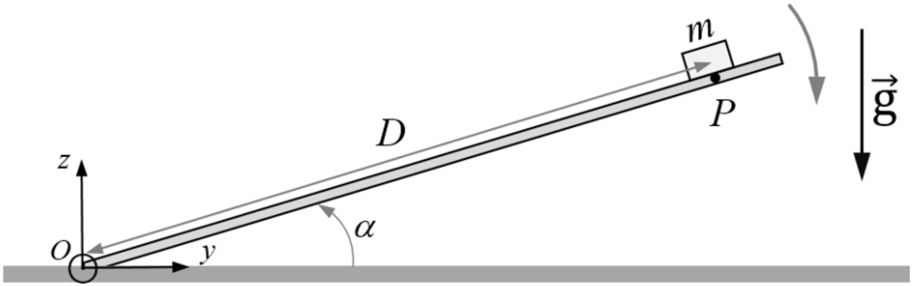
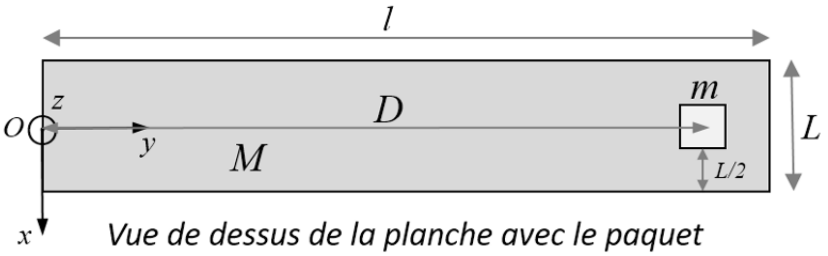
Exercises

Exercise 1 *Plank and Package*
Consider a plank of mass M with length l, width L, and negligible thickness. This plank can pivot without friction around a fixed axis along Ox. A package of mass m is placed on this plank at a distance D from point O and in the middle relative to the outer longitudinal edges (as shown in the diagram). This package is considered a point object.



Vue de profil

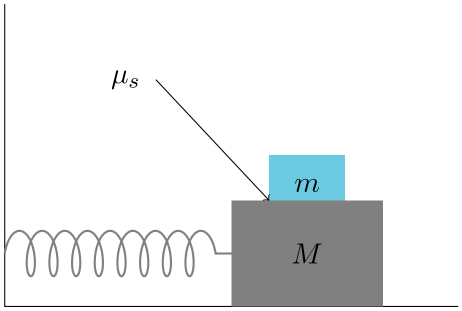


Vue de dessus de la planche avec le paquet

1) Determine the distance d_G from point O to the center of mass of the “plank + package” system.
2) Express the moment of inertia I of the “plank + package” system for rotation around an axis of rotation along Ox.
The plank is tilted at an angle α relative to the ground (horizontal). The angle α is less than the stall angle; therefore, the package does not slide due to dry friction. Then, in a second step, the plank is released ($t = 0$).
3) Calculate the angular acceleration $\dot{\omega}$ at $t = 0$ of the “plank + package” system as a function of I, m, M, g, and α.
4) Deduce the component a_z at $t = 0$ along Oz from the acceleration of point P where the package rests.
5) What is the condition on α for the package to remain on the plank at $t = 0$? Express this condition in terms of D, M, l, and m.

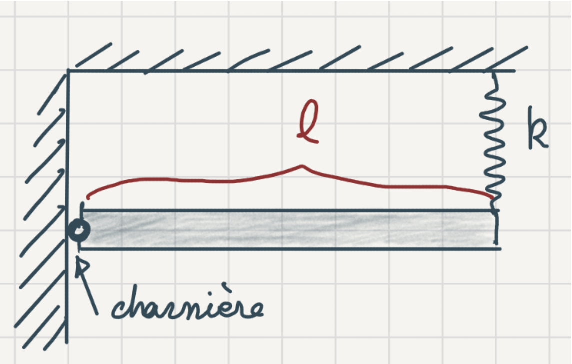
Exercise 2 *Don't let go !*

A mass M is attached to a spring and can slide without friction on a horizontal floor. It therefore performs simple harmonic oscillations at a frequency f. Above, a block of mass m is placed and dry friction is present between the masses m and M with a coefficient of static friction μ_s.



