École Polytechnique Fédérale de Lausanne

PHYS-101 (n)

Prof. Emma Tolley : Lecture notes

Adapted from Prof. Cécile Hébert's notes de cours

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Contents

Chapter 0

Mathematical Review

The analysis of physics problems requires a number of mathematical tools. In this chapter we review the mathematical basics of vectors, derivatives, integrals, and Taylor expansions which will be commonly used in the rest of the course.

0.1 Units

In this course we will use the International System of Units, abbreviated SI, the modern form of the metric system. In mechanics, we will need three fundamental units:

1. To measure **time**, we use **seconds** (s). One second is defined as the duration of 9,192,631,770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium-133 atom ($\Delta\nu_{Cs}$).

2. To measure **length**, we use **meters** (m). The meter is the length of the path traveled in vacuum by light during a time interval of 1/299,792,458 of a second. The definition changed in 1983! The book by Alonso and Finn was published in 1980 and contains the previous definition.

3. To measure **mass**, we use **kilograms** (kg). The kilogram is defined by fixing the numerical value of the Planck constant, h, to 6.62607015×10^{-34} kg m^2 s^{-1}.

0.2 Dimensional analysis

When we say "h has units of length" we can write this mathematically as:

$$\dim h = L \tag{1}$$

where use L represents length. We will also use T for time and M for mass.

Dimensional analysis is using units to check the consistency of a result, to recover a formula, or to try to guess one. The idea is that an equality can only be valid if the units are the same on both sides of the "equals" sign.

For example, a stone is dropped from a height h. The calculated fall time is:

$$t_f = \sqrt{\frac{2h}{g}} \tag{2}$$

with h the height (a length) and g the acceleration due to gravity in m/s^2.

Is this formula correct? The first thing we can do is check that the units are consistent. This does not guarantee that the result is correct, but if the units do not match, we know that it is wrong. To start, we can write the dimension of our variables:

$$\dim h = L, \qquad \dim g = L/T^2 \tag{3}$$

Then plug this into our equation for t_f:

$$\dim t_f = \dim \sqrt{\frac{2h}{g}} = \sqrt{\frac{L}{LT^{-2}}} = \sqrt{T^2} = T \tag{4}$$

So we find that t_f has units of time, as we expect.
See exercise set 0 for more examples.

1 Vectors

A vector is used to represent a quantity for which, in addition to the "value" (magnitude), it is important to know the direction and orientation. A vector is characterized by:

- its **magnitude** (or norm), the length of the vector,
- its **direction**, the line along which it lies,
- its **orientation**, the binary direction along this line.

We will use vectors to represent displacements, velocities, accelerations, and forces (in particular, force analysis is done using vectors).

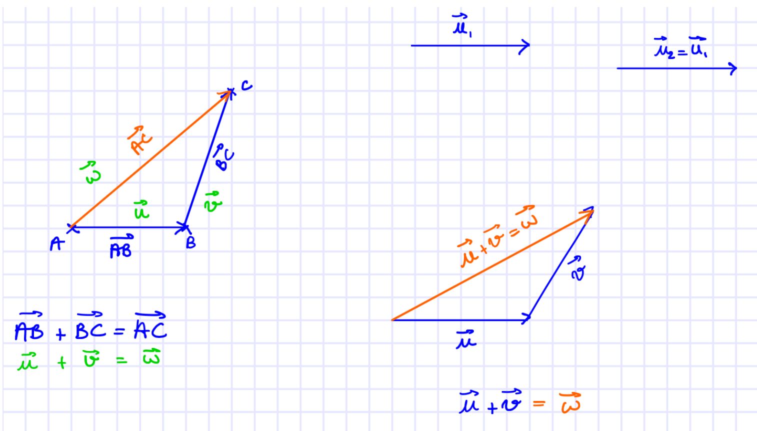


Figure 1: Example vector addition

Figure 1 demonstrates the concepts of vector equality (same magnitude, direction, and orientation) and independence from the point of attachment, particularly for vector addition. When working with forces $\vec{F}$, the point of application is an important characteristic, especially for rigid body kinematics later in the course. For instance, weight acts at the object's center of mass.

1.1 Cartesian coordinates

To manipulate vectors in space (in 3 dimensions), it is convenient to decompose them into their Cartesian components $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$.

If we place a vector starting from the origin of the coordinate system, the components of the vector are then the Cartesian coordinates of its endpoint A: see Figure 2.

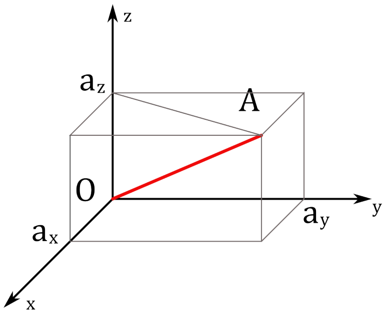


Figure 2: Here, $\vec{a} = \overrightarrow{OA}$ is composed of a_x, a_y, a_z

The vector $\vec{a}$ can be written using its components and the basis vectors of the coordinate system:

$$\vec{a} = a_x\vec{e}_x + a_y\vec{e}_y + a_z\vec{e}_z \tag{5}$$

If $\vec{e}_x, \vec{e}_y, \vec{e}_z$ are fixed in space, we can write the components of the vector $\vec{a}$ vertically, called **column notation**:

$$\vec{a} \left| \begin{array}{l} a_x \\ a_y \\ a_z \end{array} \right. \tag{6}$$

The **sum** of two vectors is calculated by summing their components:

$$\vec{a} \left| \begin{array}{l} a_x \\ a_y \\ a_z \end{array} \right. + \vec{b} \left| \begin{array}{l} b_x \\ b_y \\ b_z \end{array} \right. = \vec{a} + \vec{b} \left| \begin{array}{l} a_x + b_x \\ a_y + b_y \\ a_z + b_z \end{array} \right. \tag{7}$$

The **components** of the vector $\overrightarrow{AB}$ which starts as point A and ends at point B are obtained by subtracting the coordinates of point A from those of point B.

$$A \left| \begin{array}{l} x_A \\ y_A \\ z_A \end{array} \right. \; ; \; B \left| \begin{array}{l} x_B \\ y_B \\ z_B \end{array} \right. \quad \Longrightarrow \quad \overrightarrow{AB} \left| \begin{array}{l} x_B - x_A \\ y_B - y_A \\ z_B - z_A \end{array} \right. \tag{8}$$

1.2 Derivatives of vectors with respect to time

In physics, vectors are sometimes functions of time t:

$$\vec{a}(t) = a_x(t)\,\vec{e}_x + a_y(t)\,\vec{e}_y + a_z(t)\,\vec{e}_z \tag{9}$$

We will therefore need to compute their derivatives with respect to time. When the basis vectors $(\vec{e}x, \vec{e}y, \vec{e}_z)$ do not depend on time, we can write:

$$\frac{d\vec{a}}{dt} = \frac{da_x}{dt}\,\vec{e}_x + \frac{da_y}{dt}\,\vec{e}_y + \frac{da_z}{dt}\,\vec{e}_z \tag{10}$$

$$\text{or in column notation :} \quad \frac{d\vec{a}}{dt} \left| \begin{array}{l} da_x/dt \\ da_y/dt \\ da_z/dt \end{array} \right. \tag{11}$$

In some cases, the basis vectors will also depend on time: $\big(\vec{e}_{x'}(t), \vec{e}_{y'}(t), \vec{e}_{z'}(t)\big)$. In that case, we must apply the product rule for derivatives:

$$\frac{d\vec{a}}{dt} = \underbrace{\frac{da_{x'}}{dt}\,\vec{e}_{x'} + a_{x'}\frac{d\vec{e}_{x'}}{dt}}_{\frac{d}{dt}\left(a_{x'}\vec{e}_{x'}\right)} + \underbrace{\frac{da_{y'}}{dt}\,\vec{e}_{y'} + a_{y'}\frac{d\vec{e}_{y'}}{dt}}_{\frac{d}{dt}\left(a_{y'}\vec{e}_{y'}\right)} + \underbrace{\frac{da_{z'}}{dt}\,\vec{e}_{z'} + a_{z'}\frac{d\vec{e}_{z'}}{dt}}_{\frac{d}{dt}\left(a_{z'}\vec{e}_{z'}\right)} \tag{12}$$

1.3 Magnitude

The **magnitude** (or norm) of a vector is by definition the length of the segment it spans. For a given vector $\overrightarrow{AB}$, its magnitude is the length of the segment $[AB]$. The magnitude of $\vec{a}$ is obtained by calculating the square root of the sum of the squared components (in Cartesian coordinates):

$$\|\vec{a}\| := \sqrt{a_x^2 + a_y^2 + a_z^2}. \tag{13}$$

We may write $a = |\vec{a}|$, and thus refer to the magnitude of the velocity vector $\vec{v}$ simply as v.

1.4 Dot product (Scalar product)

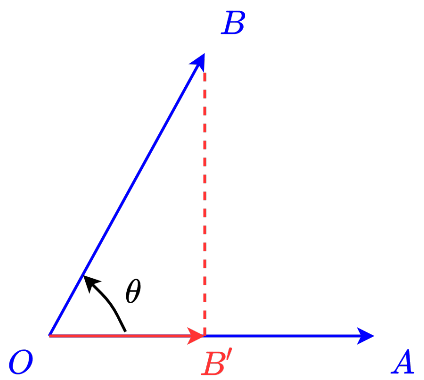


Figure 3: Dot product between $\overrightarrow{OA}$ and $\overrightarrow{OB}$, with projection $\overrightarrow{OB'}$.

Figure 3 illustrates the dot product operation between the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$:

$$\begin{aligned}\overrightarrow{OA} \cdot \overrightarrow{OB} &= \left\|\overrightarrow{OA}\right\| \left\|\overrightarrow{OB}\right\| \cos(\widehat{\overrightarrow{OA}, \overrightarrow{OB}}) \\ &= \left\|\overrightarrow{OA}\right\| \left\|\overrightarrow{OB}\right\| \cos\theta \\ &= \left\|\overrightarrow{OA}\right\| \left\|\overrightarrow{OB'}\right\|\end{aligned}$$

where $\overrightarrow{OB'}$ is the projection of the vector $\overrightarrow{OB}$ onto $\overrightarrow{OA}$. The dot product therefore allows us to **calculate the magnitude of a vector's projection**.

A few remarks about the dot product:

- For $\theta = \pi/2$, i.e., when $\overrightarrow{OA} \perp \overrightarrow{OB}$ (orthogonal), we have $\overrightarrow{OA} \cdot \overrightarrow{OB} = 0$
- When $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are colinear and in the same direction, we have $\overrightarrow{OA} \cdot \overrightarrow{OB} = \left|\overrightarrow{OA}\right| \left|\overrightarrow{OB}\right|$
- When $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are colinear and in opposite directions, we have $\overrightarrow{OA} \cdot \overrightarrow{OB} = -\left|\overrightarrow{OA}\right| \left|\overrightarrow{OB}\right|$

Referring back to Figure 2 and Eq.5, we can extract the components of a vector by taking its dot product with the basis vectors $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$:

$$a_i = \vec{a} \cdot \vec{e}_i, \quad i = x, y, z. \tag{14}$$

1.5 Cross product (Vector product)

The cross product of two non-colinear vectors $\vec{u}$ and $\vec{v}$ is defined as the unique vector $\vec{w} = \vec{u} \wedge \vec{v}$ such that:

- The vector $\vec{w}$ is **orthogonal** to both given vectors $\Longrightarrow$ direction of $\vec{w}$;
- The basis $(\vec{u}, \vec{v}, \vec{w})$ is **right-handed** $\Longrightarrow$ orientation of $\vec{w}$;
- $||\vec{w}|| = ||\vec{u}|| \cdot ||\vec{v}|| \cdot |\sin(\widehat{\vec{u}, \vec{v}})| \Longrightarrow$ magnitude of $\vec{w}$.

If $\vec{u}$ is colinear with $\vec{v}$, then $\vec{u} \wedge \vec{v} = \vec{0}$ (in particular, $\vec{u} \wedge \vec{u} = \vec{0}$). The cross product of two vectors is computed using the pairwise cross multiplication of components according to the following method:

$$\vec{a} \left| \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} \right. ; \quad \vec{b} \left| \begin{array}{c} b_1 \\ b_2 \\ b_3 \end{array} \right. \quad \Longrightarrow \quad \vec{a} \wedge \vec{b} \left| \begin{array}{c} a_2 b_3 - a_3 b_2 \\ -a_1 b_3 + a_3 b_1 \\ a_1 b_2 - a_2 b_1 \end{array} \right. \tag{15}$$

The formula obtained in Eq.15 implies that $\vec{a} \wedge \vec{b} = -, \vec{b} \wedge \vec{a}$. It is useful to know the identities of the cross product between the basis vectors of the standard right-handed Cartesian coordinate system $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$:

$$\vec{e}_x \wedge \vec{e}_y = \vec{e}_z \tag{16}$$
$$\vec{e}_y \wedge \vec{e}_z = \vec{e}_x \tag{17}$$
$$\vec{e}_z \wedge \vec{e}_x = \vec{e}_y \tag{18}$$

Note that often the alternative notation $\times$ instead of $\wedge$ is used to denote the cross product.

2 Trigonometry

In physics, we will often use angles in **radians**. A full circle measures 2π radians. This definition allows us to directly relate the arc length to the angle and radius using the formula $l = R\theta$, with θ in radians.

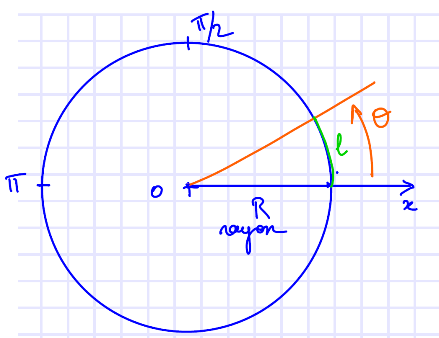


Figure 4: Unit circle and arc length. θ is defined relative to the (Ox) axis.

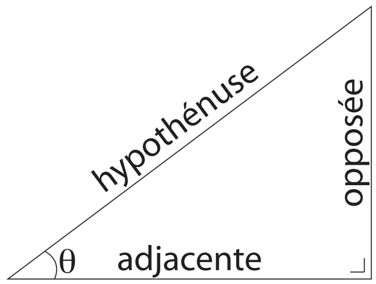


Figure 5: Trigonometry in a right triangle.

The trigonometric functions sin, cos, and tan are related to the lengths of the sides of a right triangle in the following way:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \tag{19}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \tag{20}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\text{opposite}}{\text{adjacent}} \tag{21}$$

Trigonometric identities can be useful for solving problems, particularly:

- $\sin(\pi-\theta) = \sin(\theta)$, $\cos(\pi-\theta) = -\cos(\theta)$
- $\sin(\pi/2-\theta) = \cos(\theta)$, $\cos(\pi/2-\theta) = \sin(\theta)$
- $\sin(-\theta) = -\sin(\theta)$, $\cos(-\theta) = \cos(\theta)$
- $\sin(\pi+\theta) = -\sin(\theta)$, $\cos(\pi+\theta) = -\cos(\theta)$

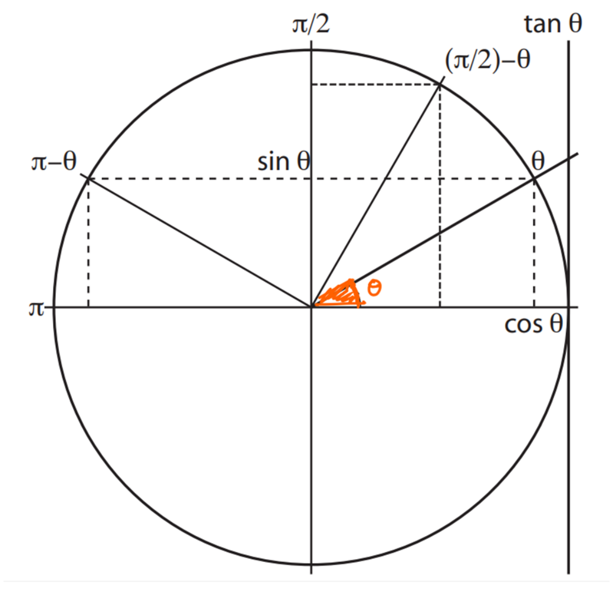


Figure 6: Trigonometry in the circle

- $\cos^2\theta + \sin^2\theta = 1$, for any angle θ.
- $\tan^2\theta = \dfrac{1}{\cos^2\theta} - 1$
- $\cos(a+b) = \cos a\cos b - \sin a\sin b$
- $\sin(a+b) = \sin a\cos b + \sin b\cos a$

It is common to use **vectors and trigonometry** together, for example when you need to find the components of a vector whose magnitude and angle with respect to a reference axis are known. In other words, projecting a given vector onto axes (not necessarily horizontal or vertical). In Figure 7, the weight vector $\vec{P}$ is decomposed into two components along the x and y axes. The angle of inclination α can be placed in several locations on the figure, allowing us to deduce: $\vec{P} = P \sin\alpha\, \vec{e}_x - P\cos\alpha\, \vec{e}_y$.

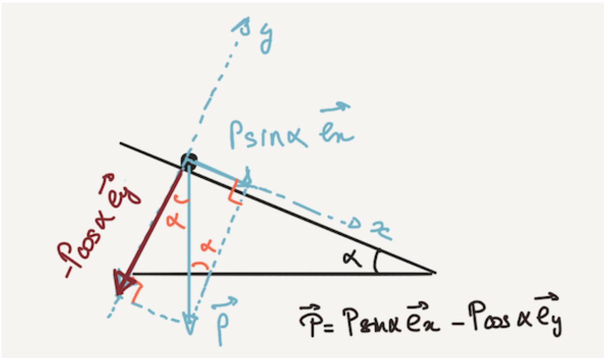


Figure 7: Projection du poids $\vec{P}$ sur les axes inclinés x et y

3 Calculus: derivatives, antiderivatives, integrals

Newton viewed calculus as the scientific description of the generation of motion, which we will see applied when we cover Newton's Laws in Chapter 3. The tools of calculus will be used extensively throughout this course: derivatives, antiderivatives, and integrals.

3.1 Derivatives

The derivative of a function represents the rate of change of the function's output with respect to its input. To start, let's consider a function $y = f(x)$ represented by the curve in Figure 8. We start at a point $(x_1, y_1 = f(x_1))$ on this curve, then move a distance Δx along the x-axis to a new point (x_1, y_2), where:

$$x_2 = x_1 + \Delta x \tag{22}$$

$$y_2 = f(x_2) = f(x_1 + \Delta x) \tag{23}$$

We moved our input value x by Δx, and the corresponding change in the function is:

$$\Delta y = f(x_1 + \Delta x) - f(x_1) \tag{24}$$

And thus the rate of change between these two points is $\Delta y/\Delta x$. This can also be characterized by an angle θ (see Figure 8), slope of the segment between points 1 and 2:

$$\tan\theta = \frac{\Delta y}{\Delta x}$$

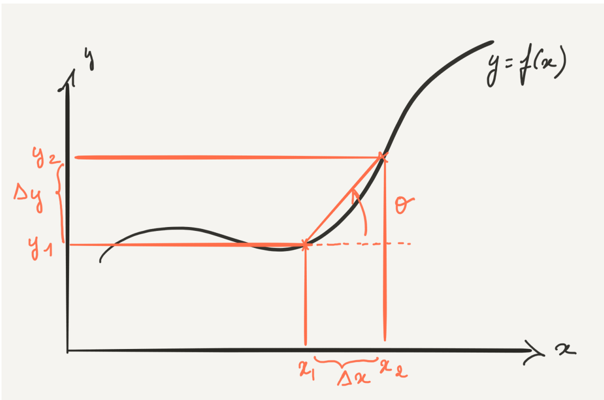


Figure 8: Here, point 1 denotes the point with coordinates (x_1, y_1) and point 2 the point with coordinates (x_2, y_2). We have $\Delta y = y_2 - y_1$ and $\Delta x = x_2 - x_1$.

The **derivative of the function** f (sometimes denoted f') at point 1 is the limit of $\tan\theta = \Delta y/\Delta x$ as point 2 approaches point 1. It is therefore the **slope of the tangent to the curve**.

$$f'(x_1) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} \tag{25}$$

Instead of using Δx we define the infinitesimal variation dx: $dx = \lim_{\Delta x \to 0} \Delta x$. We can then consider the **infinitesimal variation of the function** f given by:

$$df(x_1) = f(x_1 + dx) - f(x_1) \tag{26}$$

The derivative of f at point x_1 is equivalently:

$$f'(x_1) = \frac{f(x_1 + dx) - f(x_1)}{dx} = \frac{df}{dx}(x_1) \tag{27}$$

The following table summarizes the derivatives of some common functions:

Function	Derivative
$\cos x$	$-\sin x$
$\sin x$	$\cos x$
$\tan x$	$1/\cos^2 x$
$\ln x$	$1/x$
e^x	e^x
x^n	nx^{n-1}
x	1
constant value C	0

Table 1: Derivatives of common functions

For two functions f and g, we have the following rules:

- Product : $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$
- Composition : $(f(g(x))' = g'(x)f'(g(x))$

These two rules allow us to derive many others—for example, the derivative of f/g, which can be seen as the product of f and the composition of $1/x$ with g.

3.2 Antiderivatives (Indefinite Integrals)

The calculation of the antiderivative of $f(x)$ is the "reverse operation" of differentiation. That is, we seek a function $F(x)$ such that $F'(x) = f(x)$. Since the derivative of a constant is zero, we can add any constant to F without changing its derivative. Therefore, the antiderivative F of f is defined up to an arbitrary constant: for $\widetilde{F} = F + C$, we still have $\widetilde{F}'(x) = f(x)$. Thus $\widetilde{F}(x)$ and $F(x)$ are both antiderivatives of $f(x)$.

For example, for $f(x) = \cos x$, an antiderivative of f is $F(x) = \sin x + A$, where A is a constant of integration. If the value of F is known at a particular point, say $F(x_0) = F_0$, then we can determine $A = F_0 - \sin x_0$.

3.3 Integrals

When we want to compute the area $\mathcal{A}$ under the curve of $f(x)$ between a point $x = a$ and a point $x = z$, we use integration.

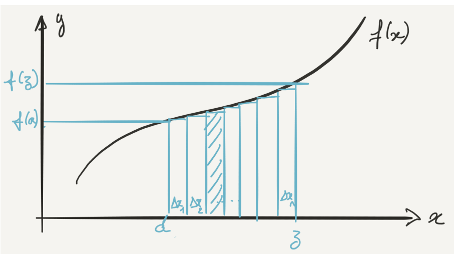


Figure 9: Visualization of the integral of a function and small rectangles of width Δx

This area can be approximated by the sum of small rectangles with height $f(x_i)$ and width Δx_i:

$$\mathcal{A} \simeq \sum_i f(x_i)\Delta x_i$$

And by letting Δx tend toward 0, we obtain the formula for the integral, where F is the antiderivative of f:

$$\mathcal{A} = \int_a^z f(x)dx = F(z) - F(a) \tag{28}$$

4 Taylor series

The Taylor series or Taylor expansion of a function allows us to replace a complicated function with an infinite sum of polynomial terms. The Taylor expansion of f around a point x_0 is given by:

$$f(x_0+\varepsilon) \simeq f(x_0) + \frac{d}{dx}f(x_0)\varepsilon + \cdots + \frac{\varepsilon^n}{n!}\frac{d^n}{dx^n}f(x_0) \tag{29}$$

This expansion is a polynomial of degree n as a function of small displacement ε, since the derivatives of f at x_0 are constants.

As an example, we compute the Taylor expansion of $f(x) = (1+x)^n$ for small x, that is, around $x_0 = 0$. The first two derivatives of f are $f'(x) = n(1+x)^{n-1}$ and $f''(x) = n(n-1)(1+x)^{n-2}$. We then have:

$$\begin{aligned}
& & f(x_0+\varepsilon) &= f(0) + \varepsilon f'(0) + \frac{\varepsilon^2}{2}f''(0) + \ldots \\
&\text{where one derives:} & f(\varepsilon) &= 1 + n\varepsilon + n(n-1)\frac{\varepsilon^2}{2} + \ldots \\
&\text{and replacing } \varepsilon \text{ with } x: & f(x) &= 1 + nx + n(n-1)\frac{x^2}{2} + \ldots \\
&\text{for } x \text{ close to 0, } x^2 \simeq 0: & f(x) &\simeq 1 + nx.
\end{aligned}$$

The following table summarizes some useful Taylor expansions around $x_0 = 0$:

Function	Truncated Taylor series
$\cos x$	$1 - x^2/2$
$\sin x$	x
$\tan x$	x
$\ln(1+x)$	x
e^x	$1+x$
$(1+x)^n$	$1+nx$
$1/(1+x)^n$	$1-nx$

Table 2: Taylor expansions at 0 of common functions

Chapter 1

Kinematics

1 Reference Frames & Coordinate Systems

The term kinematics refers to the *description of motion*. To describe motion mathematically, we need the following:

- **Reference frame**: a system of reference with respect to which motion is measured. For example: the laboratory frame, the geocentric frame (origin at the center of the Earth), the heliocentric frame (origin at the center of the Sun). The choice of reference frame depends on the problem being studied.

- **Origin** of the reference frame: a specific point that is fixed within the reference frame and relative to which the position of an object is defined.

- **Coordinate system** (basis): a system of unit vectors forming a right-handed orthonormal triad, for example $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$, used to mathematically describe and decompose motion.
  Note: the coordinate system is not necessarily fixed within the reference frame, and within a single reference frame, we can use multiple coordinate systems and switch between them.

- **Coordinates**: a set of quantities that specify the position of a point. In other words, these are the values provided to locate points in space. These coordinates are tied to the chosen basis. For example, we have Cartesian coordinates in a Cartesian system, or GPS coordinates on Earth.

2 Trajectory, Velocity, Acceleration

To begin the course, any objects under study will be represented by a physical point P. To describe the motion of P, we will use its position, velocity, and acceleration, as well as its trajectory.
The trajectory is the set of points in space through which the object (point) passes over time. It is represented by a line traced in space, and the position of the object as a function of time is commonly denoted by $P(t)$ — see Figure 1.1.

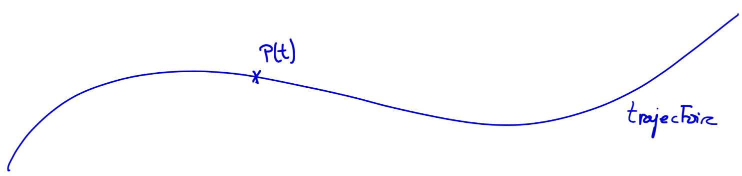


Figure 1.1: Trajectory of an object $P(t)$

The position is measured relative to the fixed origin of the reference frame (usually called O). We assume that at time t_1, the object is at P_1. The position of the object at $t = t_1$

is given by a position vector $\vec{r}(t_1) = \overrightarrow{OP_1}$. The position vector $\overrightarrow{OP}$ is therefore a time-dependent vector. Between t_1 and t_2, the object moves from $P(t_1) = P_1$ to $P(t_2) = P_2$. The vector $\overrightarrow{P_1P_2}$ is called the displacement vector $\Delta\vec{r}$ between t_1 and t_2 (see Figure 1.2). By applying vector addition rules, we get:

$$\vec{r}(t_1) + \Delta\vec{r} = \vec{r}(t_2) \tag{1.1}$$

$$\Delta\vec{r} = \overrightarrow{P_1P_2} = \vec{r}(t_2) - \vec{r}(t_1) \tag{1.2}$$

$$\Delta t = t_2 - t_1 \tag{1.3}$$

The average velocity between t_1 and t_2 is given by: $\vec{v}_{\text{moy}} = \dfrac{\Delta\vec{r}}{\Delta t}$.

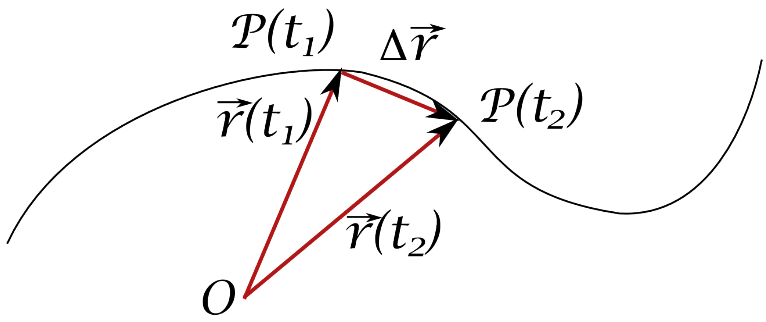


Figure 1.2: Trajectory of an object $P(t)$ and displacement vector $\Delta\vec{r}$ between t_1 and t_2.

The instantaneous velocity can be calculated using the displacement vector between $P(t)$ and $P(t+dt)$ — see Figure 1.3. As the point $P(t+dt)$ approaches $P(t)$, the displacement vector $\Delta\vec{r}$ becomes smaller, and so does the time interval dt. The displacement vector thus becomes an infinitesimal vector $d\vec{r}$.

$$\vec{v}(t) = \lim_{dt\to 0} \frac{d\vec{r}}{dt} \tag{1.4}$$

$$d\vec{r} = \vec{r}(t+dt) - \vec{r}(t) \tag{1.5}$$

$$\vec{v}(t) = \lim_{dt\to 0} \frac{\vec{r}(t+dt) - \vec{r}(t)}{dt} \tag{1.6}$$

Does this look familiar? The instantaneous velocity $\vec{v}(t)$ is the derivative of the position vector with respect to time. We can use the dot notation (indicating a time derivative): $\vec{v}(t) = \dot{\vec{r}}$, not to be confused with the dot product.

Remarks :

- When dt becomes very small ($dt \to 0$), the vector $d\vec{r}$ also becomes very small in magnitude and approaches the trajectory.

- The velocity vector $\vec{v}(t)$ is colinear with $d\vec{r}$. It is therefore tangent to the trajectory.

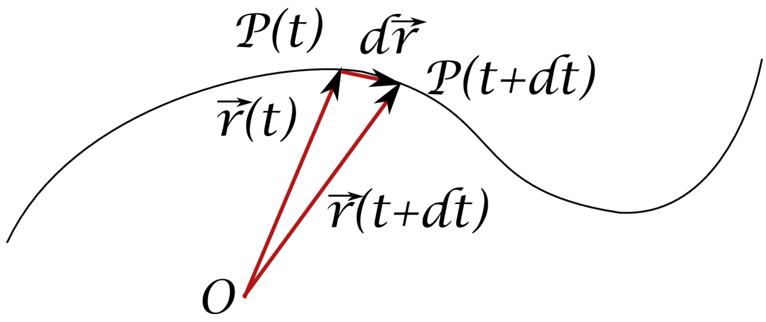


Figure 1.3: Trajectory of an object $P(t)$ and displacement vector $d\vec{r}$ between t and $t + dt$.

Acceleration corresponds to a change in the velocity vector. If the object has a velocity $\vec{v}(t_1)$ at time $t = t_1$ and $\vec{v}(t_2)$ at time $t = t_2$, then the **average acceleration** is given by:

$$\vec{a}_{\text{moy}} = \Delta\vec{v}/\Delta t \tag{1.7}$$

where $\Delta\vec{v} = \vec{v}(t_2) - \vec{v}(t_1)$ and $\Delta t = t_2 - t_1$.

By considering very short time intervals ($dt \to 0$) we obtain the **instantaneous acceleration**:

$$\vec{a}(t) = \lim_{dt\to 0} \frac{\vec{v}(t + dt) - \vec{v}(t)}{dt} \tag{1.8}$$

This is the derivative of the velocity vector with respect to time. We can again use dot notation: $\vec{a}(t) = \dot{\vec{v}}(t) = \ddot{\vec{r}}(t)$, where $\ddot{\vec{r}}(t)$ represents the second derivative with respect to time $d^2\vec{r}/dt^2$.

