 + X$ is minimum :

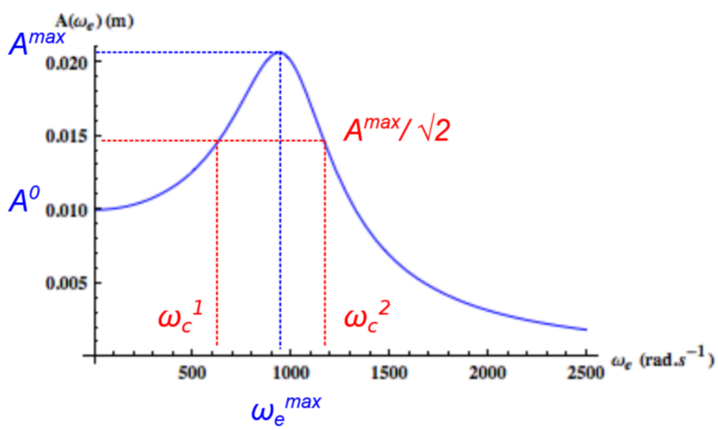
$$\frac{dg(X)}{dX} = -2Q^2(1-X) + 1 = 0 \Rightarrow X^{max} = 1 - \frac{1}{2Q^2}$$

or

$$\omega_e^{max} = \Omega_0\sqrt{1 - \frac{1}{2Q^2}}$$

and

$$A^{max} = A(X^{max}) = \frac{KI}{k}\frac{Q^2}{\sqrt{Q^2 - \frac{1}{4}}}$$



6. (1.5 pts) The cutoff frequencies are found by setting :

$$A(X) = \frac{A(X^{max})}{\sqrt{2}}$$

that is

$$\frac{KI}{k}\frac{Q}{\sqrt{Q^2(1-X)^2 + X}} = \frac{1}{\sqrt{2}}\frac{KI}{k}\frac{Q^2}{\sqrt{Q^2 - \frac{1}{4}}}$$

Which is equivalent to solving the following quadratic equation :

$$Q^2(Q^2(1-X)^2 + X) = 2(Q^2 - \frac{1}{4})$$

or

$$X^2 + \left(\frac{1}{Q^2} - 2\right)X + 1 - \frac{2}{Q^2} + \frac{1}{2Q^4} = 0$$

We find then

$$X_c = 1 - \frac{1}{2Q^2} \pm \frac{1}{Q}\sqrt{1 - \frac{1}{4Q^2}}$$

thus

$$\omega_c^1 = \Omega_0\sqrt{X_c^-}, \quad \omega_c^2 = \Omega_0\sqrt{X_c^+}$$

7. (2 pts) $\langle P^{diss}\rangle(X)$ is the average power dissipated due to friction forces. It is calculated over an oscillation period T :

$$\langle P^{diss}\rangle(X) = \frac{E^{diss}}{T} = \frac{f}{T}\int_0^T \dot{x}^2 dt$$

We have $x(t) = A\sin(\omega_e t + \varphi)$ and $\dot{x}(t) = A\omega_e \cos(\omega_e t + \varphi)$ and so :

$$\langle P^{diss}\rangle(X) = \frac{fA^2\omega_e^2}{2}\int_0^T \cos^2(\omega_e t + \varphi) dt$$

We make the change of variables $u = \omega_e t + \varphi$ and $du = \omega_e dt$. We finally get :

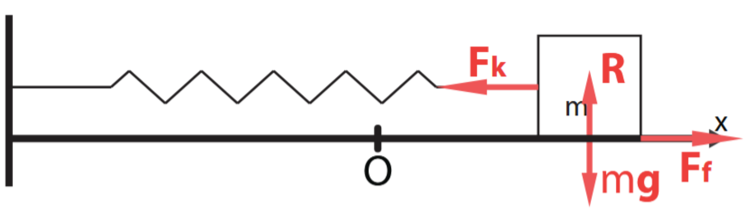
$$\langle P^{diss}\rangle(X) = \frac{fA^2\omega_e^2}{2} = \frac{K^2 I^2 Q}{2\sqrt{km}}\frac{X}{Q^2(1-X)^2 + X}$$

$\langle P^{diss}\rangle(X)$ is maximal for $X = 1$:

$$P^{max} = \frac{K^2 I^2 Q}{2\sqrt{km}}$$

N.A. : $P^{max} = 2000$ W.