What is the maximum amplitude of oscillation that the system can have so that the block of mass m does not slide on the block of mass M ?

Exercise 3 *Everything measures up*



A uniform ruler of length l and mass M is attached at one end to a hinge and at the other end to a spring with stiffness k.
The assembly is designed so that when the bar is stationary, it is horizontal.
Determine the frequency of its slight vertical oscillation movement.

Solutions

Solution 1 1) The center of mass of the plank is at a distance $l/2$ from point O, and that of the package is at D. The distance from point O to the center of mass of the "plank + package" system is given by the law of composition of centers of mass :

$$d_{cm} = \frac{1}{M+m}(Ml/2 + mD)$$

2) The moment of inertia I of the "plank + package" system is the sum of the two moments, taken on Ox. The moment of inertia of the package is $I_{pa} = mD^2$ (point object).
We must use Steiner's theorem to express that of the plank I_{pl} :

$$I_{pl} = 1/12Ml^2 + 1/4Ml^2 = 1/3Ml^2$$

Finally :

$$I_{Ox} = \frac{1}{3}Ml^2 + mD^2$$

3) We apply the angular momentum theorem for rotation around Ox.
Weight moment (the reaction moment cancels out) :

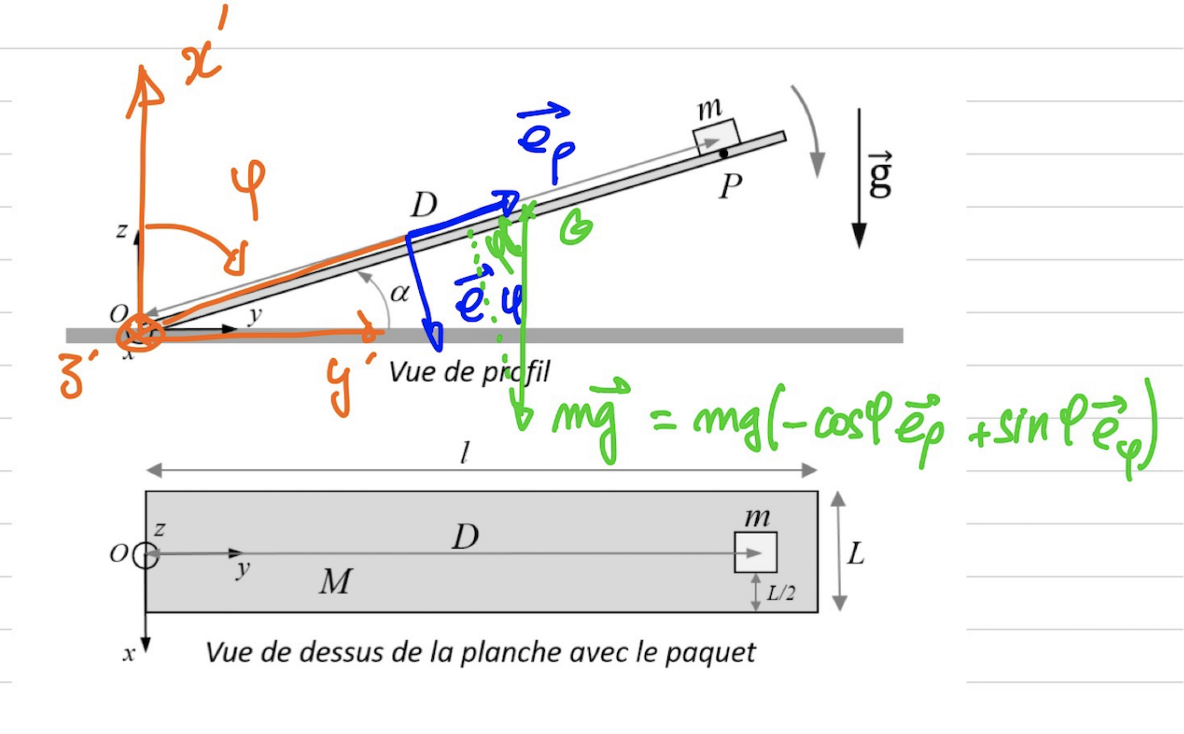
$$\vec{M}_O^{m\vec{g}} = \overrightarrow{OG} \wedge m\vec{g} = (d_{cm}\cos\alpha\vec{e}_y + d_{cm}\sin\alpha\vec{e}_z) \wedge -(M+m)g\vec{e}_z = -g(Ml/2 + mD)\cos\alpha\vec{e}_x$$