Table 1.1 summarizes the key results of this section concerning position, velocity, and acceleration.

Position	Velocity	Acceleration
$\vec{r}(t) = \overrightarrow{OP}$	$\vec{v}(t) = \dfrac{d\vec{r}}{dt} = \dot{\vec{r}}$	$\vec{a}(t) = \dfrac{d\vec{v}}{dt} = \dot{\vec{v}} = \ddot{\vec{r}}$

Table 1.1: Position, velocity, and acceleration (instantaneous) of a physical point P.

The scalar speed is the magnitude of the velocity vector: $v = \|\vec{v}\|$ and the scalar acceleration is $a = \|\vec{a}\|$. *Remark*: The magnitude of a derivative is different from the derivative of a magnitude. For the velocity vector $\vec{v}$, we have:

$$\underbrace{\frac{d\,\|\vec{v}\|}{dt}}_{\text{derivative of the magnitude}} \neq \underbrace{\left\|\frac{d\vec{v}}{dt}\right\|}_{\text{magnitude of the derivative}} \tag{1.9}$$

For example, in the case of uniform circular motion, $v = \|\vec{v}\| = \text{const}$, but the vector $\vec{v}$ changes over time (it rotates), so: $\frac{d\vec{v}}{dt} \neq \vec{0}$.

3 Cartesian coordinate systems

In this section, we focus on the Cartesian coordinate system. Consider the origin of the reference frame O and the three axes emerging from this point: (x, y, z), associated with the unit vectors $\vec{e}_x, \vec{e}_y, \vec{e}_z$, respectively. Recall that the triad $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$ forms a right-handed orthonormal basis.

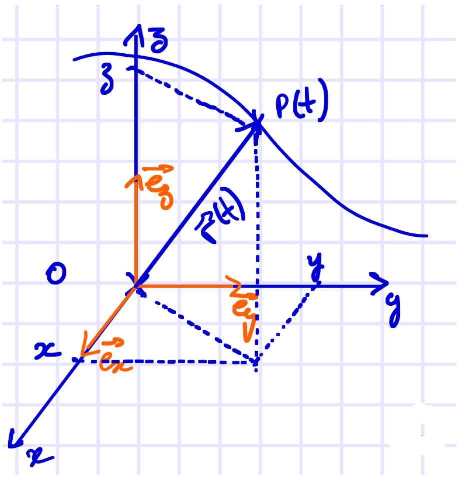


Figure 1.4: Trajectory of point P in Cartesian coordinates.

In Figure 1.4, we visualize the point P along its trajectory in space. At time t, the point $P(t)$ has coordinates (x, y, z) and the position vector $\vec{r}(t)$:

$$\vec{r}(t) = \overrightarrow{OP} = x\,\vec{e}_x + y\,\vec{e}_y + z\,\vec{e}_z \tag{1.10}$$

The **velocity vector** $\vec{v}(t)$ is given by :

$$\vec{v}(t) = \dot{\vec{r}}(t) = \frac{d}{dt}(x\,\vec{e}_x + y\,\vec{e}_y + z\,\vec{e}_z) \tag{1.11}$$

Given that the vectors $\vec{e}_x, \vec{e}_y, \vec{e}_z$ are fixed in time, we have:

$$\vec{v}(t) = \dot{x}\,\vec{e}_x + \dot{y}\,\vec{e}_y + \dot{z}\,\vec{e}_z \tag{1.12}$$

Note: The point P moves over time (along its trajectory). The coordinates x, y, z are therefore functions of time: $x(t), y(t), z(t)$, but the time dependence will often be implicit.

In the same way, we go from the velocity vector $\vec{v}$ to the acceleration vector $\vec{a}$ by differentiating once more with respect to time:

$$\vec{a}(t) = \dot{\vec{v}}(t) \quad \Longrightarrow \quad \vec{a}(t) \left| \begin{array}{l} \dot{v}_x = \ddot{x} \\ \dot{v}_y = \ddot{y} \\ \dot{v}_z = \ddot{z} \end{array} \right. \tag{1.13}$$

4 Polar coordinates

Polar coordinates are a 2-dimensional system that allows us to locate a point **in a plane**. The point P is represented by its distance ρ from the origin O and by the angle φ between the (Ox) axis and the vector $\overrightarrow{OP}$ (Figure 1.5). The polar coordinates (ρ, φ) and the Cartesian coordinates (x, y) are related by the following formula:

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{cases} \quad \text{with} \quad \rho \in [0, \infty) \quad \varphi \in [0, 2\pi) \tag{1.14}$$

The basis vectors $\vec{e}_\rho, \vec{e}_\varphi$ are not constant vectors. They are related to $\vec{e}_x, \vec{e}_y$ by:

$$\begin{cases} \vec{e}_\rho = \cos\varphi\,\vec{e}_x + \sin\varphi\,\vec{e}_y \\ \vec{e}_\varphi = -\sin\varphi\,\vec{e}_x + \cos\varphi\,\vec{e}_y \end{cases} \tag{1.15}$$

We see that if φ depends on time, then $\vec{e}_\rho$ and $\vec{e}_\varphi$ do as well.

Note: The basis vectors defined in Eq.1.15 are unit vectors (norm equal to 1); for example, for $\vec{e}_\rho$ we have:

$$\|\vec{e}_\rho\| = \sqrt{\cos^2\varphi + \sin^2\varphi} = 1$$

The time derivatives of the basis vectors $\vec{e}_\rho, \vec{e}_\varphi$ are given by:

$$\begin{cases} \dot{\vec{e}}_\rho = \dot{\varphi}\, \vec{e}_\varphi \\ \dot{\vec{e}}_\varphi = -\dot{\varphi}\, \vec{e}_\rho \end{cases} \tag{1.16}$$

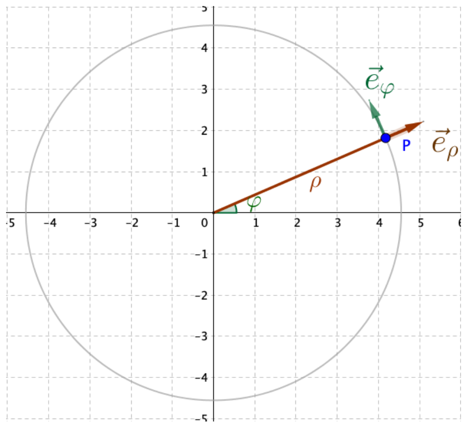


Figure 1.5: Description of point P in polar coordinates

Finally, we can express $\vec{r}, \vec{v}, \vec{a}$ in polar coordinates (i.e., using ρ, φ and $\vec{e}_\rho, \vec{e}_\varphi$). Table 1.2 summarizes these results:

Position	Velocity	Acceleration
$\vec{r}(t) = \rho\, \vec{e}_\rho$	$\vec{v}(t) = \dot{\rho}\, \vec{e}_\rho + \rho\dot{\varphi}\, \vec{e}_\varphi$	$\vec{a}(t) = (\ddot{\rho} - \rho\dot{\varphi}^2)\, \vec{e}_\rho + (2\dot{\rho}\dot{\varphi} + \rho\ddot{\varphi})\, \vec{e}_\varphi$

Table 1.2: Position, velocity and acceleration (instantaneous) in polar coordinates

5 Curvilinear Coordinates

In a curvilinear coordinate system (also called the Frenet frame), we consider the distance traveled along some trajectory: the **curvilinear abscissa** $s(t)$. The scalar speed along this trajectory is given by $v(t) = ds/dt$.

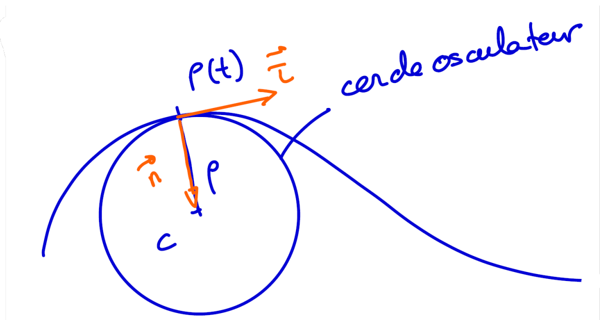


Figure 1.6: Unit vectors and osculating circle along the trajectory $P(t)$.

How can we get a vector representation of the motion? We consider an osculating circle at the point on the trajectory with radius ρ. We call $\vec{\tau}$ the unit vector tangent to the trajectory in the direction of motion, and $\vec{n}$ the unit vector pointing towards the center of the osculating circle (perpendicular to the trajectory). We can then define the velocity vector as:

$$\vec{v}(t) = v\,\vec{\tau} = \frac{ds}{dt}\,\vec{\tau} \tag{1.17}$$

And the acceleration vector as:

$$\vec{a}(t) = \frac{d}{dt}(v\vec{\tau}) = \frac{dv}{dt}\,\vec{\tau} + v\frac{d}{dt}(\vec{\tau}) = \frac{dv}{dt}\,\vec{\tau} + \frac{v^2}{\rho}\,\vec{n} \tag{1.18}$$

using $d\vec{\tau}/dt = v\vec{n}/\rho$, which follows from Equation 1.16. The first term is called the tangential acceleration $\vec{a}_\tau$ and the second the normal (or centripetal) acceleration $\vec{a}_n$.

Note: Tangential acceleration causes a change in scalar speed; normal acceleration causes a change in the direction of the velocity.

6 Cylindrical Coordinates

Cylindrical coordinates are 3-dimensional coordinate systems whose basis vectors vary depending on the position of the object. Cylindrical coordinates are simply the polar coordinate system for the plane (xy) with the addition of the z–axis.

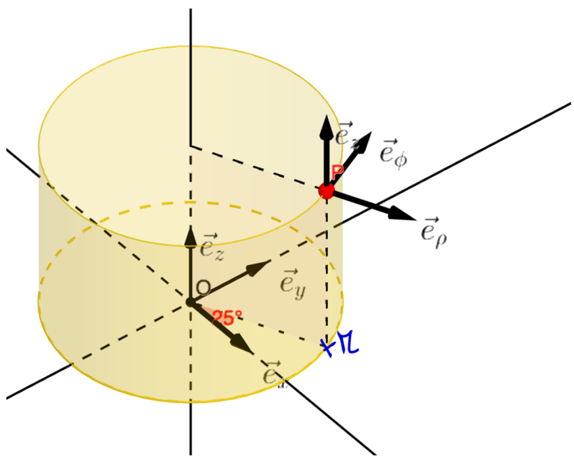


Figure 1.7: Description of point P in cylindrical coordinates

Cylindrical coordinates (ρ, φ, z) and Cartesian coordinates (x, y, z) are related as follows:

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ z = z \end{cases} \quad \text{with} \qquad \rho \in [0, \infty)\,, \quad \varphi \in [0, 2\pi)\,, \quad z \in (-\infty, \infty). \tag{1.19}$$

The basis vectors $\vec{e}_\rho, \vec{e}_\varphi, \vec{e}_z$ are related to $\vec{e}_x, \vec{e}_y, \vec{e}_z$ by :

$$\begin{cases} \vec{e}_\rho = \cos\varphi\,\vec{e}_x + \sin\varphi\,\vec{e}_y \\ \vec{e}_\varphi = -\sin\varphi\,\vec{e}_x + \cos\varphi\,\vec{e}_y \\ \vec{e}_z = \vec{e}_z \end{cases} \tag{1.20}$$

Note: The vector $\vec{e}_z$ is constant, but the vectors $\vec{e}_\rho$ and $\vec{e}_\varphi$ still vary according to Eq. 1.16. As before, we can express $\vec{r}, \vec{v}, \vec{a}$ in cylindrical coordinates (i.e., using ρ, φ, z and $\vec{e}_\rho, \vec{e}_\varphi, \vec{e}_z$). Table 1.2 summarizes these results:

Position	Velocity	Acceleration
$\vec{r}(t) = \rho\,\vec{e}_\rho + z\,\vec{e}_z$	$\vec{v}(t) = \dot{\rho}\,\vec{e}_\rho + \rho\dot{\varphi}\,\vec{e}_\varphi + \dot{z}\,\vec{e}_z$	$\vec{a}(t) = (\ddot{\rho} - \rho\dot{\varphi}^2)\,\vec{e}_\rho + (2\dot{\rho}\dot{\varphi} + \rho\ddot{\varphi})\,\vec{e}_\varphi + \ddot{z}\,\vec{e}_z$

Table 1.3: Position, velocity and acceleration (instantaneous) in cylindrical coordinates

7 Spherical Coordinates

Spherical coordinates are very useful when the physical problem has spherical symmetry. It is possible to describe the position of a point in space with respect to its distance from the origin and two rotation angles. The distance OP then represents the first coordinate of this system, denoted ρ. The angle between the vertical z-axis and the segment OP is the second coordinate, denoted θ, and finally, the angle of rotation of the point around the z-axis is the third coordinate, denoted ϕ.

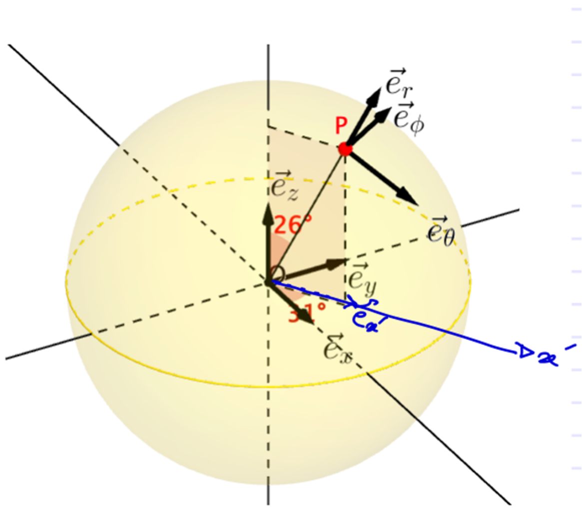


Figure 1.8: Description of point P in spherical coordinates

The spherical coordinates (r, θ, φ) are related to the Cartesian coordinates (x, y, z) as follows :

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \quad \text{with} \quad \rho \in [0, \infty)\,, \quad \theta \in [0, \pi]\,, \quad \varphi \in [0, 2\pi). \tag{1.21}$$

The basis vectors $\vec{e}_r, \vec{e}_\theta, \vec{e}_\varphi$ are related to $\vec{e}_x, \vec{e}_y, \vec{e}_z$ by :

$$\begin{cases} \vec{e}_r = \sin\theta\cos\varphi\,\vec{e}_x + \sin\theta\sin\varphi\,\vec{e}_y + \cos\theta\,\vec{e}_z \\ \vec{e}_\theta = \cos\theta\cos\varphi\,\vec{e}_x + \cos\theta\sin\varphi\,\vec{e}_y + -\sin\theta\,\vec{e}_z \\ \vec{e}_\varphi = -\sin\varphi\,\vec{e}_x + \cos\varphi\,\vec{e}_y \end{cases} \tag{1.22}$$

The time derivatives are

$$\begin{cases} \dot{\vec{e}}_r = \dot{\theta}\,\vec{e}_\theta + \dot{\varphi}\sin\theta\,\vec{e}_\varphi \\ \dot{\vec{e}}_\theta = -\dot{\theta}\,\vec{e}_r + \dot{\varphi}\cos\theta\,\vec{e}_\varphi \\ \dot{\vec{e}}_\varphi = -\dot{\varphi}\sin\theta\,\vec{e}_r - \dot{\varphi}\cos\theta\,\vec{e}_\theta \end{cases} \tag{1.23}$$

Finally, the vectors of motion $(\vec{r}, \vec{v}, \vec{a})$ are:

$$\text{position :} \quad \vec{r}(t) = r\,\vec{e}_r \tag{1.24}$$

$$\text{velocity :} \quad \vec{v}(t) = \dot{r}\,\vec{e}_r + r\dot{\theta}\,\vec{e}_\theta + r\dot{\varphi}\sin\theta\,\vec{e}_\varphi \tag{1.25}$$

$$\text{acceleration :} \quad \vec{a}(t) = a_r\,\vec{e}_r + a_\theta\,\vec{e}_\theta + a_\varphi\,\vec{e}_\varphi \tag{1.26}$$

where the components of the acceleration vector (Eq. 1.26) are:

$$a_r = \ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2\sin^2\theta \tag{1.27}$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\varphi}^2\cos\theta\sin\theta \tag{1.28}$$

$$a_\varphi = r\ddot{\varphi}\sin\theta + 2r\dot{\varphi}\dot{\theta}\cos\theta + 2\dot{r}\dot{\varphi}\sin\theta \tag{1.29}$$

8 Circular Motion in Cylindrical Coordinates

In this section, we will see how to represent a rotation using a vector quantity. To do this, we consider a point P undergoing circular motion in the (O, x, y) plane with angular velocity $\omega = \dot{\varphi}$ (where φ comes from cylindrical coordinates): Figure 1.9.

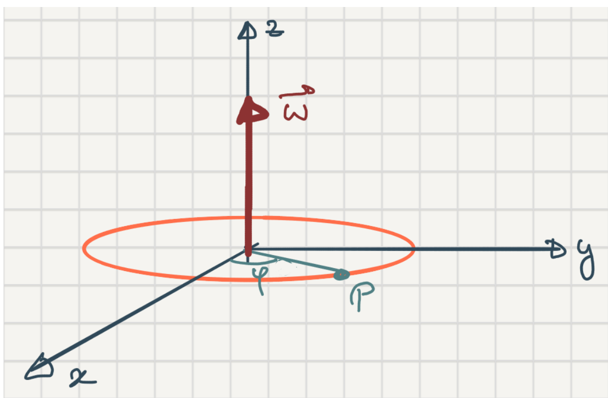


Figure 1.9: Circular motion of point P described in cylindrical coordinates.

Let $\vec{\omega}$ be the vector with magnitude ω, perpendicular to the plane containing the circle traced by P and oriented according to the right-hand rule. The vector $\vec{\omega}$ is called the **rotation vector**; it contains information about both the speed and the axis of rotation. Furthermore, the position vector $\vec{r}$ is affected by the rotation. The relationship between $\dot{\vec{r}}$ and $\vec{\omega}$ is given by:

$$\frac{d\vec{r}}{dt} = \vec{\omega} \wedge \vec{r} \tag{1.30}$$

provided that $\vec{r}$ has constant magnitude (e.g., circular motion).
Note: More generally, for any vector $\vec{u}$ of constant magnitude undergoing a rotation given by $\vec{\omega}$, we have $\dot{\vec{u}} = \vec{\omega} \wedge \vec{u}$.

Chapter 2

Accelerated Reference Frames

Newton's laws are applicable in a so-called Galilean reference frame, also called inertial reference frame. This is a frame that is either stationary or moving with constant linear velocity. However, in some cases, the reference frame best suited for describing a problem (e.g., a moving train, an orbiting planet) is not Galilean. In such cases, we must establish the relationship between the two frames of reference.

1 Introduction and Notation

Let $\mathcal{R}$ be a fixed reference frame, equipped with the Cartesian coordinate system $(O, \vec{x}_1, \vec{x}_2, \vec{x}_3)$, and let $\mathcal{R}'$ be a reference frame in motion relative to $\mathcal{R}$, equipped with the Cartesian coordinate system $(A, \vec{y}_1, \vec{y}_2, \vec{y}_3)$.

We denote by $\vec{e}_{x_i}$ and $\vec{e}_{y_i}$ the unit vectors of these two coordinate systems, respectively, with i the axis index $i \in \{x, y, z\}$. A point P can be located in both reference frames in the following way:

In $\mathcal{R}$:	$\overrightarrow{OP} = \sum_i x_i \vec{e}_{x_i}$	$\vec{v}_{\mathcal{R}}(P) = \sum_i \dot{x}_i \vec{e}_{x_i}$	$\vec{a}_{\mathcal{R}}(P) = \sum_i \ddot{x}_i \vec{e}_{x_i}$
In $\mathcal{R}'$:	$\overrightarrow{AP} = \sum_i y_i \vec{e}_{y_i}$	$\vec{v}_{\mathcal{R}'}(P) = \sum_i \dot{y}_i \vec{e}_{y_i}$	$\vec{a}_{\mathcal{R}'}(P) = \sum_i \ddot{y}_i \vec{e}_{y_i}$

The motion of $\mathcal{R}'$ in $\mathcal{R}$ can be separated into two components: a rotation and a translation. The translation gives the motion of point A in $\mathcal{R}$, and the rotation describes the rotation of the (y_j) axes relative to the (x_i) axes. We denote the rotation vector by $\vec{\omega}$. The vectors $\vec{e}_{y_i}$ therefore change in $\mathcal{R}$. Their derivative is given by

$$\frac{d}{dt}\vec{e}_{y_j} = \dot{\vec{e}}_{y_j} = \vec{\omega} \wedge \vec{e}_{y_j} \tag{2.1}$$

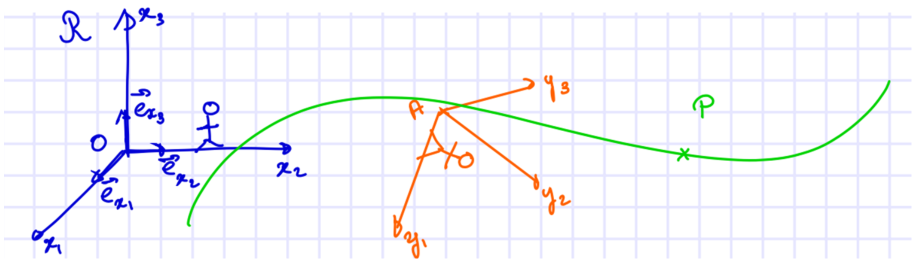


Figure 2.1:

2 Position, Velocity, and Acceleration

We consider a point P moving in $\mathcal{R}'$, equipped with the frame $(A, \vec{y}_1, \vec{y}_2, \vec{y}_3)$. Furthermore, $\mathcal{R}'$ is moving within $\mathcal{R}$: $(O, \vec{x}_1, \vec{x}_2, \vec{x}_3)$.
The **position vectors** in the two reference frames are related as follows:

$$\underbrace{\overrightarrow{OP}}_{\text{position of } P \text{ in } \mathcal{R}} = \underbrace{\overrightarrow{OA}}_{\text{position of } A \text{ in } \mathcal{R}} + \underbrace{\overrightarrow{AP}}_{\text{position of } P \text{ in } \mathcal{R}'} \tag{2.2}$$

By differentiating with respect to time, we can obtain an expression to relate the velocities:

$$\frac{d}{dt}(\overrightarrow{OP}) = \vec{v}_{\mathcal{R}}(P) = \frac{d}{dt}(\overrightarrow{OA}) + \frac{d}{dt}(\overrightarrow{AP}) = \vec{v}_{\mathcal{R}}(A) + \sum_i \frac{d}{dt}(y_i \vec{e}_{y_i}) \tag{2.3}$$

By differentiating and using the terms from the previous section in both frames, we then have the following for the **velocities**:

$$\vec{v}_{\mathcal{R}}(P) = \vec{v}_{\mathcal{R}}(A) + \vec{v}_{\mathcal{R}'}(P) + \vec{\omega} \wedge \overrightarrow{AP} \tag{2.4}$$

We proceed similarly for accelerations, by differentiating the velocity equation with respect to time:

$$\frac{d}{dt}(\vec{v}_{\mathcal{R}}(P)) = \vec{a}_{\mathcal{R}}(P) = \vec{a}_{\mathcal{R}}(A) + \frac{d}{dt}(\sum_i \dot{y}_i \vec{e}_{y_i}) + \frac{d}{dt}(\vec{\omega} \wedge \sum_i y_i \vec{e}_{y_i}) \tag{2.5}$$

Again, using the product rule and the expressions from the previous section, we then have the following for the **accelerations**:

$$\vec{a}_{\mathcal{R}}(P) = \vec{a}_{\mathcal{R}'}(P) + \vec{a}_{\mathcal{R}}(A) + \dot{\vec{\omega}} \wedge \overrightarrow{AP} + \vec{\omega} \wedge (\vec{\omega} \wedge \overrightarrow{AP}) + 2\vec{\omega} \wedge \vec{v}_{\mathcal{R}'}(P) \tag{2.6}$$

3 Analysis and Special Cases

3.1 Special Case 1

$\mathcal{R}'$ has uniform translational motion in $\mathcal{R}$:

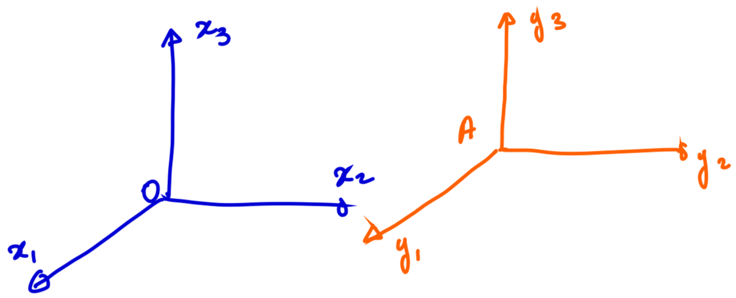


Figure 2.2: Uniform translation of the reference frame

We start from equations (2.4) and (2.6). Here, point A has uniform rectilinear motion in $\mathcal{R}$, and there is therefore no rotation of $\mathcal{R}'$ relative to $\mathcal{R}$; the axes remain parallel. These two conditions can be expressed as:

$$\vec{v}_{\mathcal{R}}(A) = \overrightarrow{\text{constant}} \tag{2.7}$$
$$\vec{a}_{\mathcal{R}}(A) = \vec{0} \tag{2.8}$$
$$\vec{\omega} = \dot{\vec{\omega}} = \vec{0} \tag{2.9}$$

So, using these results in equations (2.4) and (2.6), we deduce that:

$$\vec{v}_{\mathcal{R}}(P) = \vec{v}_{\mathcal{R}}(A) + \vec{v}_{\mathcal{R}'}(P) \tag{2.10}$$
$$\vec{a}_{\mathcal{R}}(P) = \vec{a}_{\mathcal{R}'}(P) \tag{2.11}$$

The resulting velocity is thus the composition of velocities in a translational motion. *Note:* In the next chapter, we will study Newton's laws. The fact that the acceleration is the same here proves that $\mathcal{R}'$ is a Galilean reference frame.

3.2 Special Case 2

$\mathcal{R}'$ undergoes uniform rotational motion in $\mathcal{R}$ with $A = O$ and P fixed in $\mathcal{R}'$:

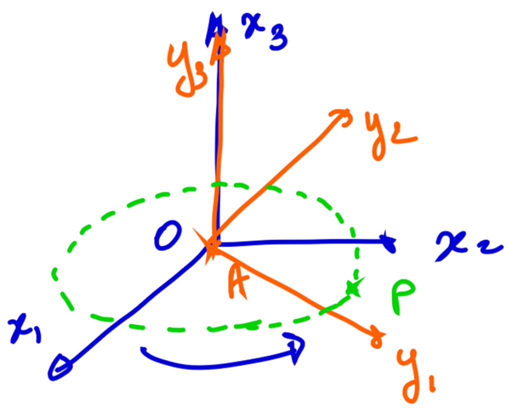


Figure 2.3: Uniform rotation of the reference frame around Ox_3

We once again start from equations (2.4) and (2.6). Here, points A and O coincide, and P is fixed in $\mathcal{R}'$, so it describes uniform circular motion. These constraints imply that:

$$\vec{v}_{\mathcal{R}'}(P) = \vec{a}_{\mathcal{R}'}(P) = \vec{0} \tag{2.12}$$
$$\vec{v}_{\mathcal{R}}(A) = \vec{a}_{\mathcal{R}}(A) = \vec{0} \tag{2.13}$$
$$\vec{\omega} = \overrightarrow{\text{constant}} \tag{2.14}$$
$$\dot{\vec{\omega}} = \vec{0} \tag{2.15}$$

Thus, using these results in equations (2.4) and (2.6), we deduce that:

$$\vec{v}_{\mathcal{R}}(P) = \vec{\omega} \wedge \overrightarrow{OP} \tag{2.16}$$
$$\vec{a}_{\mathcal{R}}(P) = \vec{\omega} \wedge (\vec{\omega} \wedge \overrightarrow{OP}) \tag{2.17}$$

Let us analyze this result (concerning the motion of point P) by taking a top-down view of the situation and introducing familiar notation:

$$\vec{e}_{\rho} = \vec{e}_{y_1} \tag{2.18}$$
$$\vec{e}_{\varphi} = \vec{e}_{y_2} \tag{2.19}$$

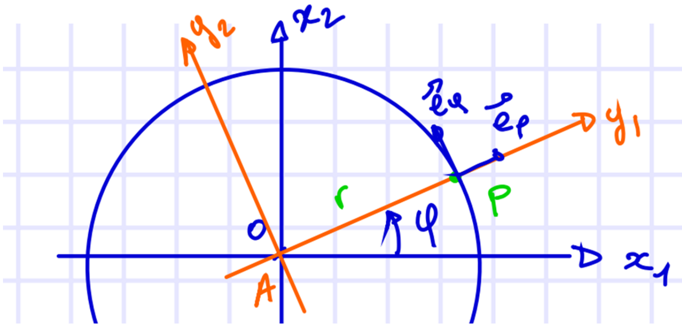


Figure 2.4:

We can then re-express $\vec{v}_{\mathcal{R}}(P)$ and $\vec{a}_{\mathcal{R}}(P)$ in terms of the coordinates $\vec{e}_\rho$ and $\vec{e}_\varphi$:

$$\vec{v}_{\mathcal{R}}(P) = \vec{\omega} \wedge \overrightarrow{OP} = \omega \vec{e}_{y3} \wedge r\vec{e}_{y_1} = r\omega\vec{e}_\varphi \tag{2.20}$$

$$\vec{a}_{\mathcal{R}}(P) = \omega \vec{e}_{y3} \wedge r\omega\vec{e}_{y_2} = -r\omega^2\vec{e}_\rho \tag{2.21}$$

The result obtained for the velocity is exactly the one previously found for velocity in polar coordinates; and for acceleration, it corresponds to the result for **centripetal acceleration**.

3.3 Special Case 3

$\mathcal{R}'$ undergoes uniform rotational motion in $\mathcal{R}$ with $A = O$ and P free:

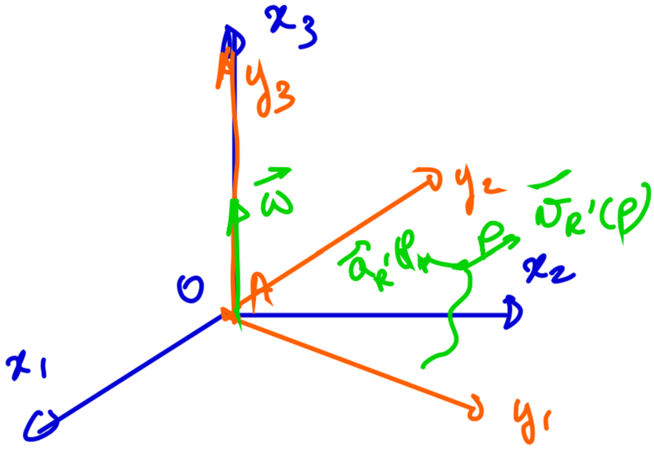


Figure 2.5: Uniform rotation of the reference frame with free P

This case is similar to the previous one, but since P is not fixed in $\mathcal{R}'$, there are fewer simplifications in equations (2.4) and (2.6): we only have $\vec{v}_{\mathcal{R}}(A) = \vec{a}_{\mathcal{R}}(A) = \vec{0}$ as well as $\vec{\omega} = \overrightarrow{\text{constant}}$ and its consequence $\dot{\vec{\omega}} = \vec{0}$. We then have:

$$\vec{v}_{\mathcal{R}}(P) = \vec{v}'_{\mathcal{R}}(P) + \vec{\omega} \wedge \overrightarrow{OP} \tag{2.22}$$

$$\vec{a}_{\mathcal{R}}(P) = \vec{a}_{\mathcal{R}'}(P) + \vec{\omega} \wedge (\vec{\omega} \wedge \overrightarrow{OP}) + 2\vec{\omega} \wedge \vec{v}_{\mathcal{R}'}(P) \tag{2.23}$$

The first term of the acceleration is called the **Relative Acceleration**, the second is the centripetal acceleration (seen in the previous special case), and the third is the **Coriolis**

acceleration. This last term reflects not only the motion of P in $\mathcal{R}'$, but also the rotation of $\mathcal{R}'$ within $\mathcal{R}$.