Solution 4 Qualitatively : The frictional force will dissipate energy. The oscillations will weaken. There will come a point when the mass will stop because the restoring force will no longer be sufficient to overcome the static friction.
We must consider the cases where the mass moves from left to right separately from the case where the mass moves from right to left. Indeed, the friction force has the same intensity, but its sign will change.
Initial conditions : (right to left motion)



The Fundamental Relation of Dynamics (FRD) gives us :

$$\sum \vec{F} = m\vec{a} \qquad \vec{R} + m\vec{g} + \vec{F}_k + \vec{F}_f = m\vec{a}$$

with the following spring and friction forces :

$$\vec{F}_k = -kx\vec{e}_x \qquad \vec{F}_f = \mu_c mg\vec{e}_x$$

Initially, we have x_0 such that $|\vec{F}_f| = \frac{1}{4}|\vec{F}_k|$, and therefore

$$\mu_c mg = \frac{1}{4}kx_0 \Rightarrow x_0 = \frac{4\mu_c mg}{k}$$

By projecting the FRD onto the axis (Ox), we obtain :

$$-kx\vec{e}_x + \mu_c mg\vec{e}_x = m\ddot{x}\vec{e}_x$$

that is,

$$m\ddot{x} + kx = \mu_c mg \Rightarrow \ddot{x} + \frac{k}{m}x = \mu_c g$$

This is an M-1 type differential equation with a second term...
solution without second term :

$$x_1(t) = A\cos\Omega_0 t + B\sin\Omega_0 t \qquad \Omega_0 = \sqrt{\frac{k}{m}}$$

particular solution : $x_2(t) = \text{const}$

$$\frac{k}{m}x_2 = \mu_c g \Rightarrow x_2 = \frac{m}{k}\mu_c g$$

complete solution

$$x(t) = A\cos\Omega_0 t + B\sin\Omega_0 t + \frac{m\mu_c g}{k}$$

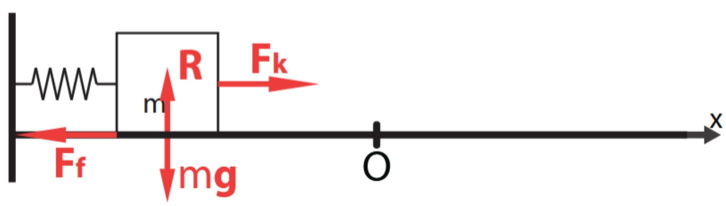
search for constants : (Init. cond. : $\dot{x}(t=0) = 0;\ x(t=0) = x_0$)

$$\dot{x}(t=0) = 0 \quad \Rightarrow \quad B = 0$$
$$x(t=0) = x_0 \quad \Rightarrow \quad A + \frac{m\mu_c g}{k} = x_0 = \frac{4\mu_c mg}{k} \Rightarrow A = \frac{3\mu_c mg}{k}$$

We draw : [box=] equation* x(t) = $3\mu_c mg_{\overline{k\cos\Omega_0 t + \frac{\mu_c mg}{k} \qquad 0<t<t_1}}$
After half a period, $\Omega_0 t_1 = \pi$, so $\dot{x}(t) = 0$: the velocity becomes zero. If the mass starts moving again, it moves from left to right.

$$x(t_1) = \frac{3\mu_c mg}{k}(-1) + \frac{\mu_c mg}{k} = -2\frac{\mu_c mg}{k}$$

We then have $|F_k| = 2\mu_c mg$, which is sufficient to set the mass in motion again.



