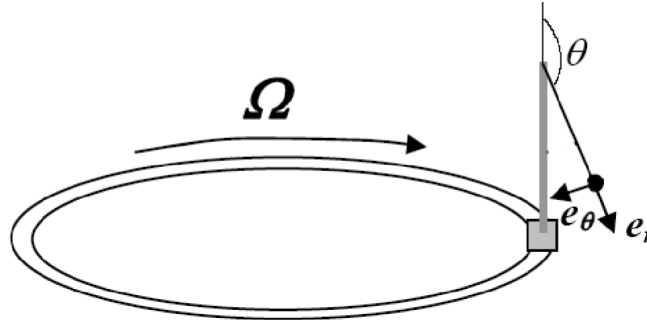


## Exercises

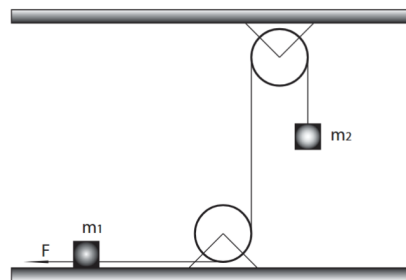
**Exercise 1** A pendulum consisting of a rod of length  $r$  with no mass and a mass  $m$  is located in a train rotating at an angular velocity  $\Omega$ . The pendulum is free to oscillate in a plane perpendicular to the trajectory.



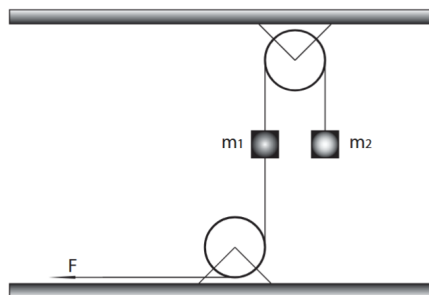
1. Placing yourself in the moving reference frame attached to the train, express the acceleration and velocity of the pendulum (use spherical coordinates as shown in the diagram).
2. Give the acceleration in the terrestrial reference frame.
3. Find the equation that determines the angle of inclination of the pendulum when there are no oscillations. (We will not attempt to solve the equation.)

**Exercise 2** Consider the following setups 1 and 2. We start with no initial velocity and let the system evolve, while keeping  $F$  constant. In both cases, we neglect friction.

- 1.



- 2.



Calculate the acceleration of each body based on  $m_1$ ,  $m_2$ , and  $F$ .

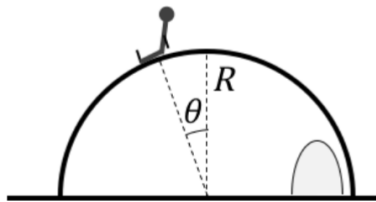
**Exercise 3** A block of mass  $m = 50$  kg slides down an inclined plane at an angle of  $30^\circ$ . It starts with no initial velocity. Its acceleration is  $a = 2$  m/s<sup>2</sup> and the plane is  $l = 10$  m long.

1. What is its kinetic energy when it reaches the bottom of the plane?
2. What is the value of the frictional forces?
3. What is the work done by the frictional forces?
4. What is the value of the coefficient of kinetic friction?

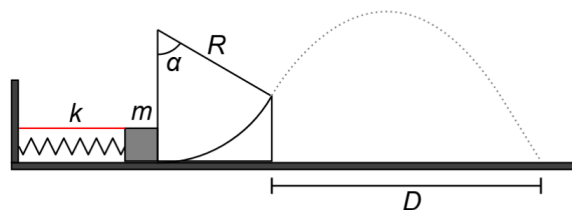
**Exercise 4** An Eskimo is sliding down the roof of his igloo. The igloo is represented as a hemisphere with radius  $R$  resting on the ground  $d$ . The position of the Eskimo on the igloo is marked by his angle  $\theta$ , defined as shown in the diagram opposite. The Eskimo slides without friction on the ice roof. Let  $m$  be the mass of the Eskimo (material point) and  $g$  the acceleration due to gravity.