Finally, the terminology in the general case is:

$$\vec{a}_{\mathcal{R}}(P) = \underbrace{\vec{a}_{\mathcal{R}'}(P)}_{\text{relative acceleration}} + \underbrace{\vec{a}_{\mathcal{R}}(A) + \dot{\vec{\omega}} \wedge \overrightarrow{AP} + \vec{\omega} \wedge (\vec{\omega} \wedge \overrightarrow{AP})}_{\text{transport acceleration}} + \underbrace{2\vec{\omega} \wedge \vec{v}_{\mathcal{R}'}(P)}_{\text{Coriolis acceleration}} \tag{2.24}$$

Chapter 3

Newton's Laws

1 Introduction

2 Mass and Momentum

Mass represents the amount of matter. It is an *extensive* quantity, meaning that a system composed of the combination of m_1 and m_2 will have a total mass equal to the sum of the two: $m_1 + m_2$. In classical mechanics, mass is conserved.

Momentum is an *extensive vector quantity* that characterizes the state of motion, and is defined as:

$$\vec{p} = m\vec{v} \tag{3.1}$$

A system composed of the combination of objects 1 and 2, having masses m_1 and m_2, and velocities $\vec{v}_1$ and $\vec{v}_2$ respectively, will have a total momentum $\vec{p}$, where, for $i \in \{1, 2\}$:

$$\vec{p} = \vec{p}_1 + \vec{p}_2 \tag{3.2}$$

$$\vec{p}_i = m_i\vec{v}_i \tag{3.3}$$

Like mass, momentum is a conserved quantity.

3 Newton's First Law

"Every object perseveres in its state of rest, or of uniform motion in a straight line, unless it is compelled to change that state by forces impressed thereon."

The so-called "state of motion" is the momentum, expressed by equation (3.1). By reformulating this law in mathematical form, we can say that if the total forces $\vec{F} = \vec{0}$, then $\vec{p} = \overrightarrow{\text{constant}}$. For this law to hold, we must be in an inertial frame of reference, called a **Galilean frame**, that is, a frame at rest or moving with uniform rectilinear motion.

A **force** is the action exerted by one body on another. There are several types:

- Fundamental Forces (arising from nature): gravity, electromagnetism, and the strong and weak nuclear forces
- Phenomenological Forces (consequences of the former): friction, tension in a rope...
- Inertial/Fictitious Forces, which are not due to the action of one body on another, but rather felt by an observer in a non-Galilean frame.

It is also possible to distinguish between **internal and external** forces. For example, in a drop of water, the molecules exert forces on each other (internal), but the drop is subject to its weight (external).

4 Newton's Second Law

"The change of motion of an object is proportional to the force impressed; and is made in the direction of the straight line in which the force is impressed."

Mathematically:

$$\Delta\vec{p} = \vec{F}\Delta t \tag{3.4}$$

If $\vec{F}$ is the force, then $\vec{F}\Delta t$ is the impulse (motive force), and we have:

$$d\vec{p} = \vec{F}dt \tag{3.5}$$

where $\vec{F}$ is the sum of all external forces. By dividing both sides by dt, we obtain the general formulation:

$$\sum \vec{F} = \frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a} \tag{3.6}$$

Where here we have assumed that m is constant. In this case, we call equation (3.6) the **Fundamental principle of dynamics**.

5 Newton's Third Law

"To every action, there is always opposed an equal reaction; or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts."

This means that for every force applied by one object on another, an equal and oppositely directed force is applied back on the first object. This is the case, for example, when a weight rests on a table, or a rock is pulled by a rope...

Note: There is no time delay between these forces—they act simultaneously as long as the objects are interacting.

This law allows us to assert that $\sum \vec{F}_{int} = \vec{0}$ because internal forces cancel in pairs, as for every internal force (from one molecule on another), an equal and opposite force exists to cancel it.

6 Force Analysis

In an inertial reference frame, $\sum \vec{F}_{ext} = m\vec{a}$. Dimensional analysis lets us conclude that $1\text{N} = 1\text{kg}\,\text{m}\,\text{s}^{-2}$. To apply Newton's second law and analyze the forces, one can proceed as follows:

- Identify the system
- Choose the reference frame
- Choose a right-handed orthonormal coordinate system
- Draw a diagram
- Project the forces
- List the initial conditions
- Solve the equations

7 Non-Inertial Reference Frame

Suppose we are in a non-inertial reference frame $\mathcal{R}'$, undergoing both translation and rotation, with $\vec{\omega} = \overrightarrow{\text{constant}}$.

In order to apply Newton's second law, one must be in an inertial (Galilean) reference frame, therefore the following equation is only valid in $\mathcal{R}$:

$$\sum \vec{F}_{ext} = m\vec{a} = m\vec{a}_{\mathcal{R}}(P) \tag{3.7}$$

However, we are interested in the acceleration as observed in our non-inertial frame $\vec{a}_{\mathcal{R}'}(P)$. We we will use equation (2.6) from the previous chapter to find $\vec{a}_{\mathcal{R}'}(P)$ as a function of $\vec{a}_{\mathcal{R}}(P)$ and additional terms.

We know that the rotation occurs at constant angular velocity, so the 4th term in equation (2.6) is zero, and we can then isolate $\vec{a}_{\mathcal{R}'}(P)$ and multiply the entire expression by m, yielding:

$$m\vec{a}_{\mathcal{R}'}(P) = m\vec{a}_{\mathcal{R}}(P) - m\vec{a}_{\mathcal{R}}(A) - m\vec{\omega} \wedge (\vec{\omega} \wedge \overrightarrow{AP}) - m * 2\vec{\omega} \wedge \vec{v}_{\mathcal{R}'}(P) \tag{3.8}$$

It is possible to use equation (3.7) to re-express the result as:

$$m\vec{a}_{\mathcal{R}'}(P) = \sum \vec{F}_{ext} - m[\vec{a}_{\mathcal{R}}(A) + \vec{\omega} \wedge (\vec{\omega} \wedge \overrightarrow{AP}) + 2\vec{\omega} \wedge \vec{v}_{\mathcal{R}'}(P)] \tag{3.9}$$

We see that the additional terms multiplied by m have the dimension of a force, but only appeared because we are working in $\mathcal{R}'$. These are called *inertial forces* or *fictitious forces*. They can be broken down into:

- the Coriolis force: $-2m\vec{\omega} \wedge \vec{v}_{\mathcal{R}'}(P)$
- the transport (or driving) force: $-m(\vec{a}_{\mathcal{R}}(A) + \vec{\omega} \wedge (\vec{\omega} \wedge \overrightarrow{AP}))$

7.1 Example 1:

Let us consider the case where a pendulum oscillates in a train, which has constant acceleration and no rotation:

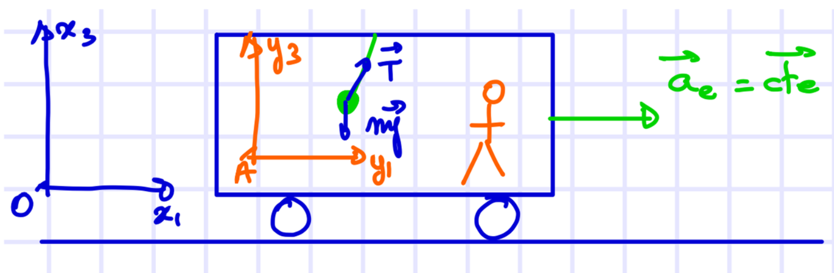


Figure 3.1: Pendulum in a train with constant acceleration, without rotation

We have here:

- $\mathcal{R}$ Galilean with (O, x_1, x_2, x_3)
- $\mathcal{R}'$ non-Galilean attached to the train car with (A, y_1, y_2, y_3)

- $\vec{\omega} = \vec{0}$
- $\vec{a}_{\mathcal{R}}(A) = \vec{a}_e = \overrightarrow{\text{constant}}$

If we now take expression (3.8) and apply these constraints, we obtain:

$$m\vec{a}_{\mathcal{R}'}(P) = m\vec{a}_{\mathcal{R}}(P) - m\vec{a}_e = \sum \vec{F}_{ext} - m\vec{a}_e = (m\vec{g} + \vec{T}) - m\vec{a}_e \tag{3.10}$$

Thus, the observer in the non-Galilean reference frame $\mathcal{R}'$ sees the pendulum subject to the external forces (weight and tension), but also to the "force" due to the train's constant acceleration.

7.2 Example 2:

Let us consider the case where an object is stationary in a uniformly rotating reference frame:

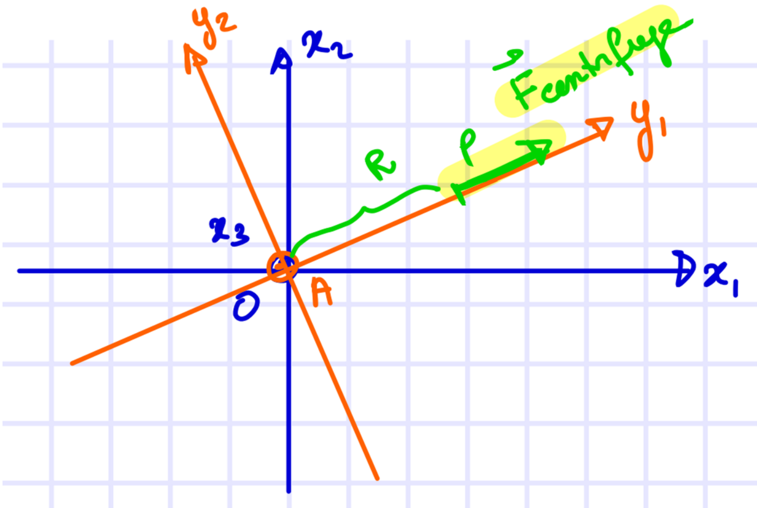


Figure 3.2: Stationary object in a uniformly rotating reference frame

We have here:

- $\mathcal{R}$ fixed and Galilean, with (O, x_1, x_2, x_3)
- $\mathcal{R}'$ non-Galilean, rotating constantly around the x_3 axis, with (A, y_1, y_2, y_3)
- A coincides with O: $\vec{v}_{\mathcal{R}}(A) = \vec{a}_{\mathcal{R}}(A) = \vec{0}$
- P stationary on (Ay_1) at a distance R from A: $\vec{v}_{\mathcal{R}'}(P) = \vec{0}$
- $\vec{\omega} = \omega\vec{e}_{y_3} = \omega\vec{e}_{x_3} = \overrightarrow{\text{constant}}$

P then describes a uniform circular motion. By applying the constraints to equation (3.8), we obtain:

$$m\vec{a}_{\mathcal{R}'}(P) = \sum \vec{F}_{ext} - m\vec{\omega} \wedge (\vec{\omega} \wedge \overrightarrow{OP}) = \sum \vec{F}_{ext} + mR\omega^2\vec{e}_{y_1} \tag{3.11}$$

P has a uniform circular motion in $\mathcal{R}$, and therefore a centripetal acceleration $\vec{a}_{\mathcal{R}}(P) = -R\omega^2\vec{e}_{y_1}$. This results in a "centrifugal force" in $\mathcal{R}'$: $mR\omega^2\vec{e}_{y_1}$

7.3 Example 3:

Let us now consider the case where the object in a uniformly rotating reference frame can move freely:

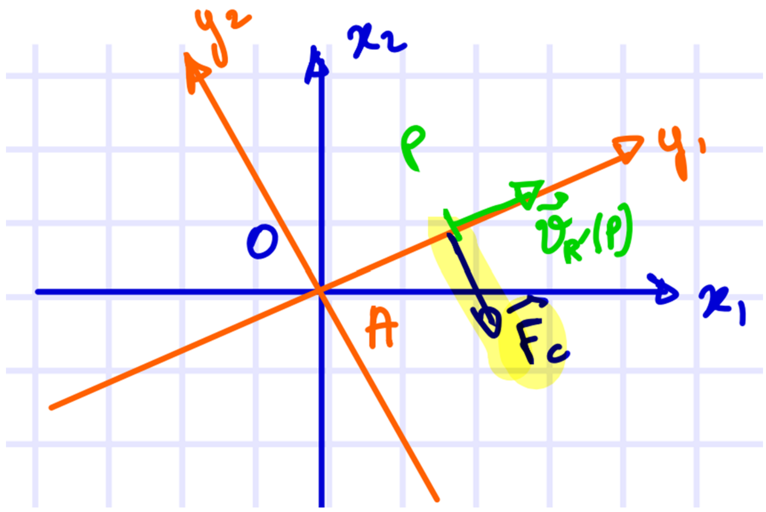


Figure 3.3: Free object in a uniformly rotating reference frame

Here, only the term in $\vec{a}_{\mathcal{R}}(A)$ cancels out because A and O remain coincident, and equation (3.8) then becomes:

$$m\vec{a}_{\mathcal{R}'}(P) = \sum \vec{F}_{ext} - m\vec{\omega} \wedge (\vec{\omega} \wedge \overrightarrow{OP}) - 2m\vec{\omega} \wedge \vec{v}_{\mathcal{R}'}(P) \tag{3.12}$$

Suppose that P moves uniformly along (Ay_1), so that $\vec{v}_{\mathcal{R}'}(P) = v_0\vec{e}_{y_1}$, then the previous expression becomes:

$$m\vec{a}_{\mathcal{R}'}(P) = \sum \vec{F}_{ext} + mR\omega^2\vec{e}_{y_1} - 2m\omega v_0\vec{e}_{y_2} \tag{3.13}$$

Chapter 4

Ballistics

1 Weight of an Object

In this section, we focus on the motion of an object subjected only to its weight. At the scale of the laboratory, the Earth is flat and the **acceleration due to the gravity of Earth** $\vec{g}$ is directed downward. The force acting on a mass m is its weight $\vec{P}$, given by:

$$\vec{P} = m\vec{g} \tag{4.1}$$

with $g = \|\vec{g}\| = 9.8$ m.s^{-2}.

Remarks :

- The mass m is an intrinsic property of the body, whereas the weight depends on the location (the gravitational acceleration g is not the same on Earth and on the Moon).
- Generally, the direction of the gravitational force points toward the center of the Earth as a good approximation. There may be slight deviations due to the Earth's rotation and inhomogeneities in the Earth's structure.
- At the scale of the laboratory, the direction of the gravitational force always points down. If the ground is not horizontal, the gravitational force vector will not be perpendicular to the ground.

2 Case of a vertical throw (1D)

We will work in the laboratory frame of reference and choose a Cartesian coordinate system (O, x, y, z) with the z-axis **pointing downward**.

At $t = 0$, we have $z(0) = 0$, $\vec{v}_0 = \vec{v}(0) = -v_0\,\vec{e}_z$.

The object is only subject to its weight $\vec{P} = m\vec{g}$. We then have:

$$m\vec{g} = m\vec{a} \implies \vec{a} = \vec{g} = g\,\vec{e}_z \tag{4.2}$$

By integrating the acceleration vector, we obtain the expressions for the velocity and the position (the motion is only along $\vec{e}_z$):

$$v_z(t) = gt - v_0 \tag{4.3}$$

$$z(t) = \frac{1}{2}gt^2 - v_0 t \tag{4.4}$$

Remark: The signs in expressions 4.3 and 4.4 change when the z-axis points upward.

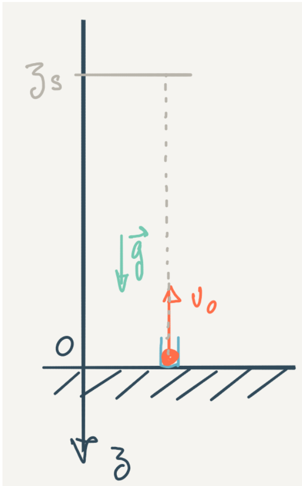


Figure 4.1: Diagram of the vertical throw & coordinate system.

3 General case

We consider the general case of the trajectory of a projectile, not necessarily launched vertically.

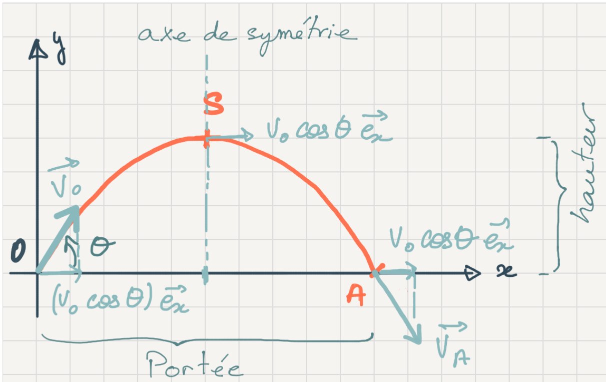


Figure 4.2: Diagram of the projectile's parabolic trajectory

The initial conditions as $t = 0$ are:

$$\vec{r}_0 = \vec{r}(0) = \vec{0} \left| \begin{array}{l} 0 \\ 0 \\ 0 \end{array} \right. \qquad \vec{v}(0) = \vec{v}_0 \left| \begin{array}{c} v_0 \cos\theta \\ 0 \\ v_0 \sin\theta \end{array} \right. \tag{4.5}$$

And, since the body is only subject to its weight $\vec{P} = m\vec{g}$, we have for the acceleration (from $m\vec{g} = m\vec{a}$):

$$\vec{a} = \vec{g} \left| \begin{array}{c} 0 \\ 0 \\ -g \end{array} \right. \tag{4.6}$$

After integration, the velocity vector $\vec{v}$ and the position vector $\vec{r}$ are given by:

$$\vec{v}(t) \left| \begin{array}{c} v_0 \cos\theta \\ 0 \\ -gt + v_0 \sin\theta \end{array} \right. \qquad \vec{r}(t) \left| \begin{array}{c} (v_0 \cos\theta)t \\ 0 \\ -\frac{1}{2}gt^2 + (v_0 \sin\theta)t \end{array} \right. \tag{4.7}$$

Remark: Equation 4.7 is called the time equation of motion, or the parametric equation of the trajectory (with t as parameter)

4 Trajectory, maximum height, impact point

Here, we seek the **trajectory** $z(x)$ (the motion is in the (x, z) plane). Taking the first component of $\vec{r}$ (Eq. 4.7), we obtain $t = x/(v_0 \cos\theta)$. Substituting this expression for t into the last component (z), we obtain:

$$z(x) = -\frac{g}{2\, v_0^2 \cos^2\theta} x^2 + \tan\theta \; x \tag{4.8}$$

The trajectory has the form $z(x) = ax^2 + bx$ with $a < 0$: it is a downward-oriented parabola passing through the origin (Figure 4.2).

We solve the problem of the **maximum height** (point S) as follows: when the object is at S, its position vector is $\vec{r}_S$. Moreover, since S is the apex/vertex of the trajectory, the velocity $\vec{v}_S$ is purely horizontal, so the vertical component is 0:

$$\vec{r}_S \left| \begin{array}{l} x_S \\ 0 \\ z_S \end{array} \right. \quad \text{et} \quad \vec{v}_S \left| \begin{array}{l} v_0 \cos\theta \\ 0 \\ 0 \end{array} \right. \tag{4.9}$$

The expression for the velocity as a function of time is known (Eq. 4.7), and by setting $\vec{v}(t_S) = \vec{v}_S$ we find:

$$t_S = \frac{v_0 \sin\theta}{g} \tag{4.10}$$

From this, we obtain the coordinates of point S, by substituting the expression for t_S into Eq. 4.7:

$$x_S = \frac{v_0^2 \cos\theta \sin\theta}{g} \tag{4.11}$$

$$z_S = \frac{1}{2}\frac{v_0^2 \sin^2\theta}{g} \tag{4.12}$$

For the **impact point** (point A), we have:

$$x_A = \frac{2\, v_0^2 \cos\theta \sin\theta}{g} \tag{4.13}$$

$$z_A = 0 \tag{4.14}$$

Remark: The symmetry of the trajectory implies that $x_A = 2, x_S$ (knowing that $x_0 = 0$).

5 Maximum range or hitting a target

In this section, we use Eq. 4.13 in order to determine the angle θ_{max} for which the projectile **will travel the farthest** (i.e. which maximizes x_A). The speed v_0 is fixed. Using the trigonometric identity $\sin 2\theta = 2\sin\theta\cos\theta$ (see Chapter 0), we have:

$$x_A(\theta) = \frac{2\, v_0^2 \cos\theta \sin\theta}{g} = \frac{v_0^2}{g}\sin 2\theta \tag{4.15}$$

The angle that maximizes this expression is the one for which $\sin 2\theta = 1$. We then find:

$$\theta_{\text{max}} = \frac{\pi}{4} = 45^\circ \tag{4.16}$$

More generally, if we want to reach a target at x_B, the equation to solve for θ is $\sin 2\theta = x_B g / v_0^2$. We then obtain the following result:

- Si $\dfrac{x_B g}{v_0^2} > 1$: no solution,
- Si $\dfrac{x_B g}{v_0^2} = 1$: unique solution $\theta = 45^\circ$,
- Si $\dfrac{x_B g}{v_0^2} < 1$: two solutions $\theta_1 \in [0, 45^\circ[$ (flat shot) et $\theta_2 \in]45^\circ, 90^\circ]$ (lob shot).

6 Time of flight

The **time of flight** is the time t_A that the projectile takes to arrive at A. We then solve:

$$\vec{r_A} \left| \begin{array}{l} (v_0 \cos\theta)\, t_A \\ 0 \\ -\frac{1}{2} g t_A^2 + (v_0 \sin\theta)\, t_A \end{array} \right. = \left| \begin{array}{l} \frac{2v_0^2}{g} \sin\theta\cos\theta \\ 0 \\ 0 \end{array} \right. \tag{4.17}$$

from which we obtain the time of flight:

$$t_A = 2\,\frac{v_0}{g}\sin\theta \tag{4.18}$$

Remark: The time of flight t_A can be expressed as a function of the maximum height z_S, since both involve a term in $v_0 \sin\theta := v_{0z}$. We then find:

$$t_A = 2\sqrt{\frac{2z_S}{g}} \tag{4.19}$$

The time of flight depends on the maximum height: the greater z_S is, the greater t_A will be as well.

7 Enveloping parabola

For a given initial speed v_0, a projectile cannot reach points outside the **enveloping parabola**. This parabola defines the region of space that is inaccessible regardless of θ.

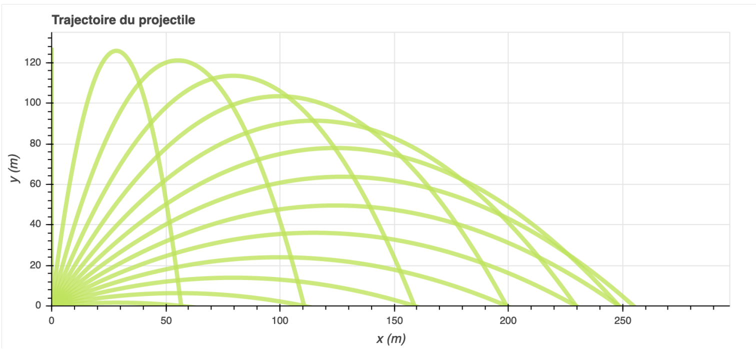


Figure 4.3: Enveloping parabola for a given v_0

8 Effect of the Earth's rotation

We consider dropping a stone from a height h from a tower located at the equator. By how much and in which direction is the stone deflected?

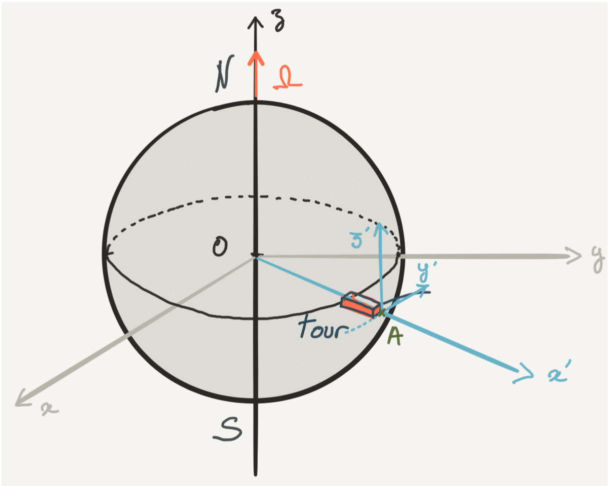


Figure 4.4: Diagram of the problem of the stone falling from a tower

To do this, we explore two ways of analyzing the problem: an intuitive way and a complete way. We define the following quantities: Ω the angular velocity of the Earth's rotation and R the Earth's radius.

Intuitive calculation:
We assume that the stone is given an initial constant velocity due to the Earth's rotation (along $\vec{e}_y$):

$$\text{stone}: \quad \vec{v}_i = (R+h)\,\Omega\,\vec{e}_y \tag{4.20}$$

$$\text{ground}: \quad \vec{v}_s = R\,\Omega\,\vec{e}_y \tag{4.21}$$

$$t_{\text{fall}} = \sqrt{\frac{2h}{g}} \tag{4.22}$$

The displacement vectors (during the fall time) $\vec{D}_{\text{stone}}$ and $\vec{D}_{\text{ground}}$ are along $\vec{e}_y$:

$$\vec{D}_{\text{stone}} = \vec{v}_i\,t_{\text{fall}} = (R+h)\,\Omega\sqrt{\frac{2h}{g}}\,\vec{e}_y \tag{4.23}$$

$$\vec{D}_{\text{ground}} = \vec{v}_s\,t_{\text{fall}} = R\,\Omega\sqrt{\frac{2h}{g}}\,\vec{e}_y \tag{4.24}$$

The deviation is therefore given by:

$$\vec{D} = \vec{D}_{\text{stone}} - \vec{D}_{\text{ground}} = h\,\Omega\sqrt{\frac{2h}{g}}\,\vec{e}_y \tag{4.25}$$

Remark: For $\Omega = 7.3 \times 10^{-5}$ rad·s^{-1} and $h = 300$ m, we find a deviation of about 17 cm.

Complete calculation: We take a frame $\mathcal{R}(O, x, y, z)$ fixed with O the center of the Earth, and a frame attached to the tower $\mathcal{R}'(A, x', y', z')$. A is the top of the tower (the point from which the stone is dropped): Figure 4.4.

In the Galilean frame $\mathcal{R}$, Newton's second law is written:

$$\Sigma\vec{F} = m\,\vec{a}_{\mathcal{R}}(P) = m\vec{g} \tag{4.26}$$

with $\vec{g} = -g\,vece_{x'}$. Moreover, the Earth's rotation vector is the same in $\mathcal{R}$ and $\mathcal{R}'$:

$$\vec{\Omega} = \Omega\,\vec{e}_z = \Omega\,\vec{e}_{z'} \tag{4.27}$$

The acceleration in the frame $\mathcal{R}$ is related to that in $\mathcal{R}'$ by the following expression:

$$\underbrace{\vec{a}_{\mathcal{R}}(P)}_{=\vec{g}} = \vec{a}_{\mathcal{R}'}(P) + \underbrace{\vec{a}_{\mathcal{R}}(A)}_{=\Omega\wedge(\vec{\Omega}\wedge\overrightarrow{OA})} + \underbrace{\dot{\vec{\Omega}}\wedge\overrightarrow{AP}}_{=0} + \vec{\Omega}\wedge(\vec{\Omega}\wedge\overrightarrow{AP}) + 2\vec{\Omega}\wedge\vec{v}_{\mathcal{R}'}(P) \tag{4.28}$$

In Eq. 4.28, we want to isolate the term $\vec{a}_{\mathcal{R}'}(P)$. We then obtain:

$$\vec{a}_{\mathcal{R}'}(P) = \vec{g} - \vec{\Omega}\wedge(\vec{\Omega}\wedge\overrightarrow{OP}) - 2\vec{\Omega}\wedge\vec{v}_{\mathcal{R}'}(P) \tag{4.29}$$

$$= -g_{\text{eff}}\,\vec{e}_{x'} - 2\vec{\Omega}\wedge\vec{v}_{\mathcal{R}'}(P) \tag{4.30}$$

Where we combine the centripetal acceleration and acceleration due to gravity into one "effective gravity" term with magnitude g_{eff}. Because Ω is small for the Earth, $g_{\text{eff}} \approx g$.

By integrating the acceleration vector in $\mathcal{R}'$, we find the velocities along the x', y', z' axes:

$$\begin{cases} \dot{x}' = -g_{\text{eff}}\,t + 2\,\Omega\,y' \approx -g_{\text{eff}}\,t \\ \dot{y}' = -2\,\Omega\,x' \\ \dot{z}' = 0 \end{cases} \tag{4.31}$$

where the simplification of $\dot{x}'$ results from the fact that the term in $\Omega y'$ is negligible compared to g_{eff}.

By integrating first $\dot{x}'$ and then $\dot{y}'$, we obtain the vector $\overrightarrow{AP}$ as a function of time:

$$\overrightarrow{AP}\left|\begin{array}{l} x'(t) = -\dfrac{1}{2}\,g_{\text{eff}}\,t^2 \\ y'(t) = \dfrac{1}{3}\,\Omega\,g_{\text{eff}}\,t^3 \\ z'(t) = 0 \end{array}\right. \tag{4.32}$$

Finally, the deviation at $t_f = \sqrt{2h/g_{\text{eff}}}$ is:

$$\vec{D} = y'(t_f)\,\vec{e}_y = \frac{2}{3}\,\Omega\,h\sqrt{\frac{2h}{g_{\text{eff}}}}\,\vec{e}_y \tag{4.33}$$

Why didn't the intuitive calculation give us the same answer? We obtained the correct order of magnitude but missed the factor 2/3. In the intuitive calculation, we were missing the effect of **conservation of angular momentum** which will be covered later in the course. Additionally, the g in Eq. 4.25 is replaced by the effective gravitational acceleration g_{eff}.

Chapter 5

Forces

1 Normal Force (fr: Réaction d'un support)

When a body is placed on a surface, the atoms of the two solids come closer together and begin to have a significant interaction. The force involved is the electromagnetic force, but its effect is modeled by phenomenological forces: **normal force** and **friction**.

The normal force is always perpendicular to the surface and denoted $\vec{R}$ or $\vec{N}$. It always points out from the surface towards the object.

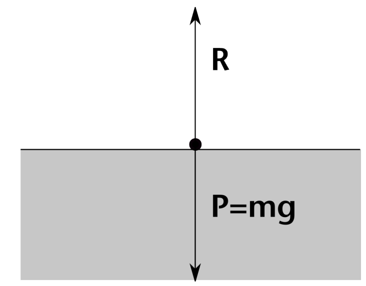


Figure 5.1: Weight and normal force of a horizontal surface.

In this example, we are interested in the relationship between weight due to gravity and the normal force: a mass m is placed on a horizontal plane.

The mass m is at rest, so $m\vec{a} = \sum \vec{F} = \vec{0}$. The forces acting are the weight $\vec{P} = m\vec{g}$ and the normal force $\vec{R}$. We then obtain:

$$m\vec{g} + \vec{R} = \vec{0} \quad \Longrightarrow \quad \vec{R} = -m\vec{g} \tag{5.1}$$

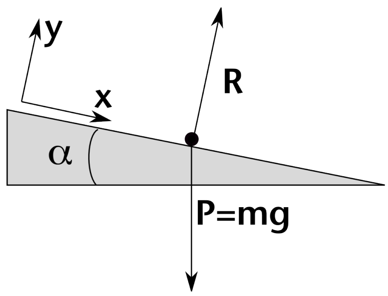


Figure 5.2: Weight and normal force of an inclined surface.

