*The only authorized items are :*
- *the official course forumula sheet*
- *one handwritten double-sided A4 sheet*
- *pens, etc.*

*The final answers to each question, as well as their justifications, must be written on the exam paper in the boxes provided for this purpose.*

## Only the answer booklet will be collected and graded. No loose sheets.

**One sheet per exercise**

*The exam consists of 3 problems, numbered 1 to 3.*
*The maximum score for this exam is 34 points + 3.5 bonus points.*

## Do not turn over before the beginning of the exam

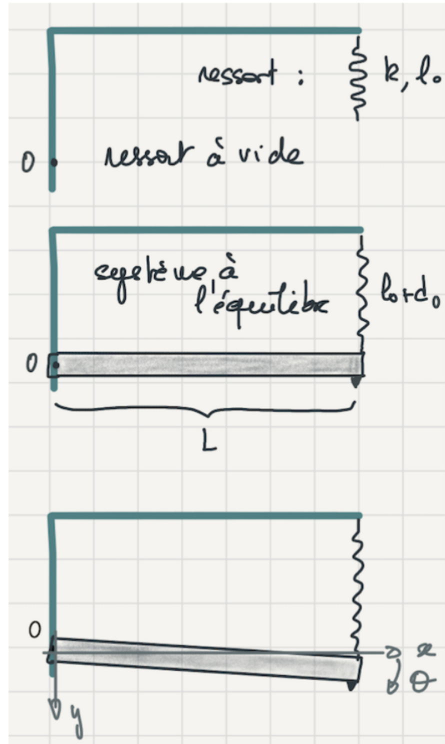
**Exercise 1 : Atomic Force Microscope (12 points)**

The atomic force microscope (AFM) allows for the observation of sample surfaces at the nanometer or even atomic scale. To do this, a tip, fixed to the end of a very thin bar, scans over the sample and the deflection of the bar is measured, providing information about the force between the tip and the sample, which can be attractive or repulsive.

In a specific mode, known as dynamic mode, which is well-suited for soft matter, the bar is made to oscillate by a periodic force, and the variation in the oscillation amplitude is observed as a function of the tip's position.

We will model an AFM tip in the following way :

A bar of length $L$ and mass $m$ is fixed at one end by a pivot and at the other to a spring with rest length $l_0$ and stiffness $k$. The parameters are such that, at equilibrium, the bar is horizontal and the spring is stretched by $d_0$. The mass of the tip is neglected compared to that of the bar. When the bar is in motion, the angle $\theta$ measured between $Ox$ and the bar remains small, so it can be assumed that the spring remains vertical, aligned with the $(Oy)$ axis, and that $\cos\theta \simeq 1$ and $\sin\theta \simeq \theta$.




a. List the forces applied to the bar and represent them on a diagram.

☞ The system is first studied at equilibrium.

b. Calculate the extension $d_0$ of the spring at equilibrium as a function of $m$, $g$, and $k$.

☞ Small oscillations around the equilibrium position are now considered. Friction is initially neglected.

c. Show that the extension of the spring is expressed as $(l - l_0) = \frac{mg}{2k} + \theta L$.

d. Assume the bar is in motion. Calculate the mechanical energy of the system as a function of the angle $\theta$ and its derivative.

e. Establish the differential equation of motion for the bar as a function of the angle $\theta$.

f. The bar is displaced from its equilibrium position by an angle $\theta_0$ and released without initial velocity. Give the expression for $\theta(t)$, as well as the period $T_0$ of the oscillations.

☞ It is now assumed that the system undergoes fluid friction, which creates a moment of force with respect to the pivot of the form $\vec{M}_O^{\vec{F}} = -b\dot{\theta}\vec{e}_z$. Furthermore, the support, now mobile, is used to exert a force on the bar of the form $\vec{F_e} = F_e \cos(\omega_e t)\vec{e}_y$ applied at the attachment point of the spring.

g. Establish the differential equation of motion for the case of small $\theta$.

h. Give the expression for $\theta(t)$ in the steady state. Recall the expression for the amplitude of the oscillations obtained as a function of the excitation frequency $\omega_e$.

i. What is the value of $\omega_e$ for which the amplitude is maximal ? Express it in terms of $k$, $m$, $L$, and $b$.