The Eskimo slides down from the top of the igloo (his initial speed is negligible).

1. The Eskimo takes off from the roof when  $\theta = \theta_d$ . Find the expression for  $\theta_d$ .
2. Express the horizontal and vertical components of its velocity when it takes off.
3. What type of motion does the Eskimo have after taking off from the roof?
4. What is the magnitude of its velocity when it lands?



**Exercise 5** Consider the following setup :



A mass  $m$  is connected to a support by a spring with stiffness  $k$  compressed by  $x$ . A string keeps the spring compressed. The frictionless track is an arc of a circle with radius  $R$ . At  $t = 0$ , the string is burned, the spring relaxes and pushes the mass. Give the answers in terms of  $g$ ,  $k$ ,  $m$ ,  $\alpha$ ,  $x$  and  $R$ .

1. What must be the minimum value of  $k$  for the mass to leave the track?

$$k_{\min} = \underline{\hspace{10cm}}$$

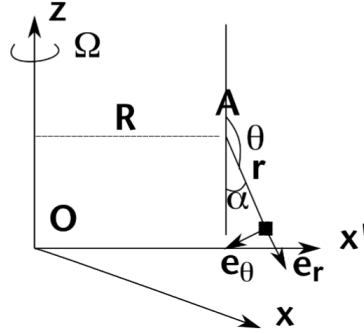
2. How far  $D$  from its takeoff point does the mass arrive?

$$D = \underline{\hspace{10cm}}$$

## Solutions

### Solution 1

- Let  $\mathcal{R}(O, \vec{x}, \vec{y})$  be the terrestrial reference frame and  $\mathcal{R}'(A, \vec{x}', \vec{y}')$  the reference frame linked to the train. We take the origin  $A$  at the attachment point of the pendulum. In  $\mathcal{R}'$ , we work in spherical coordinates.



- The constraints on the system are :  $\varphi = \dot{\varphi} = \ddot{\varphi} = 0$  and  $\dot{r} = \ddot{r} = 0$ . The velocity is given by  $\vec{v}_{R'}(P) = r\dot{\theta}\vec{e}_\theta$ . The acceleration is given by  $\vec{a}_{R'}(P) = -r\dot{\theta}^2\vec{e}_r + r\ddot{\theta}\vec{e}_\theta$ .
- The acceleration in the terrestrial reference frame  $\mathcal{R}$  is given by :

$$\vec{a}_R(P) = \vec{a}_R(A) + \vec{a}_{R'}(P) + \dot{\vec{\Omega}} \wedge \overrightarrow{AP} + \vec{\Omega} \wedge (\vec{\Omega} \wedge \overrightarrow{AP}) + 2\vec{\Omega} \wedge \vec{v}_{R'}(P).$$

Here  $A$  describes a uniform circular motion in  $R$ , so :

$$\vec{a}_R(A) = -R\Omega^2\vec{e}_x.$$

A geometrical analysis of the system gives us  $\vec{\Omega} = \Omega\vec{e}_z$  with  $\Omega = \text{const} < 0$  et  $\dot{\vec{\Omega}} = \vec{0}$ . Therefore :

$$\vec{a}_R(P) = -R\Omega^2\vec{e}_x - r\dot{\theta}^2\vec{e}_r + r\ddot{\theta}\vec{e}_\theta - \Omega^2r\sin\theta\vec{e}_x + 2\Omega r\dot{\theta}\cos\theta\vec{e}_y$$

with

$$\vec{e}_x = \cos\theta\vec{e}_\theta + \sin\theta\vec{e}_r, \quad \vec{e}_y = \vec{e}_\varphi$$

- If there is no oscillation ( $\dot{\theta} = \ddot{\theta} = 0$ ) :

$$\vec{a}_R(P) = -\Omega^2(R + r\sin\theta)(\cos\theta\vec{e}_\theta + \sin\theta\vec{e}_r)$$