In Figure 5.2, a mass m is placed on a support inclined at an angle $\vec{\alpha}$ with the horizontal. The mass m remains on the plane, so $v_y = 0$. The weight $\vec{P} = m\vec{g}$ and the normal force $\vec{R}$ have the following components:

$$\vec{R} \left| \begin{matrix} 0 \\ R > 0 \end{matrix} \right. , \qquad m\vec{g} \left| \begin{matrix} mg\sin\alpha \\ -mg\cos\alpha \end{matrix} \right. \tag{5.2}$$

The constraint $a_y = 0$ thus gives us:

$$R = mg\cos\theta \tag{5.3}$$

2 Dry friction forces

Friction is also a manifestation of complex electromagnetic interactions, which we simplify using a phenomenological model.

A friction force **opposes motion**. We distinguish two types: dry friction (one solid on another) and fluid friction (occurring within a fluid).

Dry friction takes two forms:

- When the body is stationary: **static friction**
  $\sum \vec{F} = \vec{0}$, so the friction force $\vec{F}_F$ exactly compensates the force that tries to set the object in motion, up to a limiting value. As long as $F_F \leq \mu_s R$, the body does not move. μ_s is called the coefficient of static friction.

- When the body is in motion: **dynamic/kinetic friction**
  F_F is colinear with velocity and in the opposite direction, with magnitude $F_F = \mu_c R$, where μ_c is the coefficient of kinetic friction.

Remark: In general, we have $\mu_s > \mu_c$.

3 A rolling wheel

In this section, we will use the friction model to show the following result: *When a wheel rolls without slipping, the velocity of the point of contact with the ground is zero.*

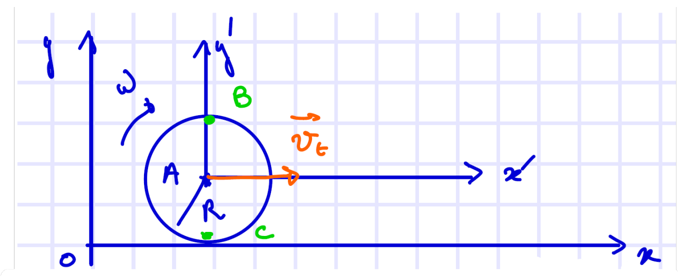


Figure 5.3: Rolling without slipping of the wheel with center A.

In the laboratory reference frame $\mathcal{R}$ with Cartesian coordinates (x, y), we consider a wheel of radius R placed on the ground: Figure 5.3. We denote A as the center of the wheel, C its point of contact with the ground, and B the point diametrically opposite to C.

Attention: During the motion of the wheel, the atoms of the wheel in contact with the ground will detach from the ground, and will no longer be a contact point. they are replaced by the next atoms on the surface of the wheel. **It is at the moment when the point is in contact with the ground that its velocity is zero.**

We also define the reference frame attached to the wheel $\mathcal{R}'$ (A, x', y') moving in $\mathcal{R}$ at the velocity $\vec{v}_t$. By definition of the point of contact (without slipping):

$$\vec{v}_{\mathcal{R}}(C) = \vec{0} \tag{5.4}$$

But, since we have a translational motion of $\mathcal{R}'$ in $\mathcal{R}$, we have:

$$\vec{v}_{\mathcal{R}}(C) = \vec{v}_{\mathcal{R}}(A) + \vec{v}_{\mathcal{R}'}(C) \quad \Longrightarrow \quad \vec{v}_{\mathcal{R}'}(C) = -\vec{v}_t \tag{5.5}$$

where we used that $\vec{v}_{\mathcal{R}}(A) = \vec{v}_t$ by definition of the translational velocity. Moreover, since the wheel rotates with constant angular velocity ω, the norm of the velocity of C in $\mathcal{R}'$ is $\|\vec{v}_{\mathcal{R}}(C)\| = R\omega$. We then find the norm of the translational velocity:

$$v_t = R\omega \tag{5.6}$$

Now, considering the point B diametrically opposite, we obtain the following results:

$$\vec{v}_{\mathcal{R}'}(B) = \vec{v}_t \tag{5.7}$$

$$\vec{v}_{\mathcal{R}}(B) = \vec{v}_{\mathcal{R}}(A) + \vec{v}_{\mathcal{R}'}(B) = 2\vec{v}_t \tag{5.8}$$

Conclusion: In $\mathcal{R}$, point C has zero velocity and point B moves at $2\vec{v}_t$.

4 Fluid friction

The fluid friction force depends on the velocity and the geometry of the object. We distinguish two limiting cases.

For **small velocities (laminar regime)**, the dependence is linear:

$$\vec{F}_F = -b_l\,\vec{v} \tag{5.9}$$

where $b_l = K\eta$, with η the viscosity coefficient and K a factor depending on the shape of the object.

At **high velocities (turbulent regime)**, the dependence is quadratic:

$$\vec{F}_F = -b_t\,v^2\frac{\vec{v}}{v} \tag{5.10}$$

where b_t is a constant, and $v = \|\vec{v}\|$.

Remark: Real cases are often intermediate between these two regimes: the dependence is neither perfectly linear nor perfectly quadratic.

Expression 5.9 (laminar regime) leads us to a differential equation that we know how to solve. We will study it according to two concrete cases where we seek the velocity $v(t)$:

1) We consider a projectile arriving with $\vec{v}_0$ in a fluid of viscosity η, with K the coefficient related to the shape of the ball. We neglect the weight of the projectile. We then have:

$$\frac{dv_x}{dt} = -\frac{K\eta}{m}v_x \tag{5.11}$$

By setting $\lambda = K\eta/m$ and with an initial velocity $v_x(0) = v_0$, we find:

$$v_x(t) = v_0 e^{-\lambda t} \tag{5.12}$$

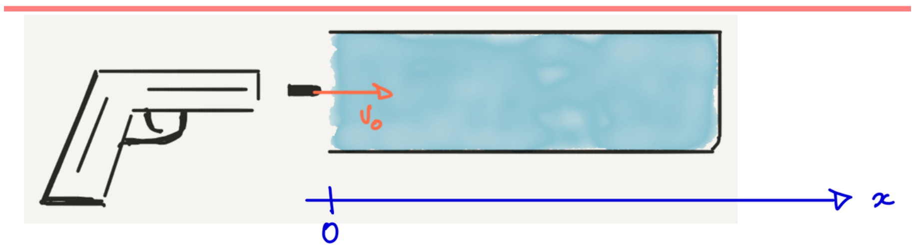


Figure 5.4: Example n°1

2) An object is dropped without initial velocity and subjected only to its weight $\vec{P} = m\vec{g}$. The viscosity of the fluid is η and the coefficient related to the shape of the object is denoted K. Along the vertical axis oriented downward, we have:

$$\frac{dv_z}{dt} = \frac{1}{m}(mg - b_l v_z) \tag{5.13}$$

The **terminal velocity** is the velocity $\vec{v}_{\text{lim}}$ at which the drag force $\vec{F}_F$ exactly balances the weight $\vec{P}$. The net force is zero and the object will move with uniform rectilinear motion. This velocity is given by:

$$mg\,\vec{e}_z - b_l v_{\text{lim}}\,\vec{e}_z = \vec{0} \qquad \Longrightarrow \qquad v_{\text{lim}} = \frac{mg}{b_l} \tag{5.14}$$

With $\lambda = K\eta/m = b_l/m$, we have $v_{\text{lim}} = g/\lambda$. This velocity is reached in the limit $t \to \infty$. We can now solve Eq. 5.12 with the condition $\lim_{t\to\infty} v_z(t) = g/\lambda$:

$$v_z(t) = \frac{g}{\lambda}\left(1 - e^{-\lambda t}\right) \tag{5.15}$$

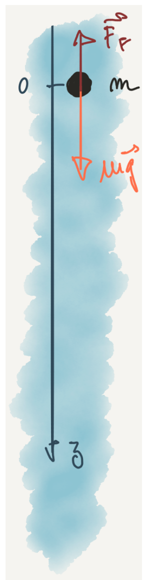


Figure 5.5: Example n°2

5 Tension in a rope

A massless, inextensible, taut rope simply transmits forces, possibly changing their direction. The tension force in a rope is a phenomenological force. It is often denoted $\vec{T}$.

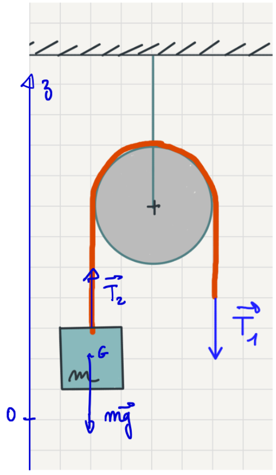


Figure 5.6: Case of a simple pulley

We present an example of force transmission using a pulley: a block of mass m is attached to a rope (massless, inextensible, taut) and a tension $\vec{T}_1$ is applied at its end. We then have $\|\vec{T}_1\| = \|\vec{T}_2\|$ (Figure 5.6). By Newton's second law, the acceleration of the mass m is:

$$ma_z\,\vec{e}_z = -mg\,\vec{e}_z + T_1\,\vec{e}_z \tag{5.16}$$

Thus, we obtain the following result:

- If $T_1 > mg$: $a_z > 0$ and the mass is accelerated upward,
- If $T_1 = mg$: $a_z = 0$ and the mass is either at rest or in uniform rectilinear motion,
- If $T_1 < mg$: $a_z < 0$ and the mass is accelerated downward.

6 Hooke's Law (restoring force of a spring)

In this section, we examine the restoring force of a spring when it is stretched or compressed. We consider a spring with non-touching coils (so it can be compressed or extended), and fixed at one of its ends. The behavior of the spring can be described as follows:

- When no force is applied to the spring, it has a natural length called the rest length, denoted l_0. This length will serve as a reference to define the compression or extension of the spring x.
- If the end of the spring is pulled, it has length $l = l_0 + x$ with $x > 0$. The spring is stretched.
- If the end of the spring is compressed, it has length $l = l_0 + x$ with $x < 0$. The spring is compressed.

The spring will exert a force that tends to bring it back to its rest length l_0. For example, when the extension is negative (compression), the spring will try to expand.

In the ideal case, the force exerted by a spring is proportional to its change in length: this is **Hooke's law**. For this, one must remain in the domain of small (reversible) deformations. In Figure 5.7, a spring is attached at one end and connected to a mass m. We take an x-axis such that the extension x is zero when the spring is at rest (at its length l_0). The restoring force of the spring is then expressed as:

$$\vec{F}_k = -kx\,\vec{e}_x \tag{5.17}$$

where k is the stiffness constant expressed in N/m. Thus, if $x > 0$, the restoring force will be directed to the left (stretching). If $x < 0$, it will be to the right (compression).

Remark: Since we have the relation $l = l_0 + x$ with l the length of the spring, Hooke's law is often written in the form: $\vec{F}_k = -k(l - l_0)\,\vec{e}_x$.

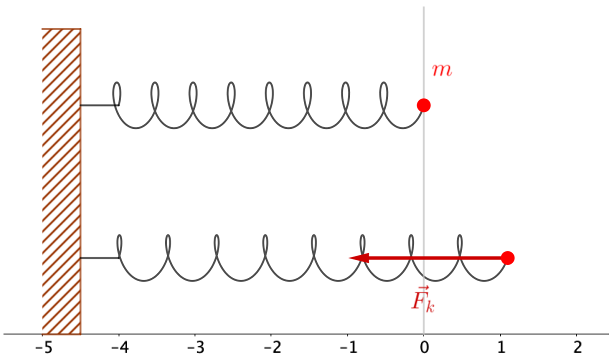


Figure 5.7: Diagram of a stretched spring and the associated restoring force $\vec{F}_k$.

7 Archimedes' Principle

A body immersed in a fluid experiences an upward buoyant force equal to the weight of the displaced volume of fluid. We denote by m_f the mass of displaced fluid: $m_f = \rho V$, where ρ is the density of the fluid and V the volume of the body. The **buoyant force** therefore has the following expression:

$$\vec{F}_A = -\rho V \vec{g} \tag{5.18}$$

We consider an object of mass m and volume V falling in a viscous fluid of density ρ. We assume fluid friction in the laminar regime and look for the terminal velocity.

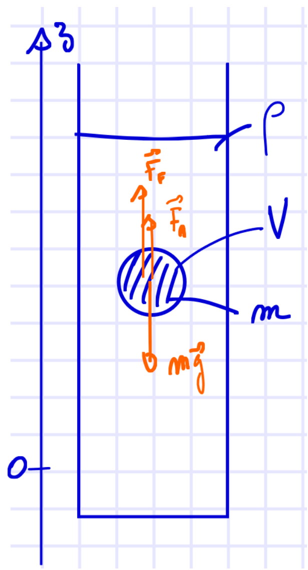


Figure 5.8: Object falling in a viscous fluid

The forces acting on the object are the weight $m\vec{g}$, the buoyant force $\vec{F}_A$, and the viscous drag force $\vec{F}_F$. The terminal velocity $\vec{v}_{\text{lim}}$ is reached when the resultant of the forces is zero. According to the Oz axis defined in Figure 5.8, we obtain:

$$m\vec{g} + \vec{F}_F + \vec{F}_A = \vec{0} \quad \Longleftrightarrow \quad -mg - b_l v_{\text{lim}} + \rho V g = 0 \tag{5.19}$$

From which we derive the terminal velocity:

$$v_{\text{lim}} = \frac{\rho V g - mg}{b_l} \tag{5.20}$$

Remark: If $mg < \rho V g$, that is, if the object is less dense than the fluid, then $v_{\text{lim}} > 0$ and the object will rise to the surface. If $mg > \rho V g$, the object will sink.

Chapter 6

Work, Energy, Conservation Principles

1 Work of a Force, Power

We consider an object moving along a trajectory at point $P(t)$. We denote by $\vec{F}$ the sum of the forces acting on the object: $\vec{F} = \sum \vec{F}^{\text{ext}}$. We have the following decomposition:

$$\vec{F} = F_\tau \, \vec{\tau} + \vec{F}_n \, \vec{n} \tag{6.1}$$

where τ and $\vec{n}$ are the basis vectors of the Frenet frame. Thus, F_τ is the component of $\vec{F}$ in the direction tangent to the trajectory.

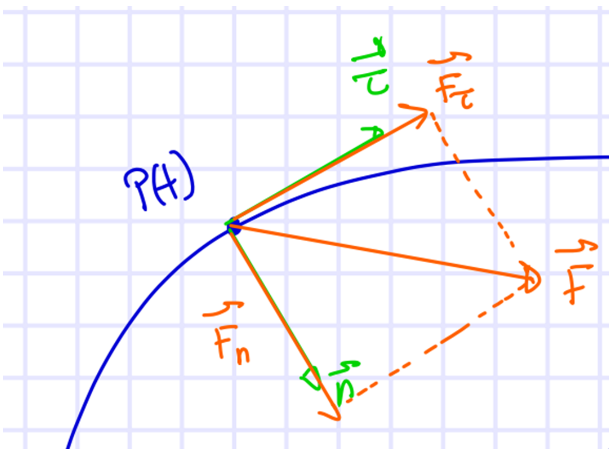


Figure 6.1:

Remark: Since the velocity $\vec{v}$ is tangent to the trajectory, it is F_τ that will vary the magnitude of $\vec{v}$.

We define the **work** of $\vec{F}$ for an infinitesimal displacement $\vec{dr}$:

$$\delta W^{\vec{F}} = \vec{F} \cdot \vec{dr} = \vec{F} \cdot dr \, \vec{\tau} = F_\tau dr \tag{6.2}$$

Thus the work between A and B is obtained by the curvilinear integral along the path AB:

$$W_{AB}^{\vec{F}} = \int_A^B \vec{F} \cdot \vec{dr} \tag{6.3}$$

Work is expressed in Joules [J]: 1 J = 1 N·m = 1 kg·m^2·s^{-2}.

Warning 1: Along the trajectory AB, $\vec{F}$ may change, as well as the direction of $\vec{\tau}$.
Warning 2: F_τ is algebraic, it can be < 0 or > 0.

By definition, the **power** P is the variation of the work W per unit time. For a work δW performed during dt, we have:

$$P = \frac{\delta W}{dt} = \vec{F} \cdot \frac{d\vec{r}}{dt} = \vec{F} \cdot \vec{v} \tag{6.4}$$

Power is expressed in Watts [W]: 1 W = 1 J·s^{-1}.

2 Kinetic Energy

We consider the case of a curvilinear motion under the action of a total force $\vec{F}_{\text{tot}}$. We use a Frenet coordinate system with unit vectors $\vec{\tau}$ and $\vec{n}$. The work performed by $\vec{F}_{\text{tot}}$ between A and B is given by:

$$W_{AB}^{\vec{F}_{\text{tot}}} = \int_A^B \vec{F}_{\text{tot}} \cdot d\vec{r} = \int_A^B m\vec{a} \cdot d\vec{r} \tag{6.5}$$

Now, in a curvilinear coordinate system, we have:

$$d\vec{r} = dr\,\tau \tag{6.6}$$

$$\vec{a} = \frac{v^2}{\rho}\,\vec{n} + \frac{dv}{dt}\,\vec{\tau} \tag{6.7}$$

The work then becomes:

$$W_{AB}^{\vec{F}_{\text{tot}}} = \int_A^B m\left(\frac{v^2}{\rho}\,\vec{n} + \frac{dv}{dt}\,\vec{\tau}\right) \cdot dr\,\vec{\tau} \tag{6.8}$$

$$= m\int_A^B dv\frac{dr}{dt} = m\int_A^B vdv \tag{6.9}$$

$$= \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2 \tag{6.10}$$

We define the **kinetic energy** by:

$$E_c = \frac{1}{2}mv^2 \tag{6.11}$$

The work obtained in Eq. 6.10 can then be rewritten as $W_{AB}^{\vec{F}_{\text{tot}}} = E_{c,B} - E_{c,A}$.

If the object is subjected to several forces $\vec{F^i}$ between A and B, we have the following relation:

$$W_{AB}^{\vec{F}_{\text{tot}}} = \int_A^B \sum_i \vec{F^i} \cdot d\vec{r} = \sum_i \int_A^B \vec{F^i} \cdot d\vec{r} = \sum_i W_{AB}^{\vec{F^i}} \tag{6.12}$$

3 Potential Energy and Mechanical Energy

By definition, **the potential energy** $E_p^{\vec{F}}(x, y, z)$ is a function of the spatial coordinates (x, y, z), is associated with the considered force $\vec{F}$, and has the dimension of an energy. For forces $\vec{F}$ such that $W_{AB}^{\vec{F}}$ is independent of the path taken between A and B, the potential energy is such that:

$$W_{AB}^{\vec{F}} = E_{p,A}^{\vec{F}} - E_{p,B}^{\vec{F}} \tag{6.13}$$

Equation 6.13 is only valid for certain forces $\vec{F}$ associated with a potential energy. These forces are called **conservative** since they conserve mechanical energy. This is the case for the gravitational force or the restoring force of a spring. For non-conservative forces, one must use Eq. 6.3.

Since the gravitational force is conservative:

$$W_{AB}^{m\vec{g}} = -mg\vec{e}_z \cdot \vec{AB} = -mg(z_B - z_A) = mgz_A - mgz_B \tag{6.14}$$

Thus, by recognition of the terms from Eq. 6.13, the potential energy in the terrestrial gravitational field is given by:

$$E_p^{m\vec{g}} = mgz \tag{6.15}$$

Proceeding similarly for the restoring force of a spring:

$$W_{AB}^{\vec{F}_\mathrm{R}} = \int_A^B \vec{F}_\mathrm{R} \cdot \vec{dr} = \int_A^B -kx\vec{e}_x \cdot dx\vec{e}_x = -k\int_A^B xdx = \frac{1}{2}kx_A^2 - \frac{1}{2}kx_B^2 \tag{6.16}$$

Thus, the potential energy of a spring is given by:

$$E_p^{\vec{F}_k} = \frac{1}{2}kx^2 \tag{6.17}$$

Note: Potential energy is defined up to a constant (the place where the reference is taken). This is not a problem since only the difference in potential energy has physical meaning.

Furthermore, we define **mechanical energy** as the sum of potential and kinetic energies:

$$E_m = E_p + E_c \tag{6.18}$$

If all the forces $\vec{F}_i$ are associated with a potential energy $E_p^{\vec{F}_i}$, we have:

$$W_{AB}^{\vec{F}_\mathrm{tot}} = \sum_i W_{AB}^{\vec{F}_i} = \sum_i \left(E_{p,A}^{\vec{F}_i} - E_{p,B}^{\vec{F}_i}\right) \tag{6.19}$$

$$= \underbrace{\sum_i E_{p,A}^{\vec{F}_i}}_{:=E_{p,A}} - \underbrace{\sum_i E_{p,B}^{\vec{F}_i}}_{:=E_{p,B}} = E_{p,A} - E_{p,B} \tag{6.20}$$

And with Eq. 6.10 which introduces kinetic energy, we obtain:

$$E_{c,B} - E_{c,A} = E_{p,A} - E_{p,B} \tag{6.21}$$

$$\iff E_{p,A} + E_{c,A} = E_{p,B} + E_{c,B} \tag{6.22}$$

$$\iff E_{m,A} = E_{m,B} \tag{6.23}$$

Conclusion: When all forces $\vec{F}_i$ are conservative, there is **conservation of mechanical energy**.

4 Link between Force and Potential Energy

A force is said to be the derivative of a potential if and only if it is a conservative force. In this case, the components of $\vec{F}$ are given by:

$$\vec{F} = \left| \begin{array}{l} -\partial E_p/\partial x \\ -\partial E_p/\partial y \\ -\partial E_p/\partial z \end{array} \right. := -\vec{\nabla} E_p \tag{6.24}$$

A force $\vec{F}$ is conservative if and only if (the conditions are equivalent):

- there exists a function $E_p(x, y, z)$ such that $W_{AB}^{\vec{F}} = E_{p,A}^{\vec{F}} - E_{p,B}^{\vec{F}}$,
- there exists a function $E_p(x, y, z)$ such that $\vec{F} = -\vec{\nabla} E_p$,
- the work of $\vec{F}$ does not depend on the path taken,
- the work of $\vec{F}$ is zero along a closed path,
- the curl of $\vec{F}$ is zero: $\vec{\nabla} \wedge \vec{F} = \vec{0}$.

5 Potential Energy and Equilibrium

In this section, we look at the link between potential energy and equilibrium position. Consider an object moving along an x-axis and place it at position x_0. Suppose that at this position we have:

$$\frac{dE_p}{dx}(x_0) = 0 \tag{6.25}$$

Then, since $\vec{F} = -\vec{\nabla} E_p$, we deduce that $F(x_0) = 0$. If the object is motionless at this point, it will remain there: this is a **position of equilibrium**. This corresponds to an extremum of the potential energy (maximum or minimum).

If it is a minimum of E_p, then we speak of a **stable** equilibrium position: the force will bring the object back to x_0 for small deviations. However, if it is a maximum, then the equilibrium position will be **unstable**.

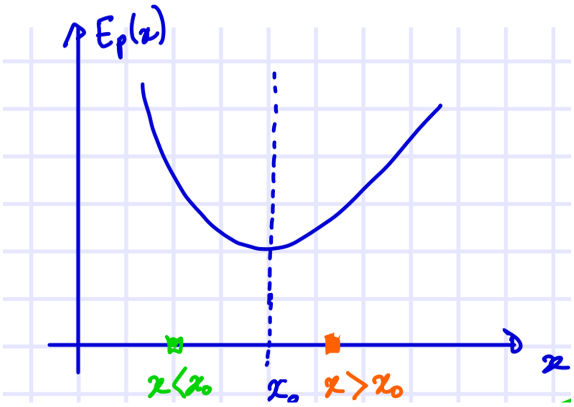


Figure 6.2: Stable equilibrium

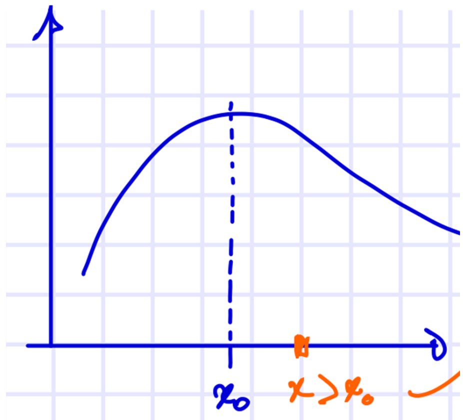


Figure 6.3: Unstable equilibrium

Stable equilibrium (minimum)	Unstable equilibrium (maximum)
$\frac{dE_p}{dx} = 0$ and $\frac{d^2E_p}{dx^2} > 0$	$\frac{dE_p}{dx} = 0$ and $\frac{d^2E_p}{dx^2} < 0$

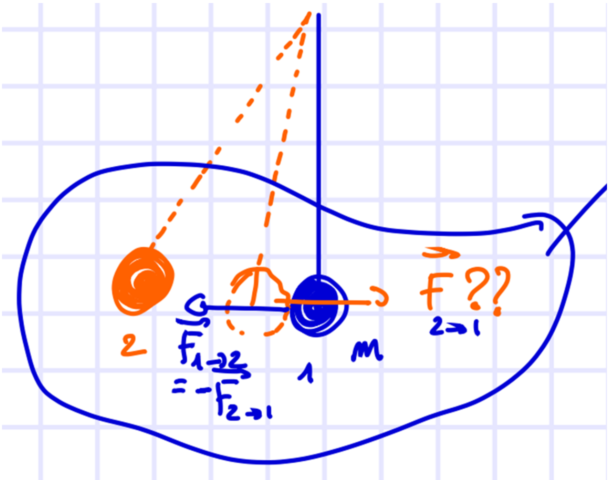
Table 6.1: Stability of positions of equilibrium

Chapter 7

Collisions & Variable systems of mass

1 Motivation

We consider the following problem: a pendulum at rest consists of a string and a mass m_1 attached to this string. A second pendulum (mass m_2, attached at the same point) is displaced from its equilibrium position and released with no initial velocity. The masses of the two pendulums will then collide.



To determine the motion of m_1, we must analyze the external forces: the weight, the tension of the string, **but especially the force exerted by mass m_2 during the collision**. The magnitude of this force and the duration of its application depend on the collision: it will therefore be very difficult to model!

Solution: We consider the system consisting of both masses together. The force of the collision will therefore be an **internal force** of the system, which will disappear when analyzing the global system. The **external forces** on the system will be the two tensions, as well as the weights of the two masses m_1, m_2.

By Newton's 3$^{\text{rd}}$ law (action-reaction), the sum of the internal forces will be zero since mass m_1 will exert an equal and opposite force on m_2:

$$\sum \vec{F}^{\text{int}} = \vec{F}_{2\to1} + \vec{F}_{1\to2} = \vec{0} \tag{7.1}$$

Finally, the collision is a very short phenomenon. We will therefore assume that, during the collision, the external forces do not have time to act and that the objects remain in the same places in space. **We will therefore not be concerned with the external forces during the collision**.

2 Center of mass; center-of-mass reference frame

Consider a system of N particles $(m_1, m_2, \dots, m_N)$ at positions $(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N)$ in a reference frame $(O, \vec{x}, \vec{y}, \vec{z})$. The **center of mass** G of the system is defined as:

$$\overrightarrow{OG} = \frac{m_1\vec{r}_1 + \cdots + m_N\vec{r}_N}{m_1 + \cdots + m_N} = \frac{\sum_\alpha m_\alpha \vec{r}_\alpha}{M} \tag{7.2}$$

where M is the total mass of the system. Thus, the position of G does not depend on the origin of the reference frame.

Remark: Despite having "mass" in its name, the center of mass is a position and has units of length.

The enter of mass is useful for expressing Newton's law for an entire system. For a particle α, the momentum is given by $\vec{p}_\alpha = m_\alpha \vec{v}_\alpha$. The **total momentum** for the system of N particles is:

$$\vec{P} = \sum_\alpha \vec{p}_\alpha = \sum_\alpha m_\alpha \vec{v}_\alpha \tag{7.3}$$

Moreover, for the **velocity of the center of mass** we have:

$$\vec{v}_G = \dot{\vec{r}}_G = \frac{d}{dt}\left[\frac{1}{M}\sum_\alpha m_\alpha \vec{r}_\alpha\right] = \frac{1}{M}\vec{p} \quad \Longrightarrow \quad \vec{p} = M\vec{v}_G \tag{7.4}$$

Finally, Newton's second law for the overall system is written as:

$$\vec{a}_G = \dot{\vec{v}}_G = \frac{1}{M}\frac{d\vec{p}}{dt} = \frac{1}{M}\sum \vec{F}^{\text{ext}} \quad \Longrightarrow \quad M\vec{a}_G = \sum \vec{F}^{\text{ext}} \tag{7.5}$$

The **center-of-mass (cm) reference frame** is the reference frame whose origin is at G and which moves with it at velocity $\vec{v}_G$. If $\sum \vec{F}^{\text{ext}} = \vec{0}$ then the cm reference frame is Galilean.

A particle α has a velocity $\vec{v}_\alpha$ in the laboratory frame $\mathcal{R}$ and $\vec{V}_\alpha$ in the cm reference frame. These velocities are related by:

$$\vec{v}_\alpha = \vec{V}_\alpha + \vec{v}_G \tag{7.6}$$

Application: Consider two particles m_1 and m_2, with velocities $\vec{v}_1$ and $\vec{v}_2$ in $\mathcal{R}$. In the cm reference frame, the particles have velocities $\vec{V}_1$ and $\vec{V}_2$ given by:

$$\vec{V}_1 = -\frac{m_2}{m_1 + m_2}(\vec{v}_2 - \vec{v}_1) \tag{7.7}$$

$$\vec{V}_2 = \frac{m_1}{m_1 + m_2}(\vec{v}_2 - \vec{v}_1) \tag{7.8}$$

and the momenta:

$$\vec{P}_1 = m_1\vec{V}_1 = -\mu(\vec{v}_2 - \vec{v}_1) \tag{7.9}$$

$$\vec{P}_2 = m_2\vec{V}_2 = \mu(\vec{v}_2 - \vec{v}_1) \tag{7.10}$$

where we have defined the **reduced mass** μ as:

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \tag{7.11}$$

3 Types of Collisions

We consider the case $\sum \vec{F}^{\text{ext}} = \vec{0}$, so the total momentum $\vec{p}_{\text{tot}}$ is conserved.
The nature of the internal forces will determine the type of collision: there are two main types

- **Elastic collision**: mechanical energy is conserved, no dissipation of energy. (e.g. bouncing ball).
- **Perfectly inelastic or soft collision**: the objects perfectly stick together after the collision. Mechanical energy is not conserved in this case.

Note: Real cases almost always lie somewhere between the two.

4 Elastic collisions

We consider particles of cylindrical or spherical shape, and we place ourselves in the cm reference frame. In this frame, the velocities are colinear and in opposite directions, but do not necessarily lie along the same line. We define the distance between the vectors as the **impact parameter** b, and when b is zero, **the collision is head-on**.

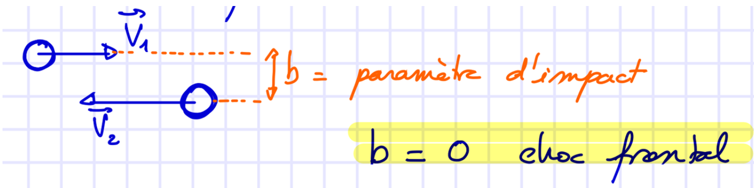


Figure 7.1: Definition of the impact parameter b. $b = 0$ corresponds to a head-on collision (choc frontal).

Remark: If $b > (R_1 + R_2)$ there is no collision between the particles.

4.1 Special case: Head-on collision (b=0) of two particles

In this case, the trajectories remain on the axis of the initial trajectories. The problem is then 1 dimension in the cm reference frame.