☞ When the tip is moved over the sample to be observed, the sample material exerts a force on the tip that increases as the tip gets closer to the surface.

j. Explain qualitatively why measuring the amplitude of the tip's oscillations provides information on the distance between the tip and the sample.

**Exercise 2 : DART Mission (12 points + 1.5 bonus points)**

In September 2022, the DART mission tested the ability of a human mission to deflect the trajectory of an asteroid, with the aim of eventually being able to deflect a "near-Earth object" (NEO), that is to say, an asteroid that could intercept the Earth's trajectory. We are going to study certain aspects of this mission.

The system chosen for the test is a binary system with a primary asteroid, Didymos, called $D_1$, spherical in shape, with radius $R_1 = 400\,\text{m}$ and mass $M_1$, around which orbits a smaller object Dimorphos, called $D_2$, spherical in shape, with radius $R_2 = 80\,\text{m}$ and mass $M_2$. $D_2$ orbits $D_1$ on a circular orbit of radius $r_0 = 1200\,\text{m}$. $R_1$, $R_2$, and $r_0$ were measured from Earth by radar. The period $T_0 = 12$ hours of $D_2$ around $D_1$ is also measured from Earth. Calculations will be performed in the reference frame $\mathcal{R}$ of $D_1$ (origin at the center of $D_1$).

The gravitational constant is given as $G = 6.67 \cdot 10^{-11}\,\text{m}^3\text{kg}^{-1}\text{s}^{-2}$. We will approximate $4\pi^2/6.67 \simeq 6$.

a. Determine the mass $M_1$ using the data.

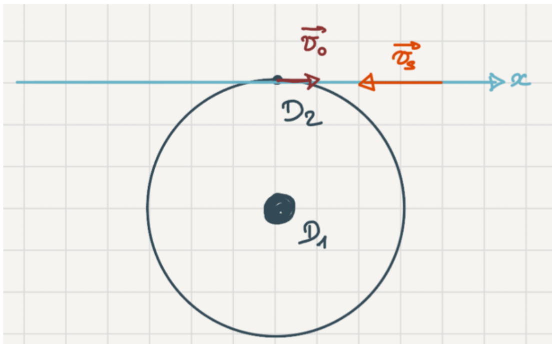
b. Determine the mass $M_2$ from $M_1$ and the radii of $D_1$ and $D_2$, assuming that the density of the two bodies is identical.

c. Numerical application : calculate $M_1$ and $M_2$.

d. Determine $v_0$, the orbital speed of $D_2$ in $\mathcal{R}$, and perform the numerical application.

☞ In the following, $M_1$ and $M_2$ will be assumed to be known (they can be taken as given).

☞ A satellite of mass $m_s = 500\,\text{kg}$, sent from Earth, must collide with $D_2$. It arrives in $\mathcal{R}$ with a velocity of magnitude $v_s = 6 \cdot 10^3\,\text{m.s}^{-1}$ and a vector tangent to the orbit of $D_2$. The collision is assumed to be perfectly head-on, with the two velocity vectors in opposite directions.




e. Calculate $v_1$, the new velocity of $D_2$ after the collision, here assumed to be perfectly inelastic.

f. Sketch on the same drawing the orbit of $D_2$ before the collision and the trajectory after the collision.

g. *Bonus question (1.5 pt) :* Calculate analytically, then numerically $\Delta v_1 = v_1 - v_0$. Use appropriate approximations to simplify the expression obtained.

☞ In fact, a reasonable hypothesis is to think that a certain amount of matter is ejected by the impact. It is assumed that the mass $m_e$ is ejected at velocity $v_e$ in a direction perfectly opposite to $\vec{v}_s$.

h. Calculate $v_2$, the new velocity of $D_2$ after the collision in this case.

i. Is $v_2$ smaller or larger than $v_1$ ? Justify.