The forces are gravity  $\vec{P} = m\vec{g}$  and tension  $\vec{T} = -T\vec{e}_r$ . Projected onto  $\vec{e}_r$  and  $\vec{e}_\theta$  and using Newton's second law, we obtain the system :

$$\begin{cases} \text{Along } \vec{e}_r : -T - mg\cos\theta = -m\Omega^2(R + r\sin\theta)\sin\theta, \\ \text{Along } \vec{e}_\theta : -mg\sin\theta = -m\Omega^2(R + r\sin\theta)\cos\theta. \end{cases}$$

The second equation gives us the expression for  $\theta$  :

$$\tan \theta = -\frac{\Omega^2}{g} (R + r \sin \theta).$$

Since this equation is quite complicated to solve analytically, we will prefer a graphical solution...

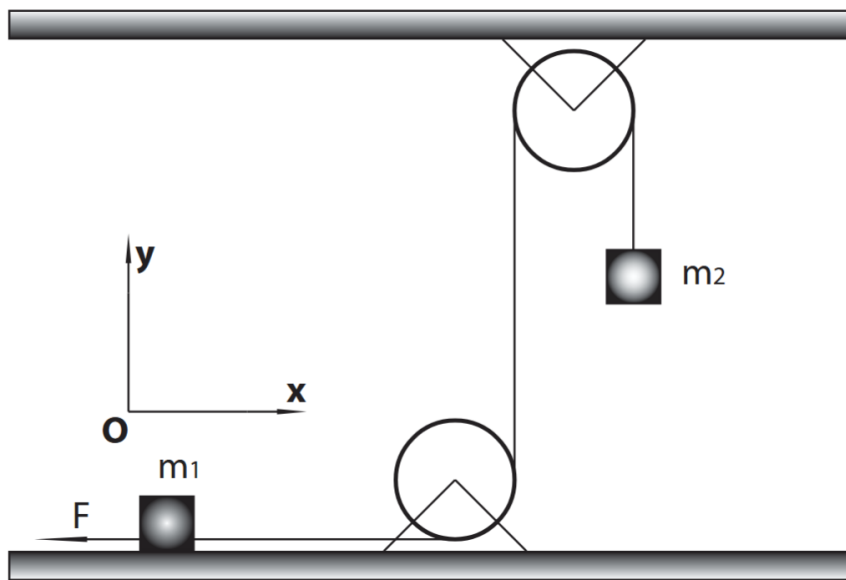
**Solution 2**

It is important to note that in any pulley system with a rigid rope, the rope transmits the tension in its entirety. In other words, the intensity of the force is the same at each end of the rope if there is no mass between them.

However, when a mass is involved, the tension is not necessarily the same, because the mass can be accelerated (and in this case,  $\sum \vec{F} \neq \vec{0}$ ).

The masses of the pulleys are neglected. . .

1. Let us consider the two systems  $m_1$  and  $m_2$  independently. The reference frame will be terrestrial with coordinates :  $(O, \vec{x}, \vec{y})$  (see diagram).



On  $m_1$  :  $\sum \vec{F} = m\vec{a}_1 = \vec{F} + \vec{P}_1 + \vec{R} + \vec{T}_1$ . The mass  $m_1$  moves along  $(O\vec{x})$  and so :

$$\begin{pmatrix} m_1 a_1 \\ 0 \end{pmatrix} = \begin{pmatrix} -F \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -m_1 g \end{pmatrix} + \begin{pmatrix} 0 \\ R \end{pmatrix} + \begin{pmatrix} T \\ 0 \end{pmatrix}$$

which gives us two equations :

$$\begin{cases} m_1 a_1 = -F + T \\ 0 = -m_1 g + R \end{cases}$$

On  $m_2$ ,  $\sum \vec{F} = m\vec{a}_2 = \vec{P} + \vec{T}_2$  :

$$\begin{pmatrix} 0 \\ m_2 a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -m_2 g \end{pmatrix} + \begin{pmatrix} 0 \\ T \end{pmatrix}$$

We thus obtain a third equation :

$$m_2 a_2 = -m_2 g + T$$

We are looking for  $a_1$  and  $a_2$ . The string is not elastic, so when  $m_2$  moves up with  $a_2$ ,  $m_1$  moves to the left with  $a_1$  :  $a_2 = -a_1$ .