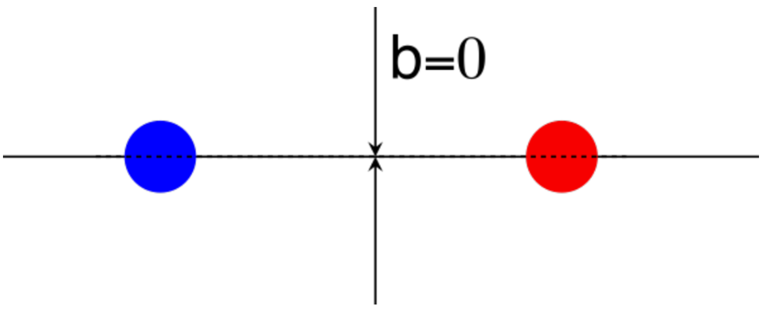


Figure 7.2: Head-on collision in the cm frame.

For an elastic collision, the momentum and kinetic energy are conserved:

$$\vec{P}_{\text{before}} = \vec{P}_{\text{after}} \quad (7.12)$$
$$E_c^{\text{before}} = E_c^{\text{after}} \quad (7.13)$$

Notation : The instant after the impact will be denoted by the index ' //
The total momentum before and after the collision is also conserved:

$$\vec{P}_{\text{tot}} = \vec{P}_1 + \vec{P}_2 = 0 \quad (7.14)$$
$$\vec{P}'_{\text{tot}} = \vec{P}'_1 + \vec{P}'_2 = 0 \quad (7.15)$$

From the last two equations, by isolating the velocities before and after impact of particle 2 and substituting them into equation (7.13), one obtains:

$$\vec{V}_1 = -\vec{V}'_1 \quad (7.16)$$
$$\vec{V}_2 = -\vec{V}'_2 \quad (7.17)$$

Therefore in the cm reference frame, the particles depart with velocities of the same magnitude and opposite directions (because $b = 0$, the direction does not change if there is no collision).

We now return to the reference frame $\mathcal{R}$ thanks to equation (7.6), and we then have for a head-on collision:

$$\vec{v}'_1 = \frac{(m_1 - m_2)\vec{v}_1 + 2m_2\vec{v}_2}{m_1 + m_2} \quad (7.18)$$
$$\vec{v}'_2 = \frac{(m_2 - m_1)\vec{v}_2 + 2m_1\vec{v}_1}{m_1 + m_2} \quad (7.19)$$

Let us study the special case where $\vec{v}_2 = \vec{0}$, the velocities after collision then become:

$$\vec{v}'_1 = \frac{m_1 - m_2}{m_1 + m_2}\vec{v}_1 \quad (7.20)$$
$$\vec{v}'_2 = \frac{2m_1}{m_1 + m_2}\vec{v}_1 \quad (7.21)$$

- If $m_1 > m_2$: The 2 particles continue in the same direction
- If $m_1 = m_2$: Particle 1 stops, 2 moves with velocity$\vec{v}_1$
- If $m_1 < m_2$: particle 1 moves in the other direction

4.2 Non-head-on collision of two cylindrical pucks

The complete calculation of non-head-on collisions is not part of the syllabus. But if you are curious, the calculation can be done easily for cylindrical or spherical objects. In this case, the velocity vector is separated into two components. The component projected onto the line joining the centers of the pucks is treated as a head-on collision, the other component corresponds to a grazing interaction and does not change (no collision). This approach is what allows the deterministic calculations of the graphical simulation tool.

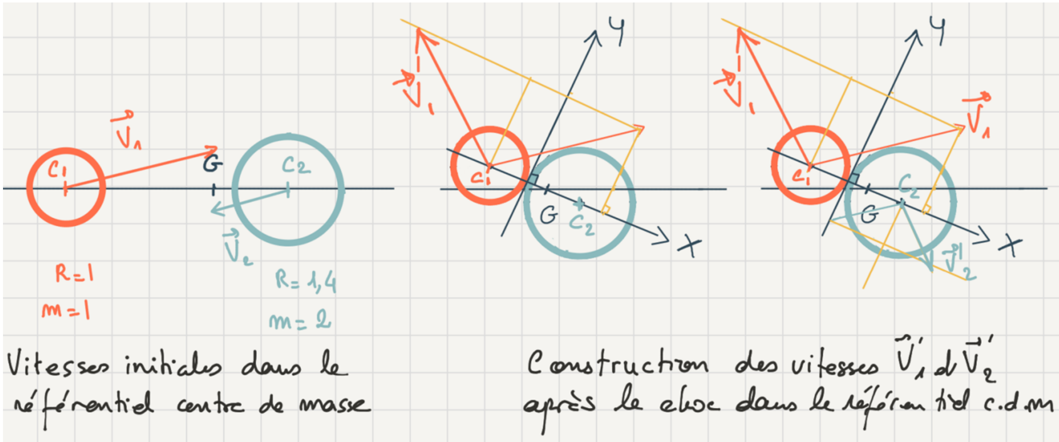


Figure 7.3: Non-head on collision

5 Inelastic collision

For an inelastic (perfectly inelastic) collision, the two particles remain stuck together after the collision (or they were stuck together before an explosion). Momentum is still conserved, but not kinetic energy: part of it is dissipated as heat.

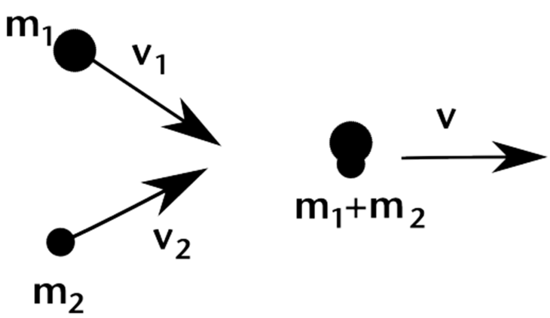


Figure 7.4: Inelastic collision

Let us consider the collision of two particles with different masses and velocities. The velocity after the collision of the two particles stuck together will be denoted $\vec{v}$. Since momentum is conserved, we have:

$$\vec{P}_{tot} = m_1\vec{v}_1 + m_2\vec{v}_2 \tag{7.22}$$

$$\vec{P}'_{tot} = (m_1 + m_2)\vec{v} \tag{7.23}$$

$$\vec{P}_{tot} = \vec{P}'_{tot} \tag{7.24}$$

We can then isolate $\vec{v}$ and deduce an expression for the velocity after the collision:

$$\vec{v} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2} \tag{7.25}$$

We notice that this is also the expression for the velocity of the center of mass G: $\vec{v}_G$ remains unchanged during the collision because $\sum \vec{F}^{\text{ext}} = \vec{0}$.

If we now consider the kinetic energy, it is not conserved, but we can calculate the difference in energy before and after the collision. Denoting the energy before and after the collision respectively as $E_{c,1}$ and $E_{c,2}$, we have:

$$E_{c,1} = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \tag{7.26}$$
$$E_{c,2} = \frac{1}{2}(m_1 + m_2)v^2 \tag{7.27}$$

We then need to calculate v^2 by taking the dot product of $\vec{v}$ with itself:

$$v^2 = \vec{v}\cdot\vec{v} = \frac{m_1^2v_1^2 + 2m_1m_2\vec{v}_1\vec{v}_2 + m_2^2v_2^2}{(m_1 + m_2)^2} \tag{7.28}$$

And we then deduce, after substituting and simplifying the square:

$$E_{c,2} - E_{c,1} = -\frac{1}{2}\frac{m_1m_2}{m_1 + m_2}(\vec{v}_1 - \vec{v}_2)^2 \tag{7.29}$$

This difference in kinetic energy is always negative or zero, indicating that if the collision occurs (negative) there is a loss of energy through transformation, and if there is no loss (zero) then the collision does not occur.

6 Variable-mass system: rocket

We will now study a variable-mass system, meaning a system where the mass changes over time. The example chosen here will be a rocket propelled by the matter ejected from its nozzles. In this case, we must go back to the origin of Newton's second law and cannot simply apply $\vec{F} = m\vec{a}$, because the mass is no longer constant!

Reference frame: Vertical z-axis pointing upward.
System: Rocket at instant t with mass $m(t)$, velocity $v(t)$)

- Rocket with initial mass $m(t = 0) = m_0$.
- The gases are ejected at constant velocity relative to the rocket ($\vec{v}_e$ = constant) and at a constant rate ($\frac{dm}{dt}$ = constant)
- After a time $(t + dt)$, the rocket has ejected a mass $-dm$ (where we take $dm < 0$)

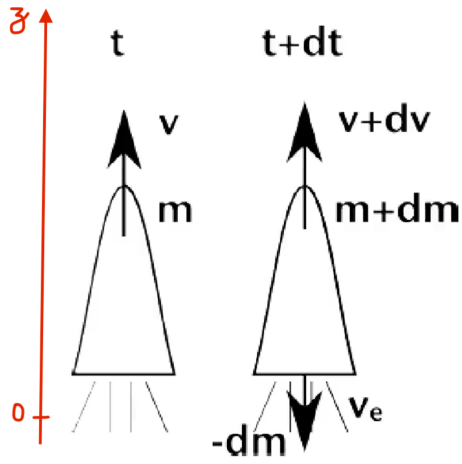


Figure 7.5: Rocket at time t and $t + dt$

Newton's second law can be written as follows:

$$\vec{F}^{\text{ext}} = \frac{d\vec{p}}{dt} = \frac{\vec{p}(t+dt) - \vec{p}(t)}{dt} \tag{7.30}$$

We then need to write the difference in momentum between t and $(t + dt)$. At time t this is straightforward:

$$\vec{p}(t) = m(t)\vec{v}(t) \tag{7.31}$$

For time $(t+dt)$, we need to consider how the system has changed. The rocket has expelled mass $-dm$, so $m(t + dt) = m(t) - (-dm) = m(t) + dm$. The rocket has velocity $\vec{v} + d\vec{v}$. The mass element ejected by the rocket has mass $-dm$ and velocity $\vec{v}_{gas}$, the speed of the ejected matter in our reference frame:

$$\vec{v}_{\text{gaz}} = \vec{v}_e + \vec{v} + d\vec{v} \tag{7.32}$$

With these elements we can write $p(t + dt)$ of the system, which consists of the rocket and its ejected matter:

$$\vec{p}(t+dt) = (m + dm)(\vec{v} + d\vec{v}) + (-dm)\vec{v}_{\text{gaz}} \tag{7.33}$$

(7.34)

We then obtain:

$$\vec{p}(t+dt) - \vec{p}(t) = m d\vec{v} - dm\vec{v}_e \tag{7.35}$$

$$\vec{F}^{ext} = \frac{\vec{p}(t+dt) - \vec{p}(t)}{dt} = m\frac{d\vec{v}}{dt} - \vec{v}_e\frac{dm}{dt} \tag{7.36}$$

Let us consider the case where **the rocket rises vertically and is far from any mass in interstellar space** ($\vec{F}^{ext} = \vec{0}$). By projecting equation (7.35) onto the vertical z-axis, and taking into account that $\vec{v}_e = -v_e\vec{e}_z$ (pointing downward), we can isolate the derivative of the velocity and integrate to obtain:

$$
\frac{dv}{dt} = -v_e \frac{\frac{dm}{dt}}{m} \tag{7.37}
$$

$$
v(t) - v(0) = -v_e \; ln \frac{m}{m_0} \tag{7.38}
$$

If we assume that the initial velocity is zero, we finally obtain an expression for the rocket's velocity as a function of time and the mass at time t:

$$
v(t) = v_e \; ln \frac{m_0}{m(t)} \tag{7.39}
$$

We then observe that $v(t)$ is large if v_e or $\frac{m_0}{m(t)}$ are large.

Similarly, for the case where **the rocket rises vertically and is subject to the gravitational field ($\vec{F}^{\textbf{ext}} = m\vec{g}$)**, we can reason in the same way: project equation (7.35) onto the z-axis, isolate the derivative of the velocity, and integrate with respect to time. We then obtain the following equations respectively:

$$
\frac{dv}{dt} = -v_e \frac{\frac{dm}{dt}}{m} - g \tag{7.40}
$$

$$
v(t) = v_e \; ln \frac{m_0}{m(t)} - gt \tag{7.41}
$$

Chapter 8

Harmonic Oscillator

0 Complex numbers

The analysis of the harmonic oscillator becomes much easier if we use complex numbers, as we'll see for the damped and driven harmonic oscillators. Here we summarize the algebra of complex numbers.

Cartesian representation: Every complex number z can be written in a Cartesian form $a + ib$, where a and b are real numbers and $i^2 = -1$. We say that a is the real part of z, and b is the imaginary part of z.

Sum: The sum of two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ is the complex number

$$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2).$$

Product: The product of z_1 and z_2 is

$$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) = a_1 a_2 + ia_1 b_2 + ib_1 a_2 + i^2 b_1 b_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2).$$

Equality: If two complex numbers are equal, the real parts and the imaginary parts are respectively equal: $a_1 + ib_1 = a_2 + ib_2$, implying that $a_1 = a_2$ and $b_1 = b_2$.

Complex conjugate: $z^* \equiv a - ib$ is the *complex conjugate* of $z = a + ib$. The quantity $|z| = \sqrt{zz^*}$ is the *magnitude* of z:

$$|z| = \sqrt{zz^*} = \sqrt{(a + ib)(a - ib)} = \sqrt{a^2 + b^2}.$$

Polar representation: Every complex number z can be written in the polar form $re^{i\theta}$. r is a real number, the *modulus*, and θ is the *argument*. To go from Cartesian to polar form we use De Moivre's theorem:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Hence,

$$re^{i\theta} = r\cos\theta + ir\sin\theta = a + ib,$$

from which it follows that

$$a = r\cos\theta, \quad b = r\sin\theta$$

and

$$r = \sqrt{a^2 + b^2}, \qquad \theta = \arctan\left(\frac{b}{a}\right).$$

We see that $r = |z|$.

1 Free Harmonic Oscillator

Let us study a model system: A mass m slides without friction on a horizontal axis. It is attached to a spring with stiffness constant k and rest length l_0 which is fixed at its other end. We pull the mass to the right and at $t = 0$ we release it without any initial velocity.

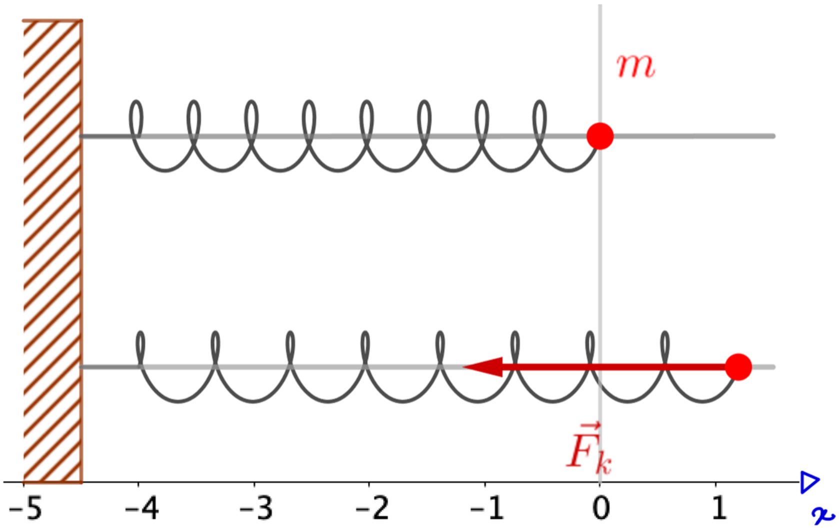


Figure 8.1: Free (non-damped) harmonic oscillator

Let us apply Newton's second law to the mass m:

– Along the vertical direction y, the weight $m\vec{g}$ and the normal force $\vec{R}$ cancel each other ($\vec{R} = -m\vec{g}$), so there is no acceleration.

– Along the horizontal direction x, the restoring force of the spring $\vec{F_k} = -kx\vec{e}_x$ creates the acceleration.

By projecting along the x axis, we obtain the equation of motion $m\vec{a} = \vec{F_k}$ where $\vec{a} = \ddot{x}\vec{e}_x$, and therefore:

$$m\ddot{x} = -kx \tag{8.1}$$

$$\ddot{x} + \frac{k}{m}x = 0 \tag{8.2}$$

Equation (8.2) is obtained by dividing by m and bringing the terms to the same side. We can then define Ω_0, the **natural angular frequency** of the system, such that:

$$\Omega_0 = \sqrt{\frac{k}{m}} \tag{8.3}$$

And thus equation (8.2) can be rewritten as:

$$\ddot{x}(t) + \Omega_0^2 x(t) = 0 \tag{8.4}$$

The equation of motion (8.4) linearly links the function $x(t)$ to its second derivative, therefore it is a **second-order Linear Differential Equation**.

The solution of this equation is harmonic (as a sum of sine and cosine), and can be written in different but equivalent forms:

$$x(t) = A * \cos(\Omega_0 t) + B * \sin(\Omega_0 t) \tag{8.5}$$
$$x(t) = C * \cos(\Omega_0 t + \phi) \tag{8.6}$$
$$x(t) = D * \sin(\Omega_0 t + \psi) \tag{8.7}$$

It is possible to derive expression (8.5) by expanding (8.6) or (8.7) using trigonometric formulas, so we will focus on this solution in this section.

To find the constants A and B for a given problem, we use the **initial conditions** for the position and the velocity of the mass at t=0. We have zero velocity at the start ($\dot{x}(t=0) = 0$) and a non-zero initial displacement ($x(t=0) = x_0$), thus:

$$x(0) = A * \cos(\Omega_0 * 0) + B * \sin(\Omega_0 * 0) = A = x_0 \tag{8.8}$$
$$\dot{x}(0) = -A\Omega_0 \sin(\Omega_0 * 0) + B\Omega_0 \cos(\Omega_0 * 0) = B\Omega_0 = 0 \tag{8.9}$$

Since the natural angular frequency is non-zero, we obtain A $= x_0$ and B $= 0$. The function x(t) reduces to:

$$x(t) = x_0 \cos(\Omega_0 t) \tag{8.10}$$

Therefore the position will oscillate with time.

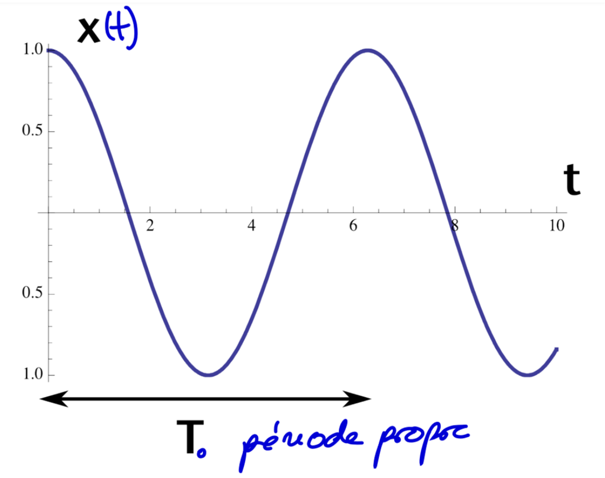


Figure 8.2: Position as a function of time. T_0 is the natural period of the system.

The motion is therefore periodic, and we can define the **Natural Period** of the system T_0 as:

$$T_0 = \frac{2\pi}{\Omega_0} = 2\pi\sqrt{\frac{m}{k}} \tag{8.11}$$

We note that it depends only on the mass and the stiffness of the spring, and is therefore independent of the initial conditions. When T_0 is independent of x_0, we call the system a **Harmonic Oscillator**.

Example: the Simple Pendulum

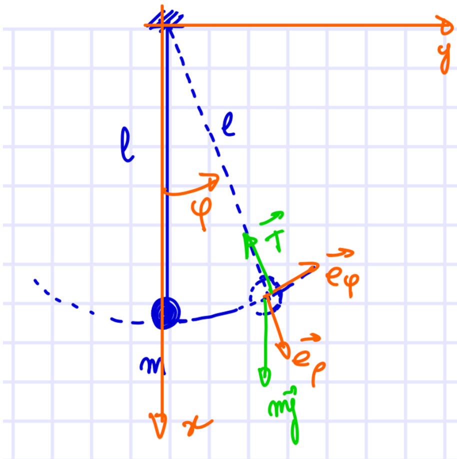


Figure 8.3: Simple pendulum

To analyze the simple pendulum we use polar coordinates, and project the forces (the weight of the mass and the tension of the string) and the acceleration as follows:

$$m\vec{g} = mg(\cos(\phi)\vec{e}_\rho - \sin(\phi)\vec{e}_\phi) \tag{8.12}$$

$$\vec{T} = -T\vec{e}_\rho \tag{8.13}$$

$$\vec{a} = (\ddot{\rho} - \rho\dot{\phi}^2)\vec{e}_\rho + (\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})\vec{e}_\phi \tag{8.14}$$

However, since the length of the string is constant we have $\rho = l$ and therefore $\dot{\rho} = \ddot{\rho} = 0$. Thus, by applying Newton's Second Law and projecting along $\vec{e}_\rho$ and $\vec{e}_\phi$, we obtain the following system of equations:

$$\text{along } \vec{e}_\rho\text{:} \quad -ml\dot{\phi}^2 = -T + mg\cos(\phi) \tag{8.15}$$

$$\text{along } \vec{e}_\phi\text{:} \quad ml\ddot{\phi} = -mg\sin(\phi) \tag{8.16}$$

The equation that interests us is (8.16) because the position of the mass with respect to time is given by the coordinates $(\rho; \phi)$ where $\rho = l$ is constant. Therefore it is entirely determined by the function $\phi(t)$. By simplifying, the equation becomes:

$$\ddot{\phi} + \frac{g}{l}sin(\phi) = 0 \tag{8.17}$$

However, equation (8.17) is a nonlinear differential equation, and therefore does not correspond to a Harmonic Oscillator.

This equation can be approximated as linear for small ϕ using the **small-angle approximation**: if $\phi \ll 1$ (small), then $\sin(\phi) \simeq \phi$. In this case, equation (8.17) can be rewritten as:

$$\ddot{\phi} + \Omega_0^2\phi = 0 \tag{8.18}$$

We define the natural angular frequency of this system as $\Omega_0 = \sqrt{\frac{g}{l}}$. The natural period $T_0 = \frac{2\pi}{\Omega_0}$ is therefore indeed independent of the initial condition ϕ_0: it is a Harmonic Oscillator.

Since equation (8.18) has the same form as equation (8.4), the solution has the same form as well. We then have (for initial conditions $\phi(0) = \phi_0$ and $\dot{\phi}(0) = 0$) the following solution:

$$\phi(t) = \phi_0 \cos(\Omega_0 t) \tag{8.19}$$

Warning: This result is valid only in the small-angle approximation!

2 Undamped Harmonic Oscillator and Energy

Let us study the kinetic, potential, and mechanical energy of the harmonic oscillator. For the basic system of a mass attached to an undamped spring (Figure 8.1), the mechanical energy is the sum of the potential energy of the spring and the kinetic energy due to the motion of the mass:

$$E_m = E_p^k + E_m = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 \tag{8.20}$$

Now the expression for $x(t)$ was obtained in the previous section, equation (8.10), and can therefore be substituted into equation (8.20). After simplification, we then obtain:

$$E_m = \frac{1}{2}kx_0^2 = \text{constant} \tag{8.21}$$

This result makes sense because the mechanical energy is conserved since there is no dissipation. However, the kinetic and potential energy themselves are not conserved; there is constant conversion from one to the other during the motion of the system.

For the one-dimensional case ($\vec{a} = \ddot{x}\vec{e}_x$) where the force derives from a potential, we can write F (projected along x) as:

$$F = -\frac{dE_p}{dx} = m\ddot{x} \tag{8.22}$$

And then, if the potential energy can be written as a function of x, we can obtain the differential equation of motion using this potential.

Remark: To have oscillations, one must be around a minimum of E_p.

For the oscillator to be harmonic, it is necessary and sufficient that the potential energy takes the following quadratic form:

$$E_p = A(x - x_0)^2 + E_{p,0} \tag{8.23}$$

with A constant and positive.

Proof: By substituting expression (8.23) into (8.22), we obtain after differentiation:

$$m\ddot{x} = -2A(x - x_0) \tag{8.24}$$

We must then perform a change of reference frame. We define $X = x - x_0$, and thus we deduce $\dot{X} = \dot{x}$ and $\ddot{X} = \ddot{x}$. Therefore, by moving everything to the same side, dividing by m, and defining $\Omega_0^2 = \frac{2A}{m}$ (only valid for A > 0), we obtain:

$$\ddot{X} + \Omega_0^2 X = 0 \tag{8.25}$$

Which is the differential equation of a harmonic oscillator (Q.E.D.).

Example: Energy of the Simple Pendulum

Let us resume the study of the simple pendulum of Figure 8.3 using energy. We take the following reference frame:

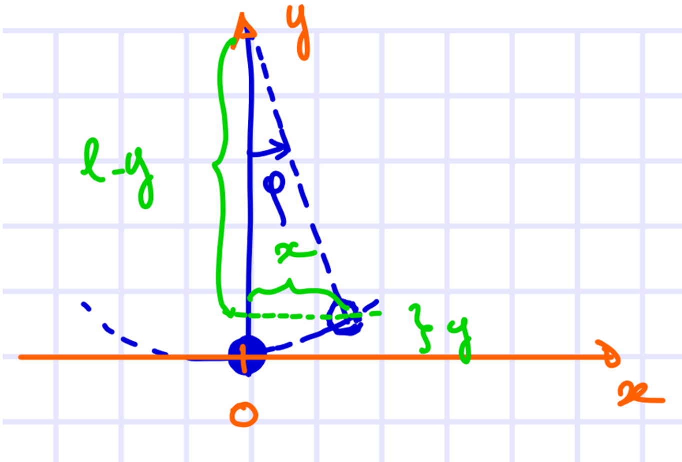


Figure 8.4: Simple pendulum

The tension does no work, the gravitational force derives from a potential ($E_p = mgy$). We need to find an expression for y as a function of x; for this we use the Pythagorean theorem and deduce for $E_p(x)$:

$$l^2 = (l - y)^2 + x^2 \tag{8.26}$$

$$y = l - \sqrt{l^2 - x^2} \tag{8.27}$$

$$E_p(x) = mg[l - \sqrt{l^2 - x^2}] = mgl\left[1 - \sqrt{1 - \left(\frac{x}{l}\right)^2}\right] \tag{8.28}$$

Expression (8.28) is clearly not a parabola of the form (8.23), therefore the oscillator is anharmonic, just as we found in the force-based study.

However, if the oscillations are small ($\frac{x}{l} \ll 1$), we can define a small $\epsilon = \left(\frac{x}{l}\right)^2$ and have:

$$E_p(x) = mgl(1 - (1 - \epsilon)^{1/2}) \tag{8.29}$$

Since $\epsilon \ll 1$, we take the Taylor expansion $(1 - \epsilon)^{1/2} \simeq 1 - \frac{1}{2}\epsilon$, and then the potential energy for small oscillations becomes:

$$E_p(x) = mgl\frac{1}{2}\epsilon = \frac{mg}{2l}x^2 \tag{8.30}$$

We therefore obtain a potential energy of the form (8.23), and thus the oscillator is harmonic for small oscillations, just as we found in the force-based study.

3 Damped Harmonic Oscillator

We consider the oscillator from section 8.1, but this time it is subject to fluid friction in the laminar regime.

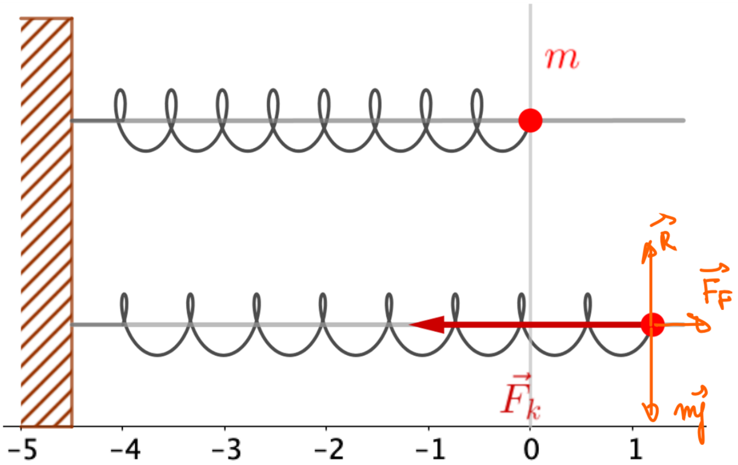


Figure 8.5: Damped Harmonic Oscillator

The friction force is written:

$$\vec{F}_f = -b_l \vec{v} = -b_l \dot{x}\hat{e}_x = -(K\eta)\dot{x}\hat{e}_x \tag{8.31}$$

This force is always in the direction opposite to the motion, which is why there is a minus sign in front of the velocity vector. We note that the type of motion will depend on the relative magnitude of the friction coefficient b_l to to the spring constant k.

Using Newton's Second Law along x (along y the weight and the normal force cancel), we obtain:

$$m\ddot{x} = -b_l\dot{x} - kx \tag{8.32}$$

$$\ddot{x} + \frac{b_l}{m}\dot{x} + \frac{k}{m}x = 0 \tag{8.33}$$

$$\ddot{x} + 2\gamma\dot{x} + \Omega_0^2 x = 0 \tag{8.34}$$

Where we have defined $2\gamma = \frac{b_l}{m} = \frac{K\eta}{m}$ and $\Omega_0 = \sqrt{\frac{k}{m}}$. This is a second-order linear differential equation, but we cannot easily solve it as we did in the undamped case because of the first derivative of x.

3.1 General method of solving such an equation

The first step is to solve the **characteristic equation**, which is a quadratic equation formed from (8.34), which takes the form:

$$\lambda^2 + 2\gamma\lambda + \Omega_0^2 = 0 \tag{8.35}$$

We obtain this equation by replacing $\ddot{x}$, $\dot{x}$, and x with λ^2, λ^1, and $\lambda^0 = 1$, respectively. The order of derivation of x is replaced by the corresponding power of lambda, while keeping the same constant coefficients in front of the variables.

Remember that for a generic quadratic equation:

$$ax^2 + bx + c = 0 \tag{8.36}$$

We have the discriminant $\Delta = b^2 - 4ac$. To solve equation (8.36), we use the discriminant $\Delta = (2\gamma)^2 - 4 * \Omega_0^2$, and we can even use the reduced discriminant $\Delta' = \frac{(2\gamma)^2}{4} - \Omega_0^2$ (2γ was defined with this factor for this particular reason of simplification). The 3 cases will then depend on the value of the reduced discriminant $\Delta' = \gamma^2 - \Omega_0^2$.

Case 1: Critical damping ($\Delta' = 0$, $\gamma = \Omega_0$)

Equation (8.36) has a single solution $\lambda = \gamma = \Omega_0$. The solution of (8.34) is then:

$$x(t) = (A + Bt)e^{-\gamma t} \tag{8.37}$$

Case 2: Strong damping ($\Delta' > 0$, $\gamma^2 > \Omega_0^2$)

Equation (8.36) has two distinct solutions given by:

$$\lambda_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - \Omega_0^2}}{1} \tag{8.38}$$

The solution of (8.34) is then:

$$x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t} \tag{8.39}$$

Since both lambdas are negative, x(t) is a sum of two decreasing exponentials.

Case 3: Under-damped regime ($\Delta' < 0$, $\gamma^2 < \Omega_0^2$)

Δ' is then complex! Let us define $\omega = \sqrt{\Omega_0^2 - \gamma^2}$, so we can write $\Delta' = \pm i\omega$. The solution of equation (8.36) is in this case $\lambda_{1,2} = -\gamma \pm i\omega$. The solution of (8.34) is then:

$$x(t) = Ae^{(-\gamma + i\omega)t} + Be^{(-\gamma - i\omega)t} = e^{-\gamma t} * (Ae^{i\omega t} + Be^{-i\omega t}) \tag{8.40}$$

The equation of motion $x(t)$ is real, but A, B and the exponentials in i are complex numbers! We must then separate real and imaginary parts, and therefore write:

$$A = a_1 + ia_2 \tag{8.41}$$

$$B = b_1 + ib_2 \tag{8.42}$$

$$e^{i\omega t} = \cos\omega t + i\sin\omega t \tag{8.43}$$

$$e^{-i\omega t} = \cos\omega t - i\sin\omega t \tag{8.44}$$

Now it only remains to substitute the above expressions into (8.39), and keep only the real part because x(t) is a displacement, and thus a real number. After expansion and isolating the real part (the imaginary part must be set to =0), we obtain:

$$x(t) = e^{-\gamma t} * (A'\cos\omega t + B'\sin\omega t) \tag{8.45}$$

It is also possible to write the solution in the following form:

$$x(t) = C'e^{-\gamma t} * \cos(\omega t + \phi) \tag{8.46}$$

Where $A' = a_1 + b_1$, $B' = a_2 + b_2$, and C' and ϕ are integration constants given by the initial conditions.