Let's go back to the FRD :

$$\sum \vec{F} = \vec{R} + m\vec{g} + \vec{F}_k + \vec{F}_f = m\vec{a}$$
$$-\mu_c mg\vec{e}_x - kx\vec{e}_x = m\ddot{x}\vec{e}_x$$
$$m\ddot{x} + kx = -\mu_c mg$$
$$\ddot{x} + \frac{k}{m}x = -\mu_c g$$

and we solve :

$$x_1(t) = A\cos\Omega_0 t + B\sin\Omega_0 t$$

$$x_2(t) = \text{const} \qquad \frac{k}{m}x_2 = -\mu_c g \Rightarrow x_2 = -\frac{\mu_c mg}{k}$$

$$x(t) = A\cos\Omega_0 t + B\sin\Omega_0 t - \frac{\mu_c mg}{k}$$

We're at $t_1 = \frac{\pi}{\Omega_0}$:

$$\dot{x}(t_1) = 0 = -A\Omega_0\sin\Omega_0 t_1 + B\Omega_0\cos\Omega_0 t_1$$

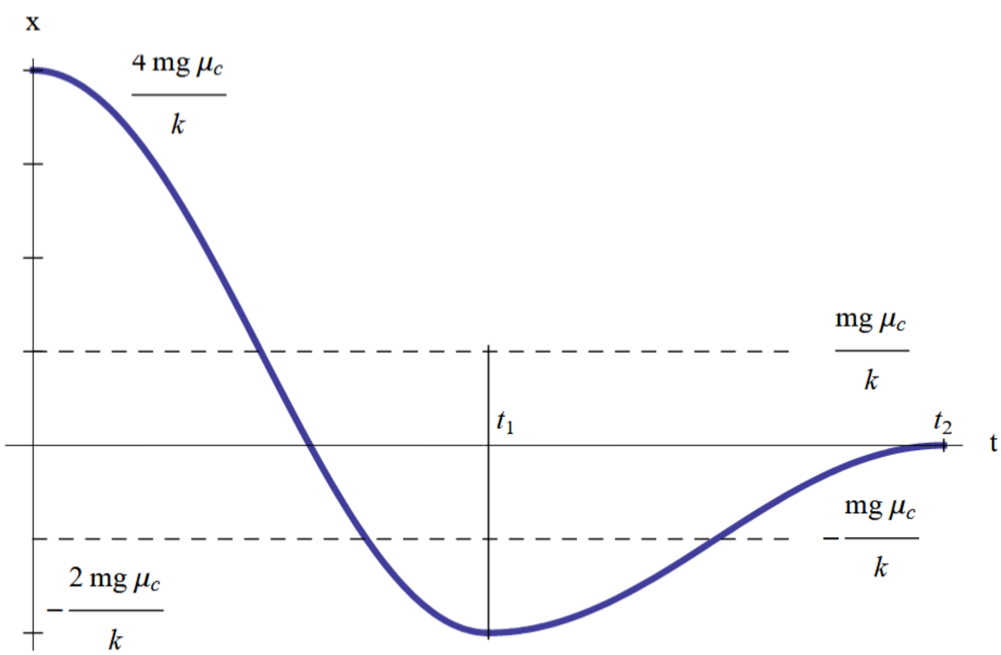
so, $B = 0$ and we get :

$$x(t) = A\cos\Omega_0 t - \frac{\mu_c mg}{k}$$

But, at $t = t_1$,

$$x(t_1) = -\frac{2\mu_c mg}{k} = -A - \frac{\mu_c mg}{k} \Rightarrow A = \frac{\mu_c mg}{k}$$

and so finally [box=] equation* x(t) = $\mu_c mg_{k\cos\Omega_0 t - \frac{\mu_c mg}{k} \qquad t_1<t<t_2}$
The mass stops again for $\Omega_0 t_2 = 2\pi \Rightarrow x(t_2) = 0$. The mass then stops there.



The first part of the movement is a half-sine wave with amplitude $\frac{3\mu_c mg}{k}$, shifted upward by $\frac{\mu_c mg}{k}$. After half a period $\left(t_1 = \frac{\pi}{\Omega_0}\right)$, it becomes a sine wave with amplitude $\frac{\mu_c mg}{k}$.