$$\vec{M}_O^{m\vec{g}} = \frac{d\vec{L}_0}{dt} = I_{Ox}\dot{\omega}\vec{e}_x = -g(Ml/2 + mD)\cos\alpha\vec{e}_x$$

With ω being the angular velocity according to (Ox). In the temporal evolution, the angle α changes over time, which is what gives the differential equation obtained in the course. Here, the calculation is simplified because we are only interested in what happens at $t = 0$.

$$\dot{\omega} = -\frac{Ml/2 + mD}{I_{Ox}}g\cos\alpha$$

4) There is a slight catch here. The proposed coordinate system does not allow for easy conversion to a polar coordinate system ; you have to figure out for yourself how to cleverly set these coordinates to properly describe the problem. We can use the following system. It corresponds to cylindrical coordinates viewed from below, and when the plank falls, φ is an increasing variable, so $\omega = \dot{\varphi}$.



The point where the package is placed describes a circular (non-uniform) motion. Its acceleration is given in cylindrical coordinates by

$$\vec{a}_P = (\ddot{\rho} - \rho\dot{\varphi}^2)\vec{e}_\rho + (\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi})\vec{e}_\varphi$$

With $\rho = D = cte$ et $\dot{\varphi} = \omega$:

$$\vec{a}_P = -D\omega^2\vec{e}_\rho + D\dot{\omega}\vec{e}_\varphi$$

The plank is released without initial velocity (and we are interested in $t = 0$), $\omega(t = 0) = 0$.

$$\vec{a}_P = D\dot{\omega}\vec{e}_\varphi$$

$$\vec{a}_P \cdot \vec{e}_z = D\dot{\omega}\,\vec{e}_\varphi \cdot \vec{e}_z = D\dot{\omega}(-\cos\alpha) = -D\frac{Ml/2 + mD}{I_{Ox}}g\cos^2\alpha$$

5) For the package to remain on the plank at $t = 0$, it is necessary that $|a_z < g|$:

$$D\frac{Ml/2 + mD}{I_{Ox}}g\cos^2\alpha < g$$

$$\frac{\frac{MlD}{2}+mD^2}{\frac{1}{3}Ml^2+mD^2}\cos^2\alpha<1$$

Solution 2

When block m is moving without falling off block M, it moves in a harmonic motion with the same frequency f and amplitude as block M. It is simply subject to the additional force of friction.

Let's consider the case where block m is in an extreme situation, about to slip at the moment when the direction of movement changes.

Newton's second law on the vertical axis gives us the value of the reaction force, which is $R = mg$. Applying Newton's law to the horizontal axis, we obtain

$$ma_{max} = \mu_s R = \mu_s mg$$

with $a_{max} = A\omega^2$ where A is the amplitude of the oscillations and ω is their angular frequency.

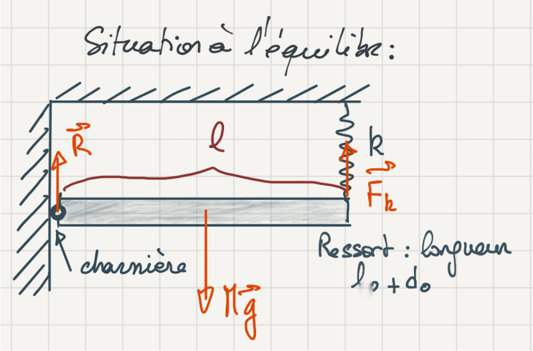
From this, we have

$$mA\omega^2 = \mu_s mg \Leftrightarrow \boxed{A = \frac{\mu_s g}{\omega^2} = \frac{\mu_s g}{(2\pi f)^2}}$$

<u>Note</u> : We focus on the moment when the block changes direction, because this is where the acceleration is strongest and therefore where the block is most likely to stall.