3.2 Analysis of the results

When we are in Case 3 where $\gamma < \Omega_0$, the function $x(t)$ has a form following (8.46), represented graphically is:

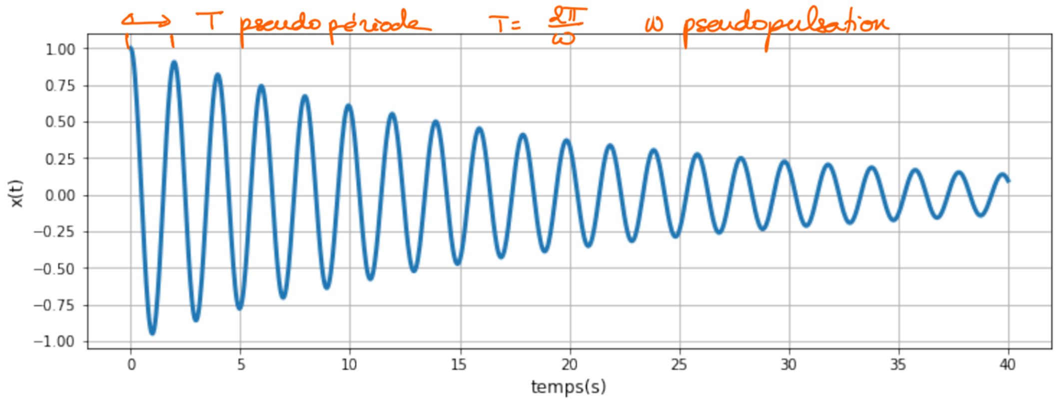


Figure 8.6: $x(t)$ in the under-damped regime.

Thanks to the **pseudoangular frequency** ω defined in the previous section as $\omega = \sqrt{\Omega_0^2 - \gamma^2}$, we can also define the time interval between two peaks as the **pseudoperiod** $T = \frac{2\pi}{\omega}$.

In Case 2 where $\gamma > \Omega_0$, the solution (8.39) represented graphically is:

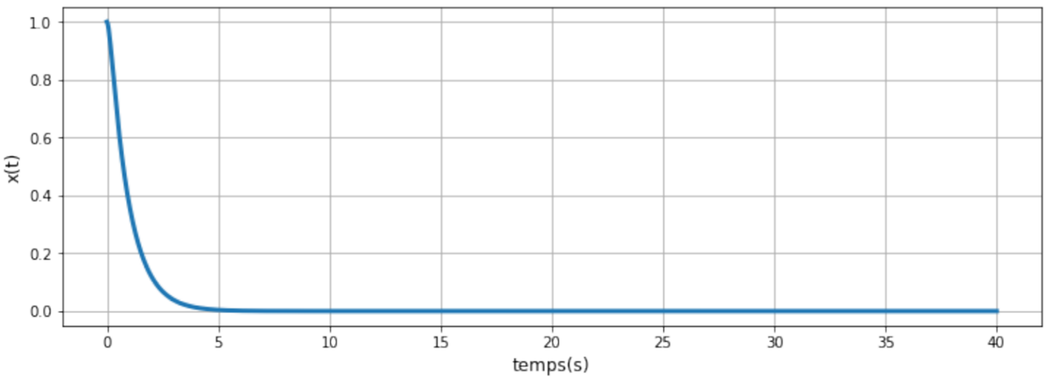


Figure 8.7: $x(t)$ in the over-damped regime.

There is no oscillation in this case, and the system returns to equilibrium more or less quickly depending on the values of λ_1 and λ_2.

Finally, for Case 1 where $\gamma = \Omega_0$, the solution (8.37) represented graphically is:

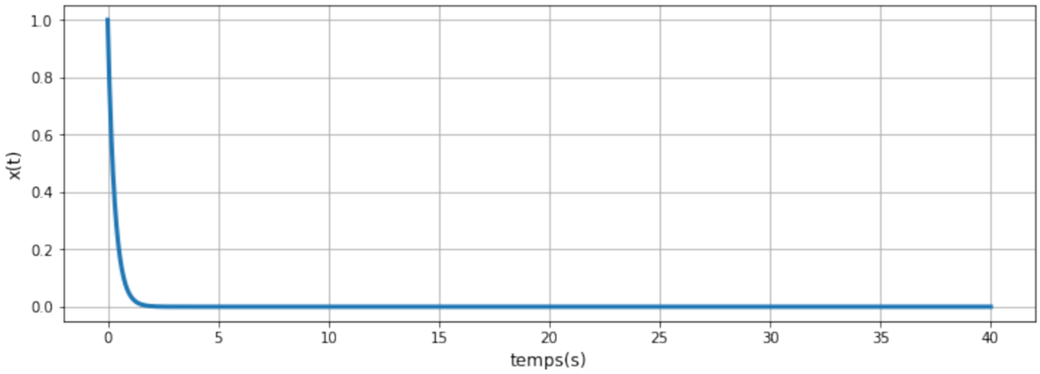


Figure 8.8: $x(t)$ in the critically-damped regime.

The return to equilibrium happens as quickly as possible.

4 Forced Oscillations

We again take the damped harmonic oscillator, but now we apply an external "driving" force that has a sinusoidal form:

$$\vec{F}_e = F_e \vec{e}_x = F_0 \cos(\omega_e t) \vec{e}_x \tag{8.47}$$

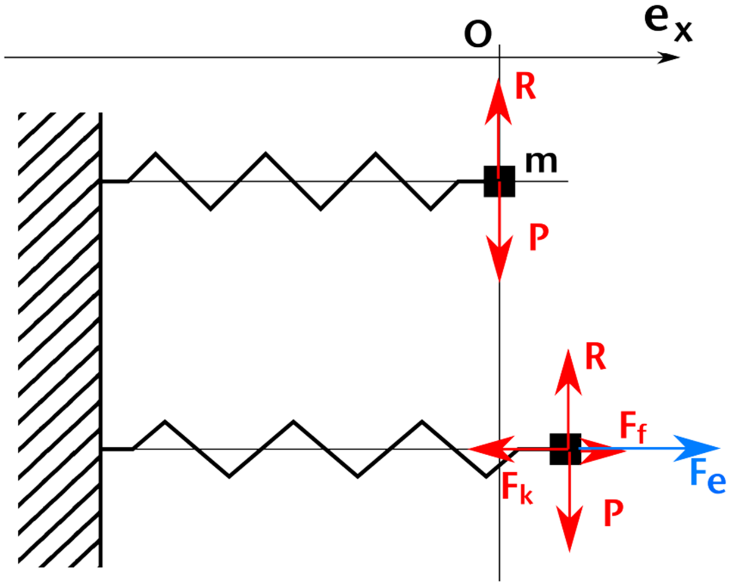


Figure 8.9: Oscillator driven by a periodic force

By applying Newton's Second Law to the system, we have the same forces as before with the addition of the driving force. We can then proceed in the same way as for the damped, non-forced case, and from equation (8.34) obtain:

$$\ddot{x} + 2\gamma\dot{x} + \Omega_0^2 x = f_0 \cos(\omega_e t) \tag{8.48}$$

where $f_0 = \frac{F_0}{m}$. This is a **second-order linear differential equation with a forcing term**. This forcing term is a function of time.

4.1 Method for solving a second-order linear differential equation with forcing term

The general method is to search for:

1) $x_2(t)$, the **general** solution of the equation **without** the forcing term (thus with integration constants): this is the solution for $f(t) = 0$, found using the procedure from section (3.1).

2) $x_1(t)$, a **particular** solution of the equation **with** forcing term (thus a function that "works").

The **general** solution of the equation **with** forcing term is:

$$x(t) = x_1(t) + x_2(t) \tag{8.49}$$

The analysis of the results from section (3.2) indicates that the general solution tends towards 0 when t tends to infinity. Therefore in this limit, which we will call the **steady state**, $x_2(t) \to 0$. In this case, we deduce that (8.49) becomes, in the steady state, $x(t) \simeq x_1(t)$.

By observing experimentally the displacement x(t), we can make hypotheses on the form of $x_1(t)$, and suppose that it will be of the form:

$$x_1(t) = A(\omega_e) * \cos(\omega_e t + \phi(\omega_e)) \tag{8.50}$$

We want to express A and ϕ as functions of ω_e and the data of our system; we must therefore solve equation (8.48) using a particular solution of the form (8.50). It will here be useful to switch to the complex domain and use the polar notation, that is:

$$\underline{a} = ae^{i\theta} = a(\cos\theta + i\sin\theta) \tag{8.51}$$

Let us study the following complex differential equation:

$$\ddot{\underline{x}} + 2\gamma\dot{\underline{x}} + \Omega_0^2\underline{x} = f_0 e^{i\omega_e t} \tag{8.52}$$

By taking the real part of the left and right-hand sides respectively, we recover equation (8.48) that we wish to study. In the complex domain we thus have a solution of the form:

$$\underline{x}_1(t) = A(\omega_e)e^{i(\omega_e t + \phi)} = A(\omega_e)e^{i\omega_e t}e^{i\phi} = \chi_0 e^{i\omega_e t} \tag{8.53}$$

We define $\chi 0 = A(\omega e)e^{i\phi(\omega_e)}$ which does not depend on time. This is a complex number whose modulus is $A(\omega_e)$ and whose argument is $\phi(\omega_e)$.

With this notation, the time derivatives of x are simplified:

$$\dot{\underline{x}} = \chi_0(i\omega_e)e^{i\omega_e t} \tag{8.54}$$
$$\ddot{\underline{x}} = -\chi_0\omega_e^2 e^{i\omega_e t} \tag{8.55}$$

We substitute expressions (8.53), (8.54) and (8.55) into equation (8.52). We can then simplify the term $e^{i\omega_e t}$ on both sides, and we are left with:

$$-\chi_0\omega_e^2 + 2i\gamma\chi_0\omega_e + \chi_0\Omega_0^2 = f_0 \tag{8.56}$$

It is then easy to factorize χ_0 and isolate it to find its expression:

$$\chi_0 = \frac{f_0}{-(\omega_e^2 - \Omega_0^2) + 2i\gamma\omega_e} = A(\omega_e)e^{i\phi(\omega_e)} \tag{8.57}$$

Now it is possible to relate A and ϕ to the specifics of the problem thanks to the above relation. It suffices to compute the modulus and argument of the left-hand side of (8.57), and we finally obtain:

$$A(\omega_e) = \frac{f_0}{\sqrt{(\omega_e^2 - \Omega_0^2)^2 + 4\gamma^2\omega_e^2}} \tag{8.58}$$

$$\tan\phi = \frac{2\gamma\omega_e}{\omega_e^2 - \Omega_0^2} \tag{8.59}$$

where $\phi \in [-\pi, 0[$.

Remark: If the driving force is sinusoidal and not cosinusoidal as in the above case, A remains the same but ϕ no longer evolves in the same interval (because the sine is a phase-shifted cosine): $\cos(\omega t) = \sin(\omega t + \pi/2)$

4.2 Analysis of the results

Establishment of the steady state

We are interested in the time response of the driven harmonic oscillator:

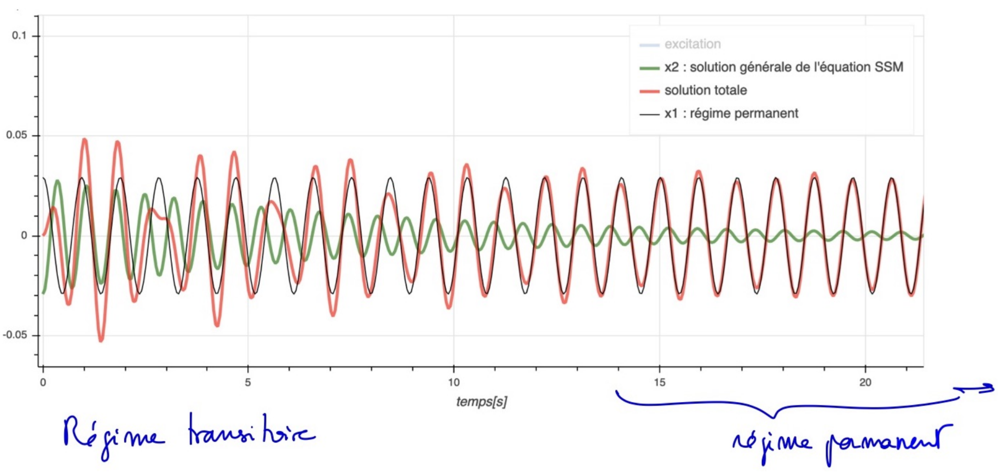


Figure 8.10: Establishment of the steady-state

We observe two well-distinguished time domains between the startup and the subsequent response. At the beginning, we are in the **Transient Regime**, in which $x_2(t)$ (the general solution of the differential equation without forcing term) has not yet converged to 0. Once $x_2(t)$ becomes sufficiently small and converges to 0, we are in the Steady State Regime. We can then graphically observe A as the amplitude of the oscillations of the total response $x(t)$, and the phase shift ϕ is the phase difference between the driving force and the response $x_1(t)$.

Analysis of the steady state

In the steady state, the response is $x(t) = x_1(t) = A(\omega_e) * \cos(\omega_e t + \phi(\omega_e))$. We know that the amplitude and the phase shift are functions of the excitation frequency, let us look at their dependence on ω_e. We also vary the friction coefficient and study the different damping regimes.

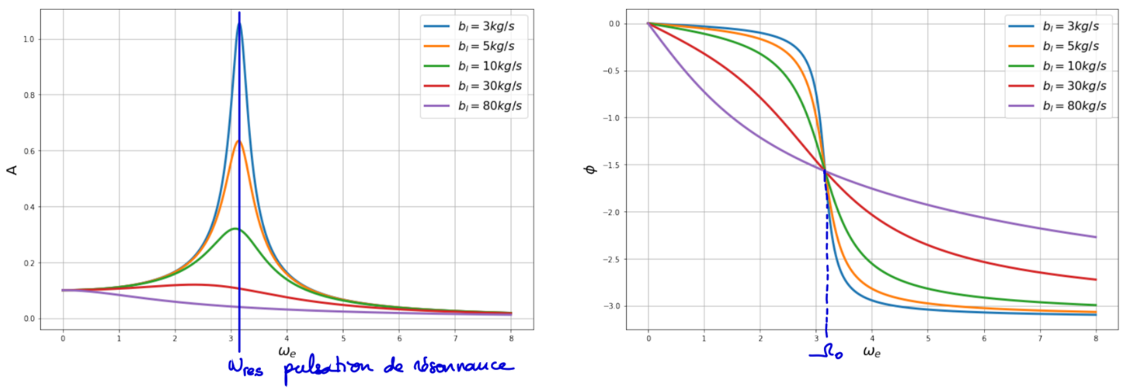


Figure 8.11: Amplitude and phase as a function of the driving frequency.

The amplitude is always positive, starts at the same initial value and goes to 0 if the driving frequency increases regardless of the friction coefficient b_l. It reaches a peak if b_l is not too large, otherwise the peak is never reached and the function is monotonically decreasing. The frequency for which A reaches its maximum is called the **Resonance Frequency** and denoted ω_{res}.

The phase shift reaches $-\frac{\pi}{2}$ for an excitation frequency equal to Ω_0, the natural frequency of the system. We note that when the resonance is pronounced (weak damping), $\omega_{\text{res}} \simeq \Omega_0$.

To calculate ω_{res} one must look for the maximum of the amplitude A. From equation (8.58), let us define the term in the square root in the denominator as a function:

$$g(\omega_e) = (\omega_e^2 - \Omega_0^2)^2 + 4\gamma^2\omega_e^2 \tag{8.60}$$

Finding the extrema of A amounts to finding the extrema of g, but the calculation is much simpler. By differentiating g with respect to ω_e and setting it equal to 0, we find 2 possible solutions:

$$\omega_e = 0 \tag{8.61}$$

$$\omega_e^2 = \Omega_0^2 - 2\gamma^2 \tag{8.62}$$

Result (8.61) corresponds to the beginning of the values of A, and means a horizontal tangent. The second extremum given by (8.62) corresponds to the frequency for which maximizes A. It is therefore the resonance angular frequency:

$$\omega_{\text{res}} = \sqrt{\Omega_0^2 - 2\gamma^2} \tag{8.63}$$

However, this result is only possible if $2\gamma^2 < \Omega_0^2$ so that the square root of expression (8.63) is positive. Therefore **there is no resonance if** $2\gamma^2 > \Omega_0^2$**!**

We see that replacing expression (8.63) into results (8.58) and (8.59) indeed gives mathematically the same thing as the observations made from figure (8.11).

Summary of the angular frequencies

- Ω_0 is the natural angular frequency of the system, i.e. the angular frequency of oscillations if there is neither damping nor a driving force.
- $\omega = \sqrt{\Omega_0^2 - \gamma^2}$ is the pseudo-angular-frequency of the damped but not forced system, when the damping is undercritical.
- ω_e is the driving/forcing angular frequency: a frequency imposed by the user in the case of a driven oscillator.
- $\omega_{\text{res}} = \sqrt{\Omega_0^2 - 2\gamma^2}$ is the resonance angular frequency: a particular value of ω_e for which the system response has a maximum amplitude in the steady state

Reminder: Period = 2π / angular frequency

Quality Factor

In order to obtain a universal description of the amplitude and phase shift curves, we define the **Quality Factor** (dimensionless) as follows:

$$Q = \frac{\Omega_0}{2\gamma} \tag{8.64}$$

With this expression, the condition to obtain a resonance becomes $Q > \frac{1}{\sqrt{2}}$. The larger Q is, the larger the maximum amplitude (in resonance) is.

We can also express A as a function of Q:

$$A(\omega_e) = \frac{A(0)Q}{\sqrt{Q^2(\frac{\omega_e^2}{\Omega_0^2} - 1)^2 + \frac{\omega_e^2}{\Omega_0^2}}} = \frac{A(0)Q}{\sqrt{Q^2(\bar{\omega_e}^2 - 1)^2 + \bar{\omega_e}^2}} \tag{8.65}$$

where $\bar{\omega_e} = \frac{\omega_e}{\Omega_0}$ is called the Reduced Pulsation.

When in resonance, we have:

$$A(\omega_{\text{res}}) = A_{\text{max}} = \frac{A(0)Q^2}{\sqrt{Q^2 - \frac{1}{4}}} \tag{8.66}$$

In particular, if $Q \gg 1$, we have $A_{\text{max}} \simeq A(0)Q$ and therefore $Q = \frac{A_{\text{max}}}{A(0)}$: Q directly gives a measure of the "quality" of the resonance.

Chapter 9

Angular Momentum & Gravitation

It is possible to describe Newton's Laws differently in order to facilitate the study of rotational dynamics. For this, we will introduce new quantities built from force and momentum.

1 Angular momentum and torque (moment of a force)

Angular momentum characterizes the rotation of an object around a point O. **Torque**, also called the **moment of a force**, makes it possible to characterize the ability of a force to produce such rotational motion.

Let us study the following situation: a point P with mass m, which has momentum $\vec{p}$ and is subject to an external force $\vec{F}$.

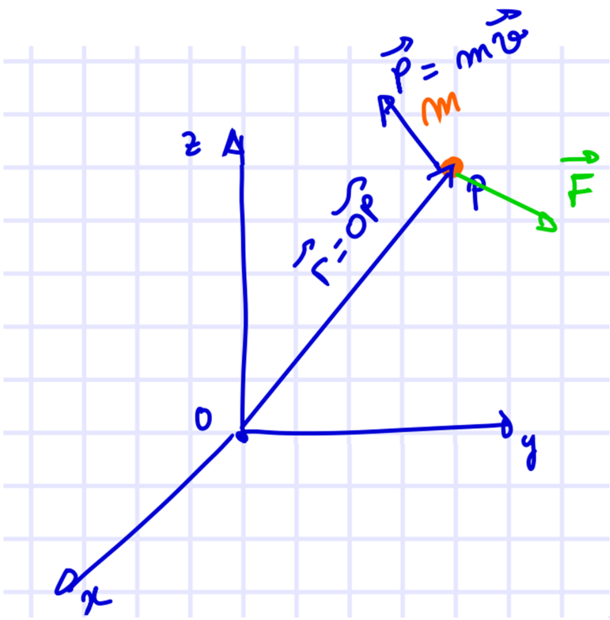


Figure 9.1: A point P with mass m, which has momentum $\vec{p}$ and is subject to an external force $\vec{F}$.

We can then define the angular momentum *with respect to point O* as:

$$\vec{L}_O = \overrightarrow{OP} \wedge \vec{p} \tag{9.1}$$

Similarly, we define the Moment of the force $\vec{F}$ *with respect to point O* as:

$$\vec{M}_O = \overrightarrow{OP} \wedge \vec{F} \tag{9.2}$$

Note: Moments and angular momentum are defined with respect to a reference point and therefore depend on that point! One may choose any fixed point in space.

If there are several forces $\vec{F}_{\text{tot}} = \sum_i \vec{F}_i$, then $\vec{M}_O = \sum_i \vec{M}_O^i = \sum_i \overrightarrow{OP} \wedge \vec{F}_i$

$\vec{L}$ depends on the linear momentum and $\vec{M}$ on the force. From Newton's Second Law we know that $\vec{F} = \frac{d\vec{p}}{dt}$. We can therefore differentiate equation (9.1) with respect to time to link it to (9.2):

$$\frac{d\vec{L}_O}{dt} = \frac{d}{dt}(\vec{OP} \wedge \vec{p}) = \vec{OP} \wedge \frac{d\vec{p}}{dt} = \vec{M}_O \tag{9.3}$$

The relation above is called the **Balance of Angular Momentum**, and it is a reformulation of Newton's Second Law.

Note: The reference point O must be the same on both sides of the equation.

Example 1: Circular Motion

Let us consider the following situation: an object of mass m moves along a circle of radius R centered at the origin O, with a velocity $\vec{v}$. Let us take cylindrical coordinates:

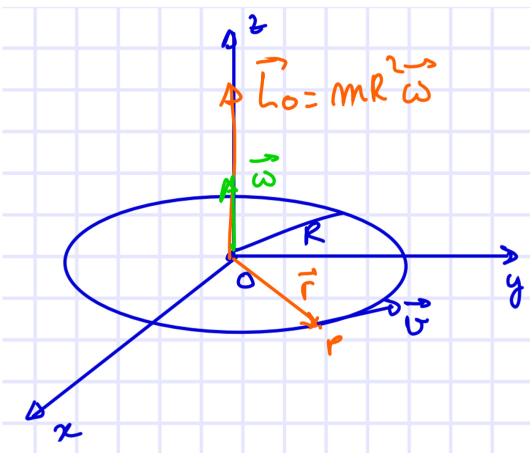


Figure 9.2: Example circular movement

Here, $\rho = R =$ constant, so the $\dot{\rho}$ term in the velocity is zero. We then have:

$$\overrightarrow{OP} = R\vec{e}_\rho \tag{9.4}$$
$$\vec{v} = R\dot{\varphi}\vec{e}_\varphi \tag{9.5}$$
$$\vec{\omega} = \dot{\varphi}\vec{e}_z = \omega\vec{e}_z \tag{9.6}$$

The angular momentum with respect to O is therefore:

$$\vec{L}_O = \overrightarrow{OP} \wedge \vec{p} = (R\vec{e}_\rho) \wedge (mR\dot{\varphi}\vec{e}_\varphi) = mR^2\dot{\varphi}\vec{e}_z = mR^2\vec{\omega} \tag{9.7}$$

Example 2: Planar Curvilinear Motion in Cylindrical Coordinates

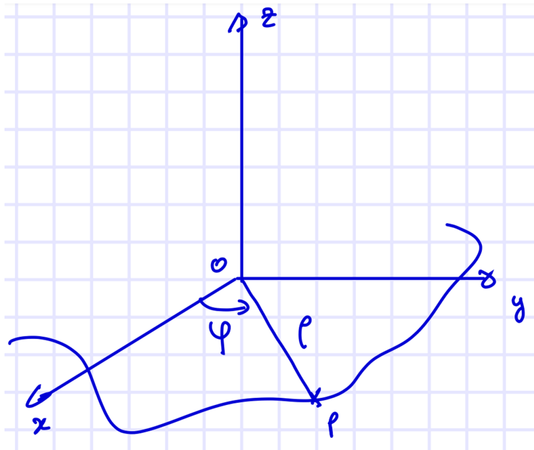


Figure 9.3: Example of curvilinear movement

In this case, ρ is no longer constant, and it is therefore necessary to include the term in $\dot{\rho}$ of the velocity:

$$\overrightarrow{OP} = \rho \vec{e}_\rho \tag{9.8}$$
$$\vec{v} = \dot{\rho}\vec{e}_\rho + \rho\dot{\varphi}\vec{e}_\varphi \tag{9.9}$$

And the angular momentum in this case is:

$$\vec{L}_O = \overrightarrow{OP} \wedge \vec{p} = (\rho\vec{e}_\rho) \wedge m(\dot{\rho}\vec{e}_\rho + \rho\dot{\varphi}\vec{e}_\varphi) = m\rho^2\dot{\varphi}\vec{e}_z = m\rho^2\vec{\omega} \tag{9.10}$$

The result is the same as in the previous example because the cross product of $\vec{e}_\rho$ with itself is $\vec{0}$.

2 Central Force

$\vec{F}$ is a **Central Force centered at O** if it is always collinear with $\overrightarrow{OP}$. Two well-known examples are the electrostatic force and the gravitational force.

For the central force $\vec{F}$ centered at O:

- The torque (moment of force) with respect to O is zero, because $\vec{F}$ and $\overrightarrow{OP}$ are collinear.
- By applying the Angular Momentum Theorem (eq. 9.3), $\vec{L}_O$ = constant since its time derivative is zero.

A central force centered at O conserves the angular momentum with respect to O.

3 Gravitational Force

Historical Reminder: Kepler's Laws

1) The planets revolve around the Sun in elliptical orbits, with the Sun occupying one of the foci.

2) The areas swept by the radius vectors are proportional to the times taken to sweep them.

3) The squares of the orbital periods of the planets around the Sun are proportional to the cubes of the major axes of their orbits.

It is by using these three laws that Newton derived the expression for the Gravitational Force. In the general case, this force is given by:

$$\vec{F}_G = -\frac{GMm}{r^2}\vec{e}_r \tag{9.11}$$

where the gravitational constant is $G = 6.674 \times 10^{-11}, m^3 kg^{-1} s^{-2}$.

Note: One can relate the constant g to G by equating the gravitational force at the Earth's surface to the weight: $g = \frac{GM}{R_T^2}$, where R_T is the Earth's radius.

Simple case: circular motion

Let us now analyze circular motion in a plane under the effect of a gravitational force, show that it is uniform, and calculate the velocity as a function of the radius of the trajectory:

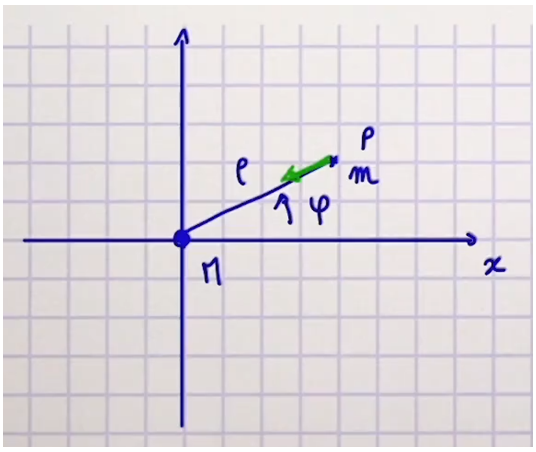


Figure 9.4: Circular motion under the force of gravity

Using the expression of the acceleration in polar coordinates and that of the gravitational force (9.11), Newton's Second Law gives:

$$\sum \vec{F} = m\vec{a} = m(-\frac{v^2}{\rho}\vec{e}_\rho + \frac{dv}{dt}\vec{e}_\varphi) = \vec{F}_G \tag{9.12}$$

By projecting onto $\vec{e}\rho$ and $\vec{e}\varphi$, the equations of motion are:

$$\text{On } \vec{e}_\rho\text{:} \quad -\frac{mv^2}{R} = -\frac{GMm}{R^2} \tag{9.13}$$

$$\text{On } \vec{e}_\varphi\text{:} \quad m\frac{dv}{dt} = 0 \tag{9.14}$$

Since the mass is non-zero, the equation along $\vec{e}\varphi$ tells us that $v =$ constant, which proves that the motion is uniform. The expression for the velocity can then be found from the equation along $\vec{e}\rho$:

$$v = \sqrt{\frac{GM}{R}} \tag{9.15}$$

The period is the time required to complete one revolution of length $2\pi R$. Thus:

$$T = \frac{2\pi R}{v} = 2\pi\sqrt{\frac{R^3}{GM}} \tag{9.16}$$

We recover Kepler's Third Law in this expression.

It is also possible to express the mass of the Sun as a function of the period:

$$M = \frac{4\pi^2 R^3}{GT^2} \tag{9.17}$$

Hence, one can determine the mass of the star if the orbital radius and the period are known.

4 Energy of the Gravitational Force

Gravitational Potential Energy

Previously, we saw that the local gravitational force (weight) can be derived from a potential energy. However, weight is only a particular case of the gravitational force, so we can seek the potential energy associated with the general $\vec{F}_G$. We wish to find an expression for the work done between A and B by the gravitational force as the difference in potential energy between A and B.

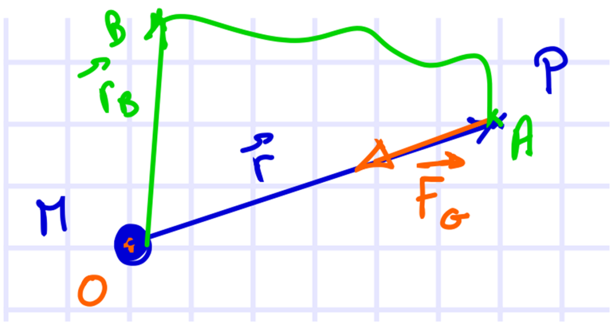


Figure 9.5:

Due to the symmetry of the gravitational force, we work in spherical coordinates, and thus return to equation 9.11.

We seek to express the work done from A to B by the gravitational force:

$$W_{AB}^{\vec{F}_G} = \int_A^B \Big(-\frac{GMm}{r^2}\Big)\vec{e}_r \cdot \vec{dr} \tag{9.18}$$

We can express $\vec{dr}$ as a function of the velocity in spherical coordinates:

$$\vec{dr} = \vec{v}dt = (\dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta + r\dot{\varphi}\sin\theta\vec{e}_\varphi)dt \tag{9.19}$$

By substituting this expression into Eq 9.18, only the term along $\vec{e}_r$ remains:

$$W_{AB}^{\vec{F}_G} = \int_A^B (-\frac{GMm}{r^2})\vec{e}_r \cdot (\dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta + r\dot{\varphi}\sin\theta\vec{e}_\varphi)dt = GMm\int_A^B (-\frac{1}{r^2})\dot{r}dt \tag{9.20}$$

By integrating:

$$W_{AB}^{\vec{F}_G} = GMm(\frac{1}{r_B} - \frac{1}{r_A}) = (-\frac{GMm}{r_A}) - (-\frac{GMm}{r_B}) \tag{9.21}$$

We then recognize the desired functional form, the difference in potential energy between A and B, and we deduce that:

$$E_P^G = -\frac{GMm}{r} \tag{9.22}$$

Escape velocity

The escape velocity is the necessary speed for an object to escape Earth's gravitational potential. We can calculate its value using the expressions derived previously.