☞ The next challenge for the DART mission is to succeed in obtaining $v_2$. This will allow the scientific team to deduce the characteristics of the impact. From Earth, it is possible to measure the new orbital period $T_2$ of the orbit, which has now become elliptical. Let $r_2$ be the shortest distance between $D_2$ and $D_1$ on this orbit. Recall Kepler's 3rd law : the square of the periods is proportional to the cube of the semi-major axes.

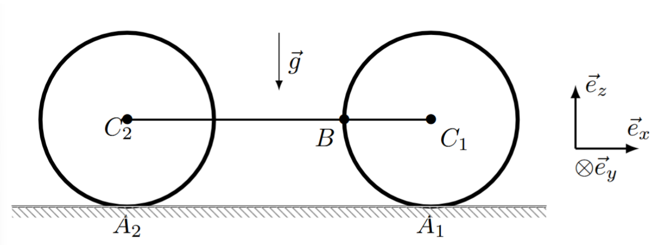
j. Calculate $r_2$ as a function of $r_0$, $T_0$, and $T_2$.

k. Determine $v_2$ as a function of $r_0$, $r_2$, $M_1$, and $G$.

l. Represent on the same diagram the effective potential energy diagrams for the trajectory before and after the impact.

**Exercise 3 : Bicycle Model (10 points + 2 bonus points)**

We wish to study the braking of a bicycle. The bicycle wheels are modeled as identical thin hollow cylinders of mass $m$ and radius $R$. A rigid rod, massless and of length $L > 2R$, connects the two circles by their centers, $C_1$ and $C_2$. The rear wheel's center is $C_2$ and the front wheel's center is $C_1$. The bicycle remains in the $(O, \vec{e}_x, \vec{e}_z)$ plane.




The bicycle is equipped only with a front brake at $B$. When the bicycle is braking, the rod exerts a constant force on the front wheel at $B$ : $\vec{F}_B = -F_B\vec{e}_z$ with $F_B > 0$. We first consider that both wheels rest on the ground and roll without slipping on it.

a. The bicycle is braking; list the external forces applied to the total system {rod + two wheels}, with their points of application, paying attention to their direction, and represent them on a drawing.

☞ The cyclist has an initial velocity $\vec{v}_0 = v_0\vec{e}_x$. From time $t = 0$, he brakes with the force $\vec{F}_B$ described above. Braking produces an acceleration of the bicycle $\vec{a}_0 = -a_0\vec{e}_x$.

b. Taking the front wheel alone as the system, express vectorially the friction force between the front wheel and the ground as a function of $a_0$, $F_B$, and $m$.

c. Taking the rear wheel alone as the system, express vectorially the friction force between the rear wheel and the ground as a function of $a_0$ and $m$.

d. Find a relation linking the two friction forces between the wheels and the ground, $m$, and $a_0$.

e. Calculate each of the two friction forces between the wheels and the ground, as well as $F_B$ as a function of $a_0$ and $m$.

f. Calculate the distance $D$ traveled before stopping as a function of $a_0$ and $v_0$.

g. Calculate the total mechanical energy of the system {rod + two wheels} as a function of the bicycle's velocity $v(t)$.

h. *Bonus question (2 points) :* Calculate the work done by the force $F_B$ during braking, as a function of $F_B$, $v_0$, and $a_0$. Deduce $F_B$. Compare with the result obtained in e.

☞ We now assume that the bicycle also has a rear brake. The cyclist now brakes hard and locks both wheels : the two wheels are therefore rigidly connected to the rod, forming a single undeformable solid. The wheels slide on the ground, with an identical kinetic friction coefficient for both wheels, denoted $\mu_c$. This braking produces an acceleration $\vec{a}_1 = -a_1\vec{e}_x$.

i. Express $a_1$ as a function of $\mu_c$ and the problem data.

j. Give the value of $a_1$ for which the rear wheel lifts off the ground under the effect of braking.