We thus have the following system :

$$\begin{cases} m_1 a_1 = -F + T \\ m_2 a_2 = -m_2 g + T \\ a_2 = -a_1 \end{cases}$$

By subtracting the first from the second, we obtain  $m_2 a_2 - m_1 a_1 = -m_2 g + F$ . Using the third, we then get  $m_2 a_2 + m_1 a_2 = -m_2 g + F$ , i.e.,  $a_2(m_1 + m_2) = F - m_2 g$ . Finally :

$$\boxed{a_2 = \frac{F - m_2 g}{m_1 + m_2}, \quad a_1 = -a_2 = \frac{m_2 g - F}{m_1 + m_2}}$$

2. On  $m_1$ ,  $\sum \vec{F}_{ext} = m\vec{a}$  implies :  $m_1 \vec{a}_1 = \vec{T}_3 + m_1 \vec{g} + \vec{T}$  with  $T_3$  the tension force in the rope below the mass. We have  $|\vec{T}_3| = F$  or more explicitly :

$$\begin{pmatrix} 0 \\ m_1 a_1 \end{pmatrix} = \begin{pmatrix} 0 \\ -F \end{pmatrix} + \begin{pmatrix} 0 \\ -m_1 g \end{pmatrix} + \begin{pmatrix} 0 \\ T \end{pmatrix}$$

which gives us a first equation :

$$m_1 a_1 = T - F - m_1 g$$

On  $m_2$ ,  $\sum \vec{F}_{ext} = m\vec{a}$  implies :  $m_2 \vec{a}_2 = \vec{T} + m_2 \vec{g}$  or :

$$\begin{pmatrix} 0 \\ m_2 a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ T \end{pmatrix} + \begin{pmatrix} 0 \\ -m_2 g \end{pmatrix}$$

yielding a second equation :

$$m_2 a_2 = T - m_2 g$$

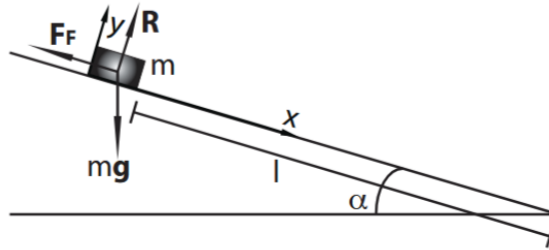
When  $m_1$  goes up,  $m_2$  comes down, so  $a_2 = -a_1$ . The system to solve is therefore :

$$\begin{cases} m_1 a_1 = T - F - m_1 g \\ m_2 a_2 = -m_2 g + T \\ a_2 = -a_1 \end{cases}$$

The second minus the first gives :  $m_2 a_2 - m_1 a_1 = F + m_1 g - m_2 g$ . Using the last line finally gives :

$$\boxed{a_2 = \frac{F + (m_1 - m_2)g}{m_1 + m_2} = -a_1}$$

**Solution 3**



1. The time to cover  $l$  from rest with acceleration  $a$  is

$$l = \frac{1}{2}at_1^2 \Rightarrow t_1 = \sqrt{\frac{2l}{a}}$$

The speed at the bottom of the incline is  $v_1 = at_1$ , hence the kinetic energy :

$$E_{k,1} = \frac{1}{2}mv_1^2 = mla = 50 \cdot 10 \cdot 2 = 1000 \text{ J}$$

2. Friction forces : project  $\sum \vec{F} = m\vec{a}$  along the incline. The resultant along the plane gives

$$-F_F + mg \sin \alpha = ma \Rightarrow F_F = mg \sin \alpha - ma.$$

With numerical values :  $F_F \approx 145 \text{ N}$ .

3. Work of the friction forces :

$$W_F = -F_F \cdot l = mla - mgl \sin \alpha = -145 \cdot 10 = -1450 \text{ J}.$$

As expected, this work is negative.