Consider an object initially at point A on the surface of the Earth ($R_A = R_T$), launched with velocity $\vec{v}_0$, and arriving at an infinite distance ($R_B = \infty$) at point B.

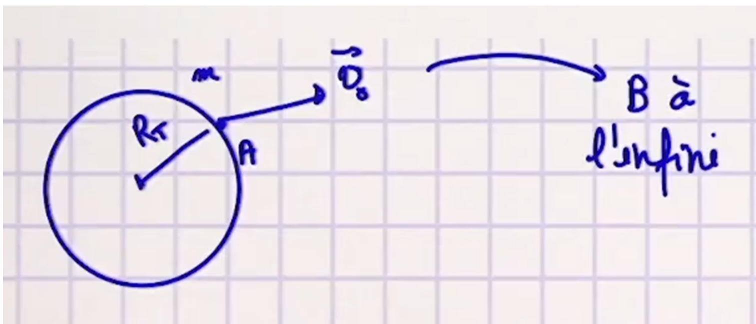


Figure 9.6:

Since the only force acting on the object is the gravitational force, which is conservative, we can apply the Conservation of Mechanical Energy, which implies:

$$E_{m,A} = E_{m,B} \tag{9.23}$$
$$E_{p,A} + E_{c,A} = E_{m,B} + E_{c,B} \tag{9.24}$$

We assume that the object just barely reaches infinity at point B, thus with zero velocity and therefore zero kinetic energy: $E_{c,B} = 0$. We use equation (9.22) for the potential terms and (6.11) for the remaining kinetic term:

$$E_{c,A} = \frac{1}{2} m v_0^2 \tag{9.25}$$

$$E_{p,A} = -\frac{GMm}{R_T} \tag{9.26}$$

$$E_{p,B} = 0 \tag{9.27}$$

We can then substitute these results into equation (9.24) and isolate the escape velocity:

$$v = \sqrt{\frac{2GM}{R_T}} \simeq 11\text{km/s} \tag{9.28}$$

Note: To escape Earth's gravitational field, the only condition is on the magnitude of the velocity. Therefore, it is possible to launch the object in any direction at this speed, and that will be sufficient for the object to escape.

Effective Potential Energy

Let us study the conservation of mechanical energy in the following case: an object of mass m moves with velocity $\vec{v}$ in a plane around a larger object of mass M, and it is subject to the gravitational force. Using polar coordinates, we have:

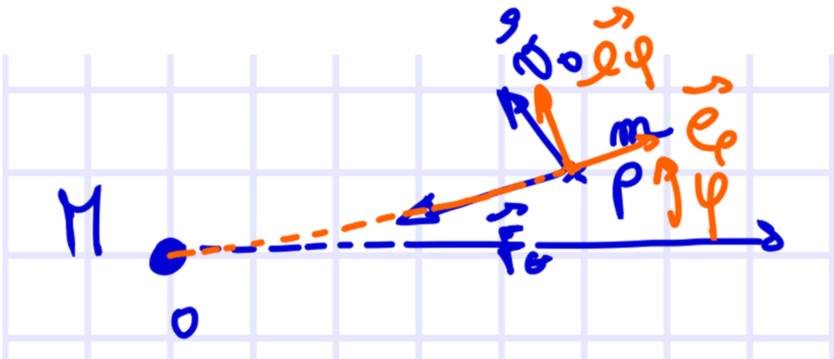


Figure 9.7:

We can express this conservation in terms of the distance separating the two objects, denoted r, which here is not necessarily constant. The conservation of mechanical energy is written:

$$E_m = E_c + E_p = \frac{1}{2} m v^2 - \frac{GMm}{r} = \text{cste} \tag{9.29}$$

We must then express v as a function of r and its derivatives. The vector $\vec{v}$ is written:

$$\vec{v} = \dot{\rho}\vec{e}_\rho + \rho\dot{\varphi}\vec{e}_\varphi + \dot{z}\vec{e}_z = \dot{\rho}\vec{e}_\rho + \rho\dot{\varphi}\vec{e}_\varphi \tag{9.30}$$

The term along $\vec{e}_z$ is zero because the motion will stay in a plane, which we choose as the (x-y) plane. We want to find v^2, the square of the magnitude of $\vec{v}$:

$$
\begin{aligned}
v^2 &= \vec{v} \cdot \vec{v} \\
&= (\dot{\rho}\vec{e}_\rho + \rho\dot{\varphi}\vec{e}_\varphi) \cdot (\dot{\rho}\vec{e}_\rho + \rho\dot{\varphi}\vec{e}_\varphi) \\
&= (\dot{\rho})^2 + (\rho\dot{\varphi})^2
\end{aligned}
$$

To obtain a relation between $\dot{\phi}$ and r, we use result (9.10) studied previously. Thus:

$$L_O^2 = |mr^2\dot{\varphi}\vec{e}_z| = m^2r^4\dot{\varphi}^2 = \text{cste} \tag{9.31}$$

$$\dot{\varphi}^2 = \frac{L_O^2}{m^2r^4} \tag{9.32}$$

$$v^2 = \dot{r}^2 + (r\dot{\varphi}^2) \tag{9.33}$$

By substituting (9.32) and (9.33) into (9.29), we then obtain:

$$E_m = \frac{1}{2}m\dot{r}^2 + \frac{L_O^2}{2mr^2} - \frac{GMm}{r} \tag{9.34}$$

In expression (9.34), the first term has the dimensions of kinetic energy, while the other two have the dimensions of potential energy. These two terms together—the gravitational potential energy and the second term from the kinetic energy—are called the **Effective Potential Energy**.

$$E_{p,\text{eff}}(r) = \frac{L_O^2}{2mr^2} - \frac{GMm}{r} \tag{9.35}$$

Let us analyze its graph:

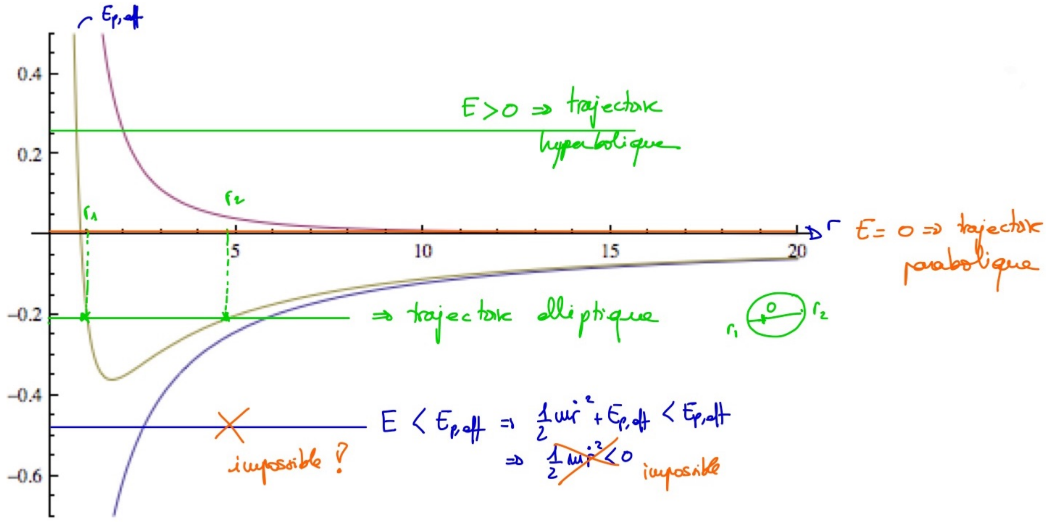


Figure 9.8: Effective potential energy as a function of distance r

It is possible to deduce from the graph the trajectory of the celestial body of mass m around the star of mass M. It depends on the value of the mechanical energy, which being constant, is represented by horizontal lines on the diagram:

- The case $E_m < E_{p,\text{eff}}$ is impossible, because in that case we would have $\frac{1}{2}m\dot{r}^2 < 0$.

- If $E_{p,\text{eff}} < E_m < 0$, there are two points of intersection in the graph. This corresponds to an elliptical trajectory, where the abscissas of the two intersection points are the lengths of the semi-major axes of the ellipse (if they are equal, the trajectory is circular).
- If $E_m = 0$, there is one point of intersection, corresponding to a minimal distance between the two bodies but no maximal distance. This is a parabolic trajectory.
- If $E_m > 0$, the resulting trajectory is a hyperbola.

Chapter 10

Rigid body mechanics

In the previous chapters we have only considered objects to be point masses. Now we will consider solid (having a certain extent) and rigid (no internal degrees of freedom; undeformable) objects. A rigid solid may undergo translational motion but also rotational motion!

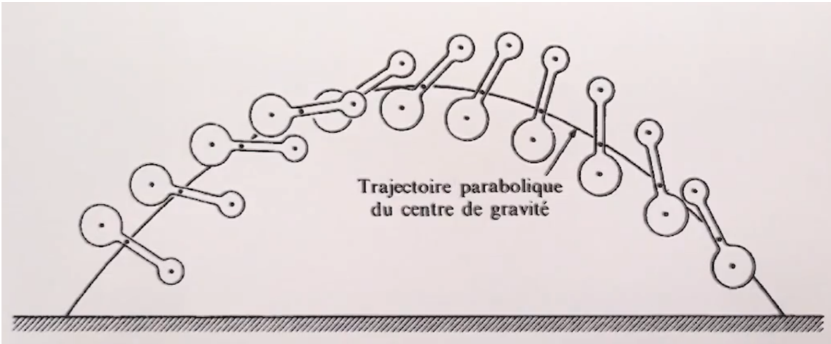


Figure 10.1: The center of mass follows a parabolic trajectory, and the solid rotates around the center of mass.

1 Center of Mass and Newton's Laws

1.1 Formation of the Laws for solids

We previously defined the center of mass (COM) for a collection of point masses and the associated Newton's laws:

$$\overrightarrow{OG} = \frac{\sum_\alpha m_\alpha \vec{r}_\alpha}{\sum_\alpha m_\alpha} = \frac{\sum_\alpha m_\alpha \vec{r}_\alpha}{M} \tag{10.1}$$

$$\sum \vec{F}^{\text{ext}} = M\vec{a}_G \tag{10.2}$$

$$\vec{P}_{\text{tot}} = M\vec{v}_G \tag{10.3}$$

We can extend these to a rigid body by moving from a discrete description to a continuous one:

$$\overrightarrow{OG} = \frac{\int_{vol} \vec{r} dm}{M} = \frac{\int_{vol} \vec{r}\rho(\vec{r})dV}{M} = \frac{\int_x \int_y \int_z \vec{r}(x;y;z)\rho(x;y;z)dV}{M} \tag{10.4}$$

And similarly for angular momentum and its theorem, which for a system of point masses is written:

$$\vec{L}_O^{\text{tot}} = \sum_\alpha \vec{L}_O^\alpha = \sum_\alpha \vec{r}_\alpha \wedge m_\alpha \vec{v}_\alpha \tag{10.5}$$

$$\sum \vec{M}_O^{\text{ext}} = \frac{d\vec{L}_O^{\text{tot}}}{dt} \tag{10.6}$$

And can be extended to the solid by writing:

$$\vec{L}_O^{\text{solid}} = \int_{vol} d\vec{L}_O = \int_{vol} \vec{r} \wedge dm\vec{v}(\vec{r}) \quad (10.7)$$

$$\sum \vec{M}_O^{\text{ext}} = \frac{d\vec{L}_O^{\text{solid}}}{dt} \quad (10.8)$$

Important: The gravitational force will always act at the center of mass G.

It is possible to determine the center of mass of an object without performing the triple integration as long as it possesses one or more axes of symmetry:

1. If the solid has **one** axis of symmetry, the center of mass lies on that axis.
2. If the object has **several** axes of symmetry, the center of mass will be at the intersection of those axes.

1.2 Specific solids

For two-dimensional objects (disks, plates), one can write $M = \rho_S * S$ with ρ_S the surface mass density (in kg/m^2).

For one-dimensional objects (a bar), one can write $M = \rho_l * l$ with ρ_l the linear mass density (in kg/m).

It is also possible, due to the geometry of certain objects, for the center of mass to lie outside of the object itself, for example for a ring (or a hollow cylinder). Its center of mass is located at the geometric center of the object (where there is no material), at the intersection of its axes of symmetry.

1.3 Composite objects

If two solids are superimposed or otherwise combined, one finds the center of mass of the "total" object using the centers of mass and masses of the two individual objects, treating them as point masses. Thus:

$$\overrightarrow{OG} = \frac{m_1\overrightarrow{OG}_1 + m_2\overrightarrow{OG}_2}{m_1 + m_2} \quad (10.9)$$

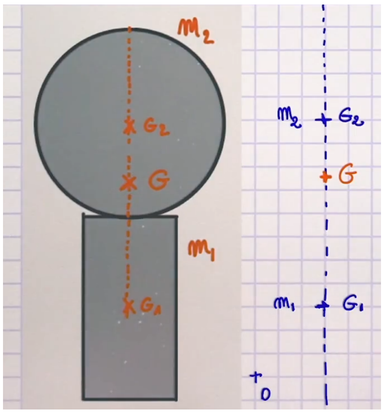


Figure 10.2: Centre of Mass of a Superposition of two solids

1.4 Solid with a Hole

To calculate the COM of a solid with a hole, we can proceed by taking two superimposed objects:

- The plug of mass m_1 with COM denoted $\overrightarrow{OG}_1$, which would fill the hole
- The perforated object of interest, of mass M and with COM denoted $\overrightarrow{OG}$, which we want to determine

By then using the formula for the COM of the total superposed object (the non-perforated base object, called 2, with mass m_2 and COM denoted $\overrightarrow{OG}_2$), we have:

$$\overrightarrow{OG}_2 = \frac{M\overrightarrow{OG} + m_1\overrightarrow{OG}_1}{M + m_1} = \frac{M\overrightarrow{OG} + m_1\overrightarrow{OG}_1}{m_2} \tag{10.10}$$

We can then deduce the desired COM from this expression by isolating $\overrightarrow{OG}$:

$$\overrightarrow{OG} = \frac{m_2\overrightarrow{OG}_2 - m_1\overrightarrow{OG}_1}{M} \tag{10.11}$$

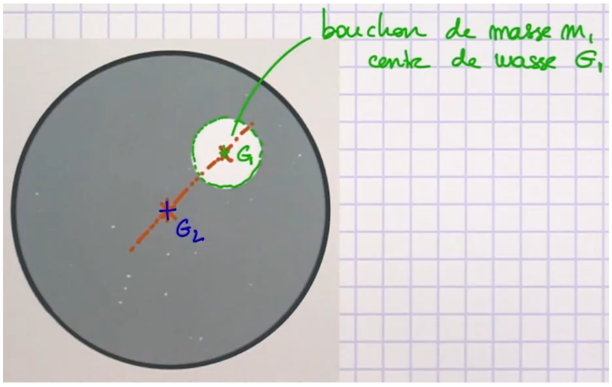


Figure 10.3: Center of Mass of an Object with a Hole

We note that finding the center of mass of an object with a hole amounts to superposing the non-perforated object and a hole of "negative mass."

2 Statics

In statics we analyze a stationary solid, and we seek the conditions under which it remains stationary. Two conditions must then be satisfied (no translation and no rotation of the object):

$$\vec{F}^{\text{ext}} = \vec{0} \tag{10.12}$$

$$\vec{M}_0^{\text{ext}} = \vec{0} \tag{10.13}$$

With O a fixed point of the reference frame.

Example: Beam (non-homogeneous) on 2 supports

Let a non-homogeneous beam (COM not centered) rest on 2 supports A and B. We know the mass, the position of the COM, and we define the distances to the supports $AG = x_1$ and $GB = x_2$. The objective is to determine the forces at the supports F_A and F_B.

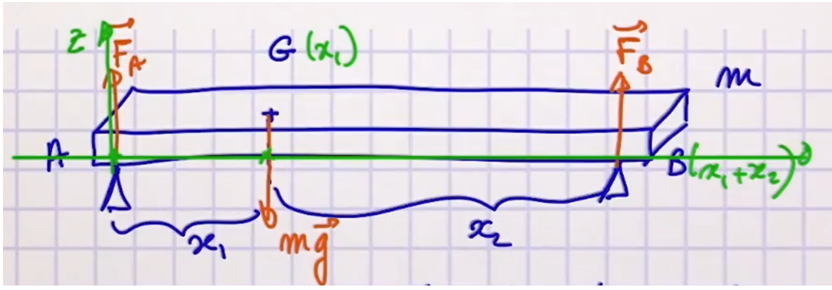


Figure 10.4: Non-homogeneous beam on 2 supports

By applying the two conditions cited above, and taking point A for the sum of moments (so as to use the known distances and have only one unknown), the system is:

$$\sum F^{\text{ext}} = \vec{F}_A + \vec{F}_B + m\vec{g} = (F_A + F_B - mg)\vec{e}_z = \vec{0} \tag{10.14}$$

$$\sum \vec{M}_A = \overrightarrow{AG} \wedge m\vec{g} + \overrightarrow{AB} \wedge \overrightarrow{F_B} = (mgx_1 - (x_1 + x_2)F_B)\vec{e}_y = \vec{0} \tag{10.15}$$

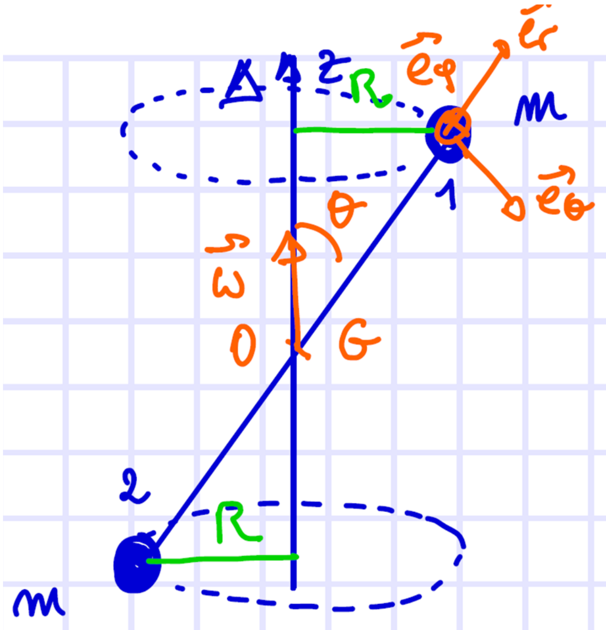
By projecting the sum of forces on the axis z (vertical) and the sum of moments on the axis y, one can isolate the supporting forces:

$$F_A = mg\frac{x_2}{x_1 + x_2} \tag{10.16}$$

$$F_B = mg\frac{x_1}{x_1 + x_2} \tag{10.17}$$

3 (Kinetic) Energy of Rotation

Let us first consider the following system: A rigid rod connects two equal masses each with mass m (its center of mass is therefore at the midpoint of the rod), and it rotates around an axis of rotation (Gz) at constant speed. The masses then each describe uniform circular trajectories.



Using spherical coordinates, we define:

$$\vec{\omega} = \omega \vec{e}_z \tag{10.18}$$
$$\overrightarrow{OP}_1 = r\vec{e}_r \tag{10.19}$$
$$r = l/2 \tag{10.20}$$
$$R = r \sin\theta \tag{10.21}$$

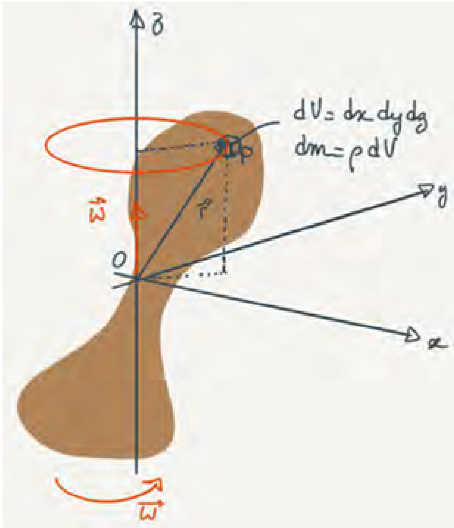
We also deduce the expressions for velocity and thus for kinetic energy (which are equal) for the respective masses, as well as for the entire system:

$$\text{Velocity of a mass: } \vec{v} = R\omega\vec{e}_\varphi \tag{10.22}$$
$$\text{Kinetic energy of a mass } E_{c,m} = \frac{1}{2}mv^2 = \frac{1}{2}mR^2\omega^2 \tag{10.23}$$
$$\text{Kinetic energy of the total system:} E_{c,rot} = 2 * \frac{1}{2}mv^2 = \frac{1}{2}(2m)R^2\omega^2 \tag{10.24}$$

We note that the total kinetic energy can be divided into two parts:

1. an expression that depends simply on the geometry of the object: $\frac{1}{2}(2m)R^2$
2. an expression that depends on the rotational speed: ω^2

Let us now generalize this to an arbitrary solid:



An arbitrary point P in the solid behaves like the point mass in the previous case. It is occupying an infinitesimal volume dV with mass dm. This element dm has a rotational energy from which, using the previous case, we deduce the rotational energy of the solid:

$$\text{Mass centered at P:} dm = \rho * dV \tag{10.25}$$

$$\text{Energy: } E_{c,dm} = \frac{1}{2}dm(R\omega)^2 = \frac{1}{2}\rho(\vec{r})R^2\omega^2 dV \tag{10.26}$$

$$\text{Energy of the system: } E_{c,rot} = \int_{vol}\frac{1}{2}\rho(\vec{r})R^2\omega^2 dV = \frac{1}{2}\omega^2\int_{vol}\rho(\vec{r})R^2 dV \tag{10.27}$$

We introduce the notion of **Moment of Inertia** of the solid with respect to the Oz axis as:

$$I_z = \int_{vol}\rho(\vec{r})R^2 dV \tag{10.28}$$

This allows us to rewrite **the (kinetic) rotational energy** as:

$$E_{c,rot} = \frac{1}{2}I_z\omega^2 \tag{10.29}$$

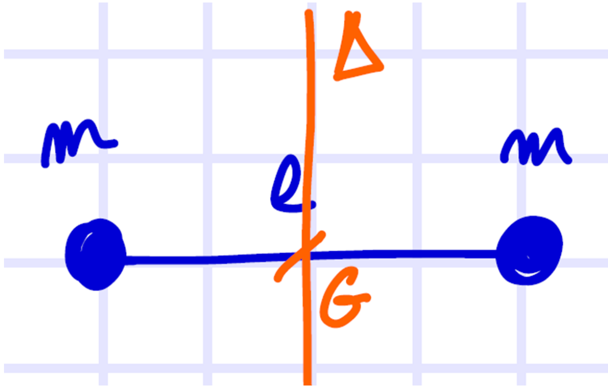
We notice the structure of this expression, which is reminiscent of translational kinetic energy: I plays the role of mass, while ω plays the role of velocity.

4 Moment of Inertia of a Solid with Respect to an Axis

The moment of inertia depends on both the geometry of the solid and the specific axis of rotation.

4.1 Simple rod with two masses

Consider a massless rod, of length l, with 2 masses m at each end. By calculating its moment of inertia with respect to a rotation axis perpendicular to the rod, at the center of mass, we obtain:

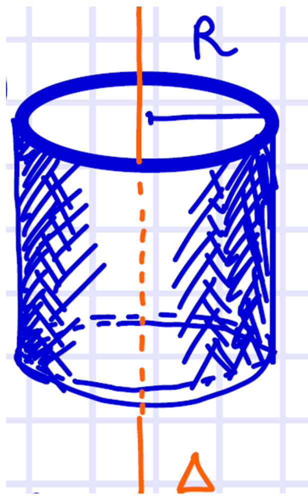


$$I_\Delta = m\Big(\frac{l}{2}\Big)^2 + m\Big(\frac{l}{2}\Big)^2 = \frac{1}{2}ml^2 \tag{10.30}$$

4.2 Thin hollow cylinder

The hollow cylinder has a radius R and a total mass M. Its moment of inertia with respect to its axis of symmetry is:

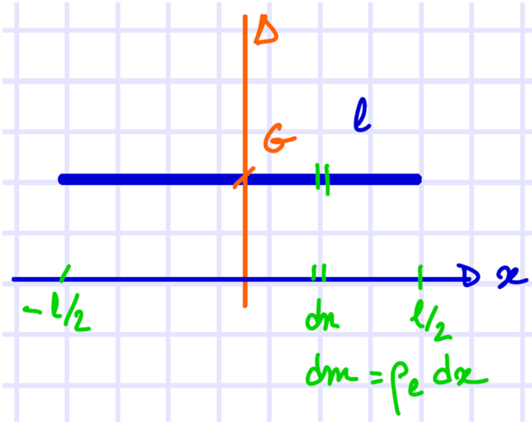
$$I_\Delta = MR^2 \tag{10.31}$$



4.3 Rod: Linear Mass Density

Let us now study a thin rod of mass M distributed along its length L. Its linear mass density is defined as $\rho_L = M/L$. For a moment of inertia around rotation axis Δ perpendicular to the rod at its center of mass, we define an x-axis with the origin at the center of mass G, and then compute:

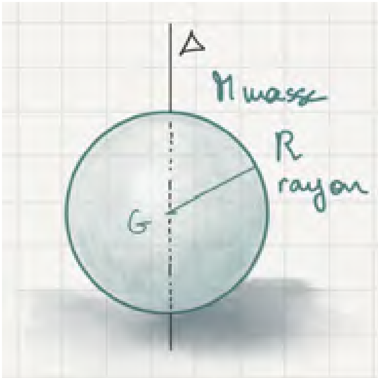
$$I_\Delta = \int_{-l/2}^{l/2} \rho_L * x^2 dx = \frac{1}{12}ML^2 \tag{10.32}$$


Δ
ℓ
G
x
-ℓ/2
dx
ℓ/2
dm = ρℓ dx

4.4 Common Homogeneous Solids

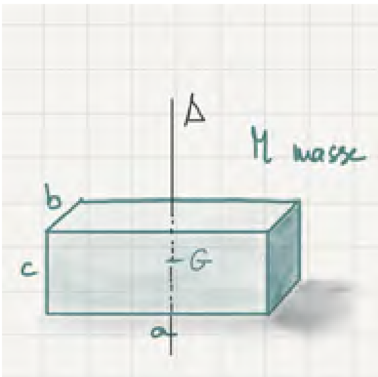
Sphere

$$I_\Delta = \frac{2}{5} M R_S^2 \tag{10.33}$$



Rectangular Parallelepiped

$$I_\Delta = \frac{1}{12} M (a^2 + b^2) \tag{10.34}$$



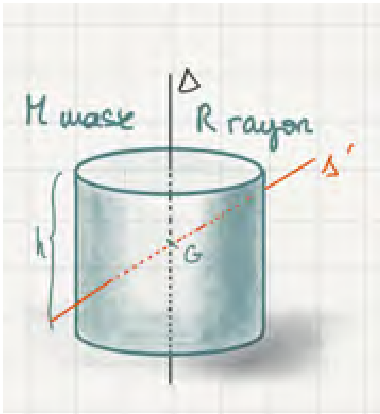
Remark: For a thin rectangular plate, the expression above can be reused because for this axis, the moment of inertia does not depend on c.

Cylinder

Depending on the chosen axis:

$$I_\Delta = \frac{1}{2} M R^2 \tag{10.35}$$

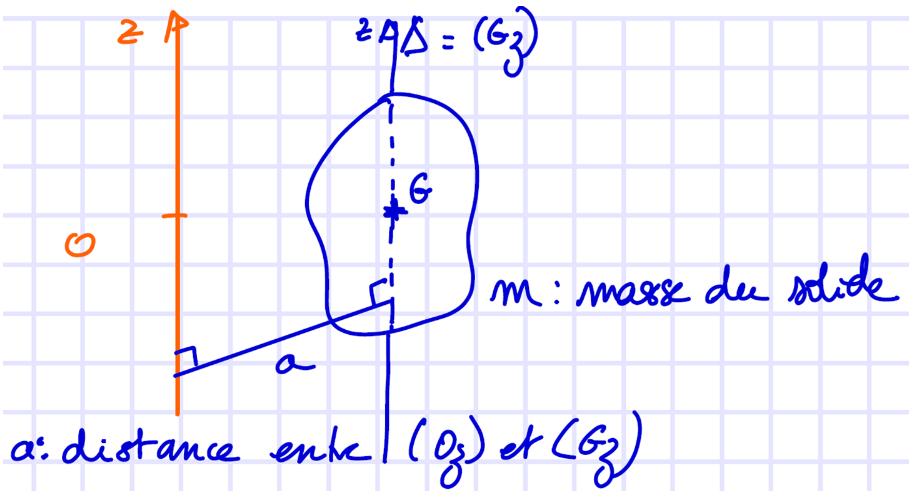
$$I_{\Delta'} = \frac{1}{4} M (R^2 + \frac{h^2}{3}) \tag{10.36}$$



4.5 Steiner's Theorem

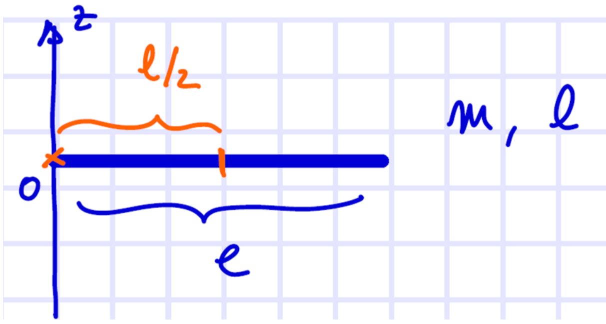
Let two parallel axes (Oz) and (Gz) be separated by a distance a, with G bring the center of mass of the object. If I_{Gz} is the moment of inertia with respect to (Gz), then I_{Oz}, the moment of inertia with respect to (Oz), is:

$$I_{Oz} = I_{Gz} + ma^2 \tag{10.37}$$



Example: Homogeneous Rod Rotating Around One End

The moment of inertia of a homogeneous rod (uniformly distributed linear mass) around an axis perpendicular to its center of mass is known. By applying Steiner's theorem, we can calculate the moment of rotation around one end.



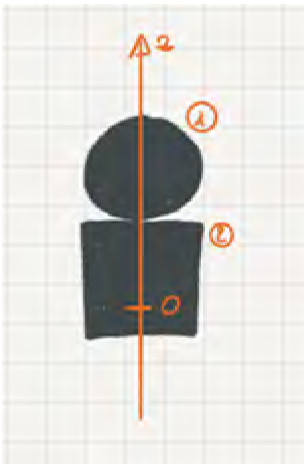
$$I_{Oz} = \frac{1}{12}ml^2 + m\left(\frac{l}{2}\right)^2 \tag{10.38}$$

$$I_{Oz} = ml^2\left(\frac{1}{12} + \frac{1}{4}\right) = \frac{1}{3}ml^2 \tag{10.39}$$

4.6 Composite Solids and Solids with Holes

The moment of inertia is calculated by integration. Since the integral is additive, it is possible to compute the moment of inertia of composite solids by adding the moments of inertia of the solids that compose them:

$$I_{\text{Oz}}^{\text{tot}} = I_{\text{Oz}}^{(1)} + I_{\text{Oz}}^{(2)} \tag{10.40}$$



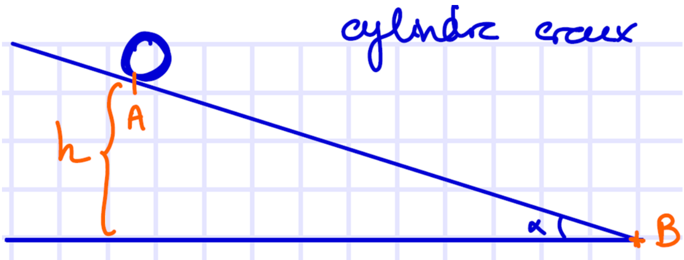
Similarly, the moment of inertia of a solid with a hole is obtained by subtracting from the non-perforated solid the moment of inertia of the plug corresponding to the hole:

$$I_{\text{Oz}}^{\text{bouché}} = I_{\text{Oz}}^{rel} + I_{\text{Oz}}^{bouchon} \tag{10.41}$$

$$I_{\text{Oz}}^{rel} = I_{\text{Oz}}^{\text{bouché}} - I_{\text{Oz}}^{bouchon} \tag{10.42}$$

5 Application

A hollow cylinder of mass M and radius R rolls without slipping down an inclined plane. It is released with no initial velocity at point A at height h, and we seek its velocity at point B.