Solution 3

At the equilibrium position, the bar is horizontal. To keep the bar in place, the sum of the moments is $\vec{0}$, and therefore the spring exerts a force ; it is not at rest ! It has a length of $l_0 + d_0$.



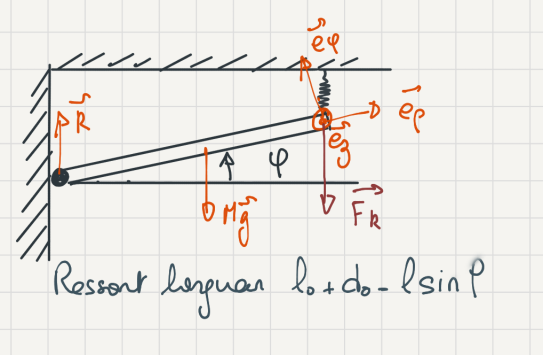
We begin by calculating the spring elongation at the equilibrium position ($\sum \vec{M}_{F_{ext}} = \vec{0}$) :

$$\overrightarrow{OO} \wedge \vec{R} + \overrightarrow{OG} \wedge m\vec{g} + \overrightarrow{OA} \wedge \vec{F}_k = \vec{0}$$

$$\frac{l}{2}\vec{e}_y \wedge (-mg)\vec{e}_z + l\vec{e}_y \wedge kd_0\vec{e}_z = \vec{0}$$

$$\Rightarrow \frac{1}{2}lmg = kld_0$$

$$\boxed{d_0 = \frac{mg}{2k}}$$



Calculation of oscillations around this elongation :

$$\sum \vec{M}_0^{ext} = \frac{\mathrm{d}\vec{L}_0}{\mathrm{d}t}$$

$$\overrightarrow{OO} \wedge \vec{R} + \overrightarrow{OG} \wedge m\vec{g} + \overrightarrow{OA} \wedge \vec{F}_k = \frac{\mathrm{d}\vec{L}_0}{\mathrm{d}t}$$

$$\frac{l}{2}\vec{e}_\rho \wedge (-\sin\varphi\vec{e}_\rho - \cos\varphi\vec{e}_\varphi)mg + l\vec{e}_\rho \wedge (\sin\varphi\vec{e}_\rho + \cos\varphi\vec{e}_\varphi)kx = \frac{\mathrm{d}\vec{L}_0}{\mathrm{d}t}$$

where x is the elongation of the spring relative to the $\underline{\underline{\text{resting position}}}$ of the spring (without mass), i.e.

$$x = -l\sin\varphi + d_0$$

Finally,

$$\begin{aligned}\sum \vec{M}_0^{ext} &= -\frac{l}{2}mg\cos\varphi\vec{e}_z + kl\cos\varphi(-l\sin\varphi + d_0)\vec{e}_z \\ &= \left[-\frac{l}{2}mg\cos\varphi - l^2k\cos\varphi\sin\varphi + kld_0\cos\varphi\right]\vec{e}_z = -kl^2\cos\varphi\sin\varphi\vec{e}_z\end{aligned}$$

As

$$\vec{L}_0 = I_0\dot{\varphi}\vec{e}_z = \frac{ml^2}{3}\dot{\varphi}\vec{e}_z$$

The law of moments gives us (with the small angle approximation) :

$$\sum \vec{M}_0^{ext} = \frac{\mathrm{d}\vec{L}_0}{\mathrm{d}t}$$

$$-kl^2\varphi\vec{e}_z = \frac{ml^2}{3}\ddot{\varphi}\vec{e}_z$$

$$\ddot{\varphi} + \frac{3k}{m}\varphi = 0$$

Thus,

$$\boxed{f = \frac{1}{2\pi}\sqrt{\frac{3k}{m}}}$$