4. The projection of the forces on an axis perpendicular to the plane, and the fact that the movement lies in the plane gives :

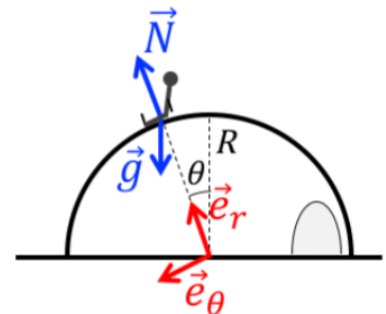
$$F_F = \mu_c R = \mu_c mg \cos \alpha \Rightarrow \mu_c = \frac{F_F}{mg \cos \alpha} = \tan \alpha - \frac{a}{g \cos \alpha}$$

Numerically,  $\mu_c \approx 0.34$ .

**Solution 4**

a) The Eskimo is subject to his weight  $m\vec{g}$  and the reaction force of the roof  $\vec{N}$ . There is no tangential component to the reaction force of the roof because there is no friction force.  $\vec{N}$  is collinear to the radial vector  $\vec{e}_r$ ,  $\vec{N} = N\vec{e}_r$ .

The lift-off condition is  $N = 0$  and so we need to find  $N$ .



Projection of Newton's second law onto  $\vec{e}_r$  :

$$ma_c = N - mg \cos \theta$$

$a_c$  being the centrifugal acceleration. For a circular motion :

$$a_c = -\frac{v^2}{R} = -R\dot{\theta}^2$$

Newton's second law projected along  $\vec{e}_r$  is therefore written :

$$-mR\dot{\theta}^2 = N - mg \cos \theta \Rightarrow N = mg \cos \theta - mR\dot{\theta}^2$$

Let us use the conservation of mechanical energy to express the term  $mR\dot{\theta}^2$  :

$$\begin{aligned} mgR &= mgR \cos \theta + \frac{1}{2}mv^2 = mgR \cos \theta + \frac{1}{2}mR^2\dot{\theta}^2 \\ \Rightarrow mR\dot{\theta}^2 &= 2mg(1 - \cos \theta) \end{aligned}$$

In the end,  $N$  is expressed as :

$$N = mg \cos \theta - 2mg(1 - \cos \theta) = mg(3 \cos \theta - 2)$$

The condition for the Eskimo to take off is  $N = 0 \Rightarrow \cos \theta = \frac{2}{3}$

$$\theta_d = \arccos \frac{2}{3}$$

b) The magnitude of the velocity when he takes off is given by the conservation of mechanical energy :

$$mgR = mgR \cos \theta_d + \frac{1}{2}mv_d^2 \Rightarrow v_d^2 = 2Rg(1 - \cos \theta_d) = 2Rg \left(1 - \frac{2}{3}\right) = \frac{2}{3}Rg \Rightarrow \|v_d\| = \sqrt{\frac{2Rg}{3}}$$

Its horizontal components  $v_{dx}$  and vertical components  $v_{dy}$  are :

$$\begin{aligned} v_{dx} &= \cos \theta_d \|v_d\| = \frac{2}{3} \sqrt{\frac{2Rg}{3}} \\ v_{dy} &= \sin \theta_d \|v_d\| = \sqrt{1 - \left(\frac{2}{3}\right)^2} \sqrt{\frac{2Rg}{3}} = \frac{\sqrt{5}}{3} \sqrt{\frac{2Rg}{3}} \end{aligned}$$

c) The Eskimo is subjected only to his weight after taking off. His motion is uniformly accelerated ( $\vec{a} = \vec{g}$ ) and his trajectory is parabolic.

d) Since the entire motion occurs without friction (sliding and parabolic path), we can directly state the conservation of mechanical energy between the top of the igloo and the point of impact :

$$mgR = \frac{1}{2}mv_a^2 \Rightarrow \|v_a\| = \sqrt{2Rg}$$

**Solution 5**

1. (2 pts) By considering the conservation of energy, the initial elastic potential energy  $E_k = \frac{1}{2}kx^2$  is transformed at the takeoff point into kinetic energy  $E_c = \frac{1}{2}v_0^2$  and gravitational potential energy  $E_p = mgR(1 - \cos \alpha)$

$$\frac{1}{2}kx^2 = \frac{1}{2}mv_0^2 + mgR(1 - \cos \alpha)$$

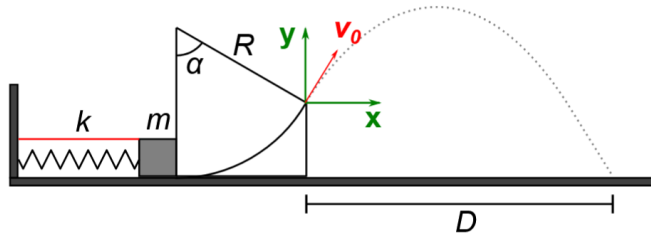
For the mass to leave the track, the takeoff velocity  $v_0$  must be non-zero, in other words, it is necessary that  $E_k \geq E_p$  :

$$\frac{1}{2}kx^2 \geq mgR(1 - \cos \alpha)$$

that is

$$k_{lim} = \frac{2mgR(1 - \cos \alpha)}{x^2}$$

2. (3 pts) Once the mass has taken off, the problem becomes ballistic. The takeoff angle is obviously  $\alpha$ .



It follows then :

$$\vec{a} = \begin{pmatrix} 0 \\ -g \end{pmatrix} \quad \vec{v} = \begin{pmatrix} v_0 \cos \alpha \\ v_0 \sin \alpha - gt \end{pmatrix} \quad \vec{r} = \begin{pmatrix} v_0 \cos \alpha t \\ v_0 \sin \alpha t - \frac{1}{2}gt^2 \end{pmatrix}$$

The mass arrives at the landing point  $B = \begin{pmatrix} D \\ -h \end{pmatrix}$  at time  $t_f$ . We therefore have two equations with two unknowns :

$$\begin{cases} D = v_0 \cos \alpha t_f \\ -R(1 - \cos \alpha) = v_0 \sin \alpha t_f - \frac{1}{2}gt_f^2 \end{cases}$$

Since we are looking for  $D$ , we are going to eliminate  $t_f$ . On  $Oy$ , we have the second-order equation

$$v_0 \sin \alpha t_f - \frac{1}{2}gt_f^2 + R(1 - \cos \alpha) = 0$$

which we solve with the discriminant :

$$\Delta = (v_0 \sin \alpha)^2 + 2gR(1 - \cos \alpha)$$

$$t_f = \frac{-v_0 \sin \alpha \pm \sqrt{\Delta}}{-g}$$

The positive solution for  $t_f$  is therefore :

$$t_f = \frac{v_0 \sin \alpha + \sqrt{(v_0 \sin \alpha)^2 + 2gR(1 - \cos \alpha)}}{g}$$

that is

$$t_f = \frac{v_0 \sin \alpha}{g} \left[ 1 + \sqrt{1 + \frac{2gR(1 - \cos \alpha)}{v_0^2 \sin^2 \alpha}} \right]$$

and thus :

$$D = v_0 \cos \alpha t_f = \frac{v_0^2 \cos \alpha \sin \alpha}{g} \left[ 1 + \sqrt{1 + \frac{2gR(1 - \cos \alpha)}{v_0^2 \sin^2 \alpha}} \right]$$

The velocity  $v_0$  is calculated from the conservation of energy :

$$\frac{1}{2}kx^2 = \frac{1}{2}mv_0^2 + mgR(1 - \cos \alpha) \Rightarrow v_0 = \sqrt{\frac{kx^2}{m} - 2gR(1 - \cos \alpha)}$$

and therefore, in the end :

$$D = \frac{\left(\frac{kx^2}{m} - 2gR(1 - \cos \alpha)\right) \cos \alpha \sin \alpha}{g} \left[ 1 + \sqrt{1 + \frac{2gR(1 - \cos \alpha)}{\left(\frac{kx^2}{m} - 2gR(1 - \cos \alpha)\right) \sin^2 \alpha}} \right]$$