It is subject to the weight due to gravity, the normal force, and the friction force at the surface. However, only gravity does work because the velocity at the point of contact is zero (rolling without slipping), and the normal force acts perpendicular to the trajectory: $W = W_P + W_R + W_f = W_P$. The mechanical energy of the center of mass G at a given point is written:

$$E_m(G) = E_p + E_{\text{c,rot}} + E_{\text{c,t}} = E_p + \frac{1}{2}I_G\omega^2 + \frac{1}{2}Mv_G^2 \tag{10.43}$$

$$I_G = MR^2 \tag{10.44}$$

$$v_G = R\omega \Leftrightarrow \omega = \frac{v_G}{R} \tag{10.45}$$

Let us take as reference for the potential energy the base of the inclined plane, so that $E_p(A) = Mgh$ and $E_p(B) = 0$. By conservation of mechanical energy between A and B (because the weight is a conservative force):

$$E_m(A) = E_m(B) \Leftrightarrow Mgh = \frac{1}{2}I_G\omega_B^2 + \frac{1}{2}Mv_B^2 = \frac{1}{2}MR^2 * \frac{v_B^2}{R^2} + \frac{1}{2}Mv_B^2 \tag{10.46}$$

$$v_B = \sqrt{gh} \tag{10.47}$$

The expression resembles that obtained for an object that slides without rolling: $v_B = \sqrt{2gh}$. The velocity here is smaller because for the same initial potential energy, part of it is used to make the object rotate and give it rotational energy.

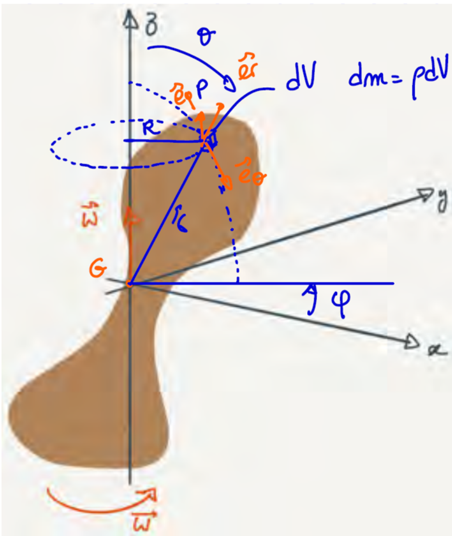
6 Angular Momentum of a Solid

Let us now study the angular momentum of a general solid and no longer of a simple particle. The quantity was defined by equation (9.1), and the angular momentum theorem by (9.3).

General Case: arbitrary solid in rotation

Let a solid be rotating around an axis passing through G. We choose the axis (Gz) so that it is the axis of rotation. The goal will be to compute the angular momentum of this solid associated with the rotation.

Consider an infinitesimal element of the solid centered at point P, with volume dV and mass $dm = \rho dV$. Let us study the problem in spherical coordinates.



The infinitesimal angular momentum in the neighborhood of the chosen point P is written:

$$d\vec{L}_G = \vec{r} \wedge d\vec{p} = \vec{r} \wedge dm * \vec{v}(P) \tag{10.48}$$

$$\vec{v}(P) = R\omega\vec{e}_\varphi = r\sin\theta\ \omega\vec{e}_\varphi \tag{10.49}$$

$$\rightarrow d\vec{L}_G = r\vec{e}_r \wedge dm * r\sin\theta\ \omega\vec{e}_\varphi = r^2 dm \sin\theta\omega(-\vec{e}_\theta) \tag{10.50}$$

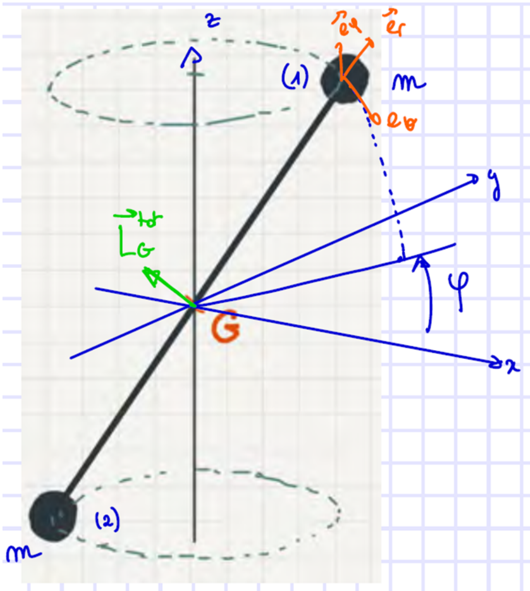
For the entire solid, we then write:

$$\vec{L}_G = \int_{\text{vol}} r^2 dm \sin\theta\ \omega(-\vec{e}_\theta) \tag{10.51}$$

For each point P of the solid, the resulting infinitesimal angular momentum will each have a different direction. Therefore this computation is an integration of many vector elements with different directions, and it is far too complex to intuitively obtain $\vec{L}_G$ in the general case.

Simple Case: dumbbell in rotation

Let us simplify the problem by studying a dumbbell (massless rod with small mass attached at each end).



The total angular momentum will be the sum of the angular momentum of each mass. We can write:

$$\vec{L}_G^{(1)} = \vec{r} \wedge m\vec{v} = r\vec{e}_r \wedge m(r\omega \sin\theta \vec{e}_\varphi) = -mr^2\omega \sin\theta \vec{e}_\theta \tag{10.52}$$

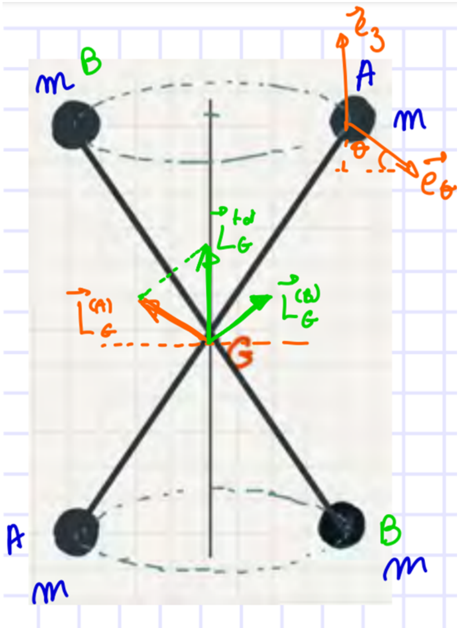
$$\vec{L}_G^{(2)} = -\vec{r} \wedge m(-\vec{v}) = \vec{L}_G^{(1)} \tag{10.53}$$

$$\vec{L}_G^{\text{sol}} = \vec{L}_G^{(1)} + \vec{L}_G^{(2)} = -(2m)r^2\omega \sin\theta \vec{e}_\theta = -Mr^2\omega \sin\theta \vec{e}_\theta \tag{10.54}$$

The obtained angular momentum is perpendicular to the axis of the rod, passing through the center of mass G. While rotating, the solid traces a cone, and **the direction of the angular-momentum vector also changes, describing a cone** with a different angle relative to the horizontal (angle θ).
If the angular momentum changes, its derivative is nonzero, and therefore it is necessary to apply a torque to maintain the rotation (consistent with the angular momentum theorem seen previously).
In general the angular momentum is not parallel to the axis of rotation of the solid, unless some symmetry exists in the mass distribution. Maintaining such an axis of rotation requires applying a torque.

Examples of Symmetric Cases

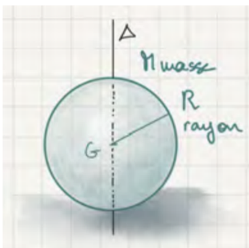
Let us now double dumbbell. We have the first dumbbell with two masses (A), and have added a second identical dumbbell (B), symmetric with respect to the axis of rotation.

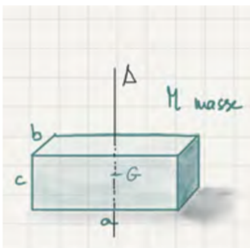


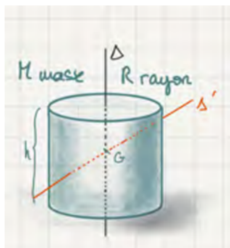
The total angular momentum of the solid will be $\vec{L}_G^{\text{tot}} = \vec{L}_G^A + \vec{L}_G^B$, collinear with $\vec{e}_z$ due to the symmetry of the problem. Using the expression for solid (A) projected onto the z-axis by a dot product and $M = 4m$ the total mass, we then have:

$$\vec{L}_G^{\text{tot}} = |L_G^{\text{tot}}|\vec{e}_z = (2\vec{L}_G^A \cdot \vec{e}_z)\vec{e}_z = 4m(r^2 \sin^2\theta)\omega\vec{e}_z = MR^2\omega\vec{e}_z = I_G\vec{\omega} \tag{10.55}$$

Here the angular momentum is parallel to the vector $\vec{\omega}$ because the problem is symmetric! It is possible to generalize this expression to standard solids:





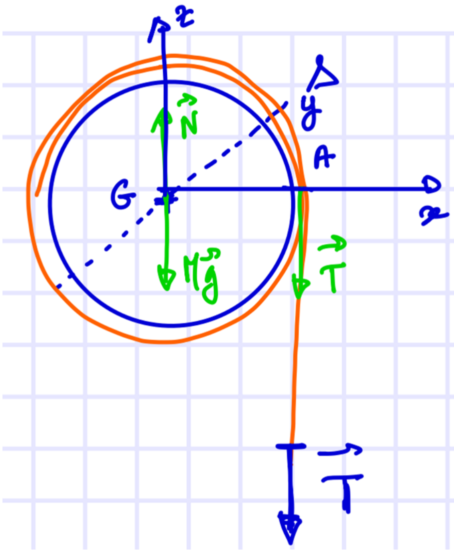


In all three cases, the solid can be cut along two planes that allow us, by gathering the four quadrants into four infinitesimal masses, to recover the previous situation (double symmetric dumbbell).

There therefore exist symmetry axes such that for a rotation around such an axis, $\vec{L}_G = I_{Gz}\vec{\omega}$: these are called **principal axes of inertia. Every solid has at least 3 principal axes of inertia**.

Example: Pulley and Rope

Let us consider a pulley modeled as a homogeneous solid disk fixed to the axis Δ passing through G, of mass M and radius R. A massless rope winds around it. During a time t_0, a tension $\vec{T}$ is applied to the rope, and the system is initially at rest: we want to determine the length of rope unwound after time t_0.



Taking into account all the forces present (weight of the pulley, tension of the rope, reaction at the axis) and paying attention to their respective points of application (at G or A respectively). We then apply the angular momentum theorem at the center of mass G:

$$\sum \vec{M}_G^{\text{ext}} = RT\vec{e}_y = \frac{d\vec{L}_G}{dt} \tag{10.56}$$

Since the axis Δ is a principal axis of inertia, the angular momentum can be written as $\vec{L}_G = I_{Gy}\omega\vec{e}_y = \frac{1}{2}MR^2\omega\vec{e}_y$. In this expression, only ω depends on time, the other terms are constant. Returning to the angular momentum theorem and projecting onto the y-axis, we can then isolate and integrate twice:

$$\frac{1}{2}MR^2\dot{\omega} = RT \tag{10.57}$$

$$\dot{\omega} = \frac{2T}{MR} = cste \tag{10.58}$$

$$\omega = \frac{2T}{MR}t + \omega(t=0) = \frac{2T}{MR}t \tag{10.59}$$

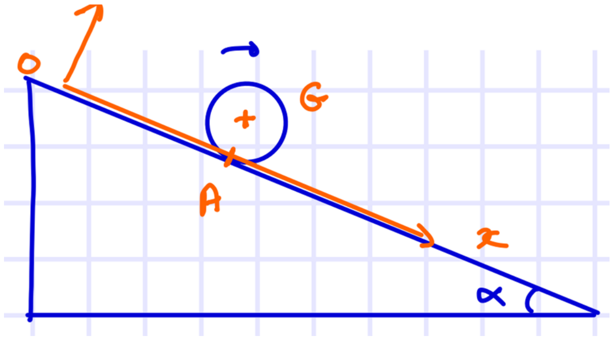
$$\theta = \frac{T}{MR}t^2 \tag{10.60}$$

The length of rope unwound is then written:

$$L = \theta_f * R = \frac{T}{M}t_0^2 \tag{10.61}$$

7 Rolling Solid

Let us now study a cylinder rolling without slipping along an inclined plane with angle α.



We have seen so far that it is possible to apply the angular momentum theorem to solve the problem at a fixed point. The 3 points of interest here are O, the origin of the coordinate system, A, the point of contact between the solid and the plane, and G, the center of mass of the cylinder. O is fixed; A and G are not!

Furthermore, we must be able to compute the angular momentum at the chosen point of application. This is possible for the center of mass G or for a point of the solid with zero velocity:

- for G: $\vec{L}_G = I_{Gz}\vec{\omega}$ if Gz is a principal axis of inertia
- for A: $\vec{L}_A = I_{Az}\vec{\omega}$ if A is a point of the solid with zero velocity and Az is a principal axis of inertia
- for O: neither a point of the solid with zero velocity nor the center of mass. We cannot write $\vec{L}_O = I_{Oz}\vec{\omega}$

It is therefore a problem to apply the angular momentum theorem in this example: **the fixed-point constraint in the reference frame is too restrictive.**

Let us study its application at an arbitrary non-fixed point A in the reference frame $\mathcal{R}$. Returning to the integral expression of angular momentum, and using the known relations ($\vec{p}^{\text{tot}} = M\vec{v}_{\mathcal{R}}(G)$; $\sum \vec{M}_O = \sum \overrightarrow{OP}\text{app} \wedge \vec{F}^{\text{ext}}$; $\vec{L}_O = \int \text{vol}\overrightarrow{OP} \wedge dm, \vec{v}_{\mathcal{R}}(P)$) :

$$\vec{L}_A = \int_{\text{vol}} \overrightarrow{AP} \wedge dm\vec{v}_{\mathcal{R}}(P) = \int_{\text{vol}} (\overrightarrow{AO} + \overrightarrow{OP}) \wedge dm\vec{v}_{\mathcal{R}}(P) \tag{10.62}$$

$$\vec{L}_A = \overrightarrow{AO} \wedge \int_{\text{vol}} dm\vec{v}_{\mathcal{R}}(P) + \int_{\text{vol}} \overrightarrow{OP} \wedge dm\vec{v}_{\mathcal{R}}(P) = -\overrightarrow{OA} \wedge \vec{p}^{\text{tot}} + \vec{L}_O \tag{10.63}$$

$$\frac{d\vec{L}_A}{dt} = -\vec{v}_{\mathcal{R}}(A) \wedge \vec{p}^{\text{tot}} - \overrightarrow{OA} \wedge \sum \vec{F}^{\text{ext}} + \sum \vec{M}_O \tag{10.64}$$

$$\frac{d\vec{L}_A}{dt} = -M\vec{v}_{\mathcal{R}}(A) \wedge \vec{v}_{\mathcal{R}}(G) + \sum (\overrightarrow{AO} + \overrightarrow{OP}_{\text{app}}) \wedge \vec{F}^{\text{ext}} \tag{10.65}$$

From this we deduce the **general formula for an arbitrary point A**:

$$\frac{d\vec{L}_A}{dt} = \sum \vec{M}_A - M\vec{v}_{\mathcal{R}}(A) \wedge \vec{v}_{\mathcal{R}}(G) \tag{10.66}$$

The second term will be zero in three cases:

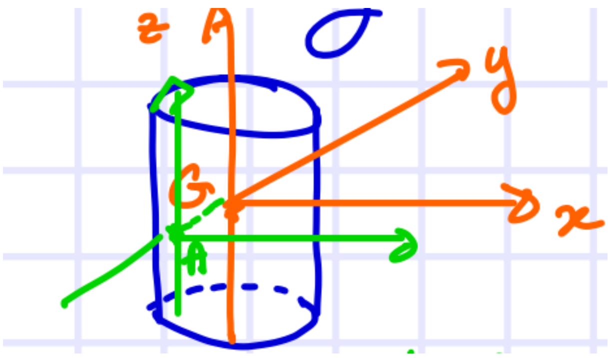
- if A is fixed in $\mathcal{R}$
- if A is the center of mass
- if $\vec{v}_{\mathcal{R}}(A)$ is colinear with $\vec{v}_{\mathcal{R}}(G)$

Returning now to the case of the cylinder and the inclined plane: the reference point A **in the frame $\mathcal{R}$ has a nonzero velocity and is colinear with that of G** (different from the zero velocity of the material point in contact with the plane), because at each instant t the point A changes position. Therefore, we can eliminate the second term and apply $\frac{d\vec{L}_A}{dt} = \sum \vec{M}_A$ directly. This is also possible, as discussed earlier, for the center of mass G, but not for point O (due to the problem of computing $\vec{L}_O$)!

Furthermore, knowing the principal axis of inertia is important in order to compute the angular momentum at the chosen point. Under what condition, then, are the axes (O, x, y, z) **principal axes of inertia** for a point O?

We assume that this is the case if (G, x, y, z) are principal axes of inertia, and if O lies on one of them. In this case, for a rotation around (Oz): $\vec{\omega} = \omega\vec{e}_z$ and $\vec{L}_O = I_{Oz}\vec{\omega}$.

In our example, the point A lies on the principal axis Oy, and therefore the axes (A, x, y, z) are also principal axes.



8 Inertia Tensor (outside the syllabus)

In the general case, when the rotation does not occur around a principal axis of inertia, the angular momentum is written as the product between a tensor (matrix) and a vector:

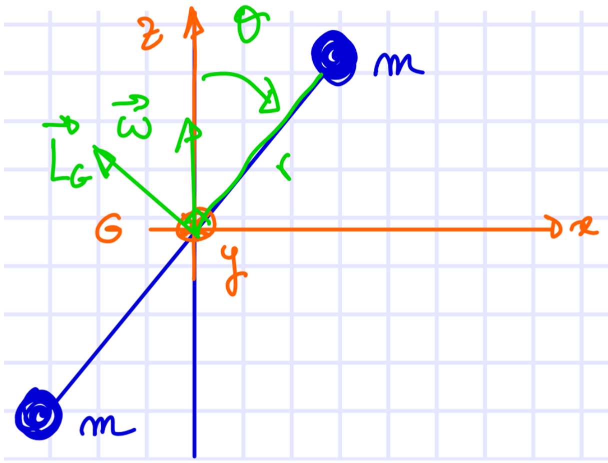
$$\vec{L}_G = \underline{I}_G\vec{\omega} \tag{10.67}$$

With $\underline{I}_G$ the inertia tensor, which depends on the chosen origin and axes. Its expression is:

$$\underline{I}_G = \begin{bmatrix} \int(y^2+z^2)dm & -\int xydm & -\int xzdm \\ -\int xydm & \int(x^2+z^2)dm & -\int yzdm \\ -\int xzdm & -\int yzdm & \int(x^2+y^2)dm \end{bmatrix} \tag{10.68}$$

Example

For the example of the simple non-symmetric dumbbell, the calculation allows us to obtain in (G, x, y, z):



$$\vec{\omega} = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} \tag{10.69}$$

$$\underline{I}_G = \begin{bmatrix} 2m(r\cos\theta)^2 & 0 & -2mr^2\sin\theta\cos\theta \\ 0 & 2mr^2 & 0 \\ -2mr^2\sin\theta\cos\theta & 0 & 2m(r\sin\theta)^2 \end{bmatrix} \tag{10.70}$$

$$\vec{L}_G = \underline{I}_G\vec{\omega} = \begin{pmatrix} -2m\omega r^2\sin\theta\cos\theta \\ 0 \\ 2m\omega r^2\sin^2\theta \end{pmatrix} \tag{10.71}$$

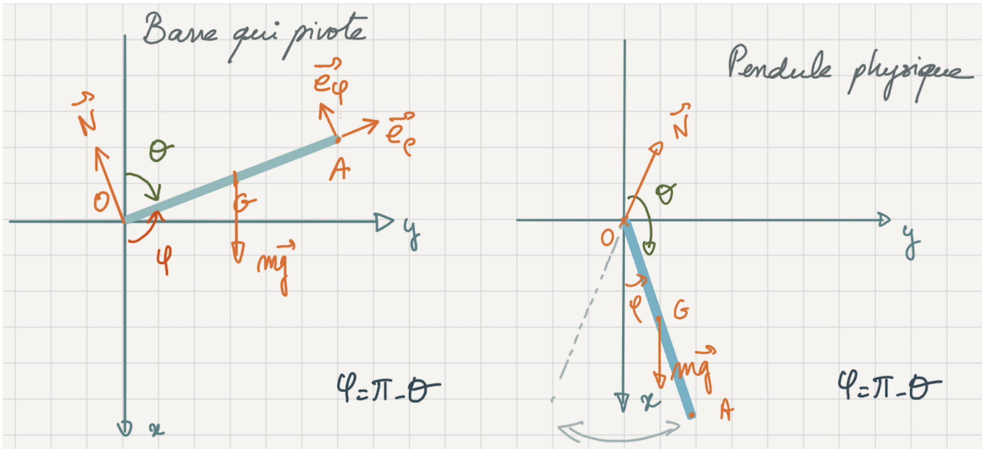
Chapter 11

Applications of Rigid Body Mechanics

1 Falling Bar and Physical Pendulum

1.1 Problems and Generalities

A bar that falls from a vertical position is, physically, an example close to the physical pendulum, which consists of a bar oscillating around an equilibrium position instead of a mass suspended by a string.



Depending on the problem, it will be more appropriate to choose the angle θ (which varies between 0 and 2π) or the angle φ (varying between $-\pi$ and π), between the vertical and our object. This choice is made according to the particularities of the problem:

- **Fall:** The solid falls from the vertical until it reaches the ground (horizontal) and no longer moves. The angle θ is more appropriate here because it will vary only between 0 and 90 degrees.

- **Pendulum:** Here the solid oscillates over the two lower quadrants, and it is then much more appropriate to choose the angle φ around which the solid moves, with $\phi = 0$ the equilibrium position.

Note that the two angles are related by the relation $\varphi = \pi - \theta$, and the angle φ is the one associated with the cylindrical coordinates (O, ρ, φ, z), taken with respect to the vertical axis Ox pointing downward.

Regarless of the choice between ϕ or θ, the forces present are the weight $m\vec{g}$ and the normal force $\vec{N}$. At point O, the moment of $\vec{N}$ is zero. Moreover, the pivot at point O, with the axis (Oz) as a principal axis of inertia, we can write:

$$\vec{L}_O = I_O\vec{\omega} \tag{11.1}$$

$$I_O = I_G + m\frac{l}{2}^2 = \frac{ml^2}{12} + \frac{ml^2}{4} = \frac{1}{3}ml^2 \tag{11.2}$$

$$\sum \vec{M}_O = \frac{d\vec{L}_O}{dt} = \overrightarrow{OG} \wedge m\vec{g} \tag{11.3}$$

Let us study the motion using the two different angles.

Using θ:

$$\vec{\omega} = -\dot{\theta}\vec{e}_z \tag{11.4}$$

$$\overrightarrow{OG} \wedge m\vec{g} = \frac{lmg}{2}\sin(\pi - \theta)(-\vec{e}_z) = -\frac{lmg}{2}\sin\theta\vec{e}_z \tag{11.5}$$

$$\frac{d\vec{L}_O}{dt} = \frac{1}{3}ml^2(-\ddot{\theta})\vec{e}_z = -\frac{lmg}{2}\sin\theta\vec{e}_z \tag{11.6}$$

$$\ddot{\theta} - \frac{\mathbf{3}}{\mathbf{2}}\frac{\mathbf{g}}{\mathbf{l}}\sin\theta = \mathbf{0} \tag{11.7}$$

Using φ:

$$\vec{\omega} = -\frac{d}{dt}(\pi - \varphi)\vec{e}_z = \dot{\varphi}\vec{e}_z \tag{11.8}$$

$$\overrightarrow{OG} \wedge m\vec{g} = \frac{lmg}{2}\sin(\varphi)(-\vec{e}_z) = -\frac{lmg}{2}\sin\varphi\vec{e}_z \tag{11.9}$$

$$\frac{d\vec{L}_O}{dt} = \frac{1}{3}ml^2\ddot{\varphi}\vec{e}_z = -\frac{lmg}{2}\sin\varphi\vec{e}_z \tag{11.10}$$

$$\ddot{\varphi} + \frac{\mathbf{3}}{\mathbf{2}}\frac{\mathbf{g}}{\mathbf{l}}\sin\varphi = \mathbf{0} \tag{11.11}$$

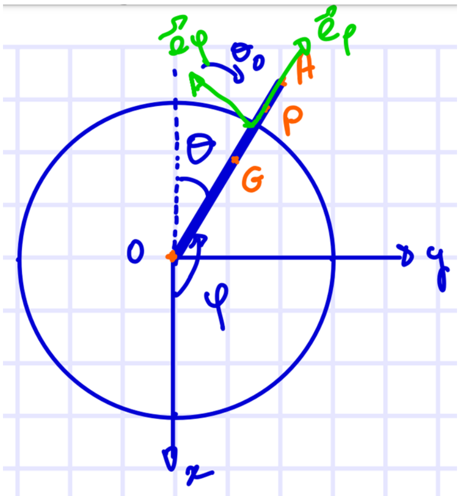
The obtained equations inform us about the most appropriate choice for each case:

- **Pendulum:** If one wants to obtain a harmonic oscillator equation, it is necessary to use the small-angle approximation. With φ, the approximation is made near 0, and we can directly approximate $\sin\varphi$ by φ. However, for θ, the approximation must be made near π, and therefore requires the Taylor expansion of the sine function around this nonzero equilibrium point. *Thus, it is more appropriate to choose φ.*

- **Falling bar:** Here, regardless of the chosen description, we cannot apply the small-angle approximation and reduce the motion to a harmonic oscillator: the solution is found through a numerical resolution (seen in Analysis IV). *We choose θ because it parameterizes the problem with an angle that is an increasing function of time.*

1.2 Falling Bar

We release the bar at the initial angular position $\theta(0) = \theta_0$ and with zero velocity $\dot{\theta}(0) = 0$. A point P on the bar, at a distance d from O, describes a non-uniform circular motion. Our goal is to determine the acceleration of point P.



Let us return to the equation of motion in terms of θ and the relations between the angles:

$$\varphi = \pi - \theta \Rightarrow \dot{\varphi} = -\dot{\theta} \Rightarrow \ddot{\varphi} = -\ddot{\theta} \tag{11.12}$$

$$\ddot{\theta} - \frac{3}{2}\frac{g}{l}\sin\theta = 0 \tag{11.13}$$

Using the known expression of acceleration in polar coordinates, the angular relations, substituting $\rho = d$ at point P and replacing $\ddot{\theta}$ using the previous differential equation, we obtain:

$$\vec{a}_\rho = -\rho\dot{\varphi}^2\vec{e}_\rho + \rho\ddot{\varphi}\vec{e}_\varphi \tag{11.14}$$

$$\vec{a}_\rho = -d\dot{\theta}^2\vec{e}_\rho - d\ddot{\theta}\vec{e}_\varphi \tag{11.15}$$

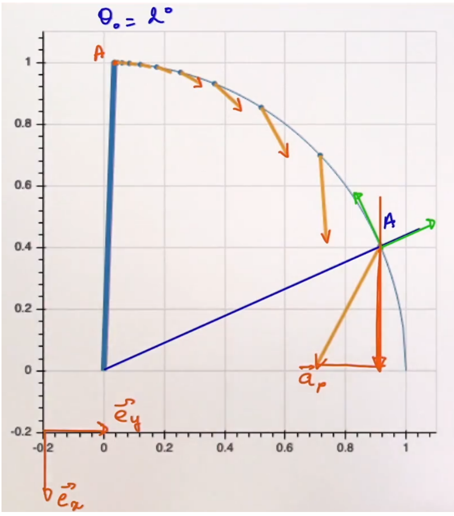
$$\vec{a}_\rho(t) = -d\dot{\theta}^2(t)\vec{e}_\rho - d(\frac{3g}{2l}\sin\theta(t))\vec{e}_\varphi \tag{11.16}$$

The two quantities $\dot{\theta}$ and $\sin\theta$ have been obtained numerically, as a function of time, using the Runge-Kutta algorithm (Analysis IV). Thus, using a table of given values, we can determine the acceleration of point P.

Experimentally:

- we choose an initial position θ_0
- we measure at regular intervals Δt the position θ and the velocity $\dot{\theta}$ of point P
- for each position, we calculate $\vec{a}_\rho$ using the equation of motion determined above and the measured position and velocity

We will now examine the projection of the acceleration on the vertical axis (Ox), at the point $P = A$ at the end of the bar: this amounts to calculating the dot product $\vec{a}_\rho(t) \cdot \vec{e}_x$.



Using the relations derived in the chapter on polar coordinates, trigonometric formulas, and the expression of the acceleration calculated above, **for a point on the bar at an arbitrary distance** d we obtain:

$$\vec{e}_\rho \cdot \vec{e}_x = (\cos\varphi \vec{e}_x + \sin\varphi \vec{e}_y) \cdot \vec{e}_x = \cos\varphi = \cos(\pi - \theta) = -\cos\theta \tag{11.17}$$

$$\vec{e}_\varphi \cdot \vec{e}_x = (-\sin\varphi \vec{e}_x + \cos\varphi \vec{e}_y) \cdot \vec{e}_x = -\sin\varphi = -\sin(\pi - \theta) = -\sin\theta \tag{11.18}$$

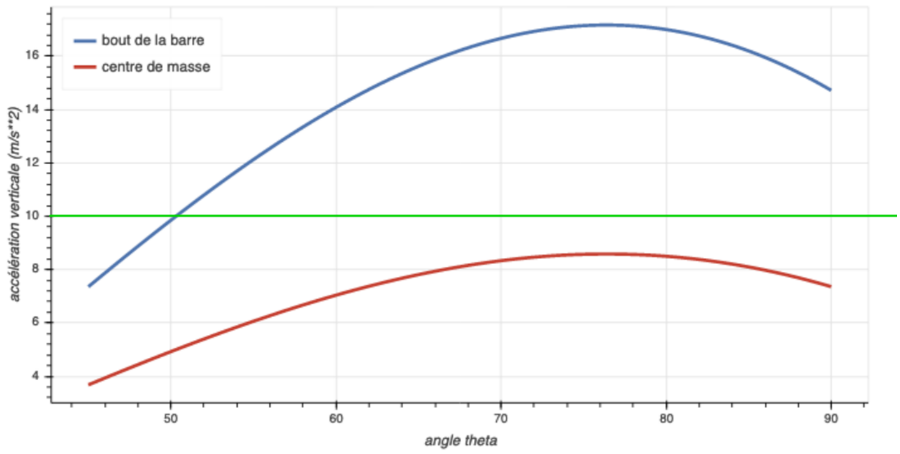
$$\vec{a}_\rho \cdot \vec{e}_x = d\dot{\theta}^2 \cos\theta + d\frac{3g}{2l}\sin^2\theta \tag{11.19}$$

Let us consider two particular points:

A, the end of the bar $\vec{a}_\rho \cdot \vec{e}_x(d = l) = l\dot{\theta}^2 \cos\theta + \frac{3}{2} g \sin^2\theta$ (11.20)

G, the center of mass of the bar $\vec{a}_\rho \cdot \vec{e}_x(d = l/2) = \frac{l}{2}\dot{\theta}^2 \cos\theta + \frac{3}{4} g \sin^2\theta$ (11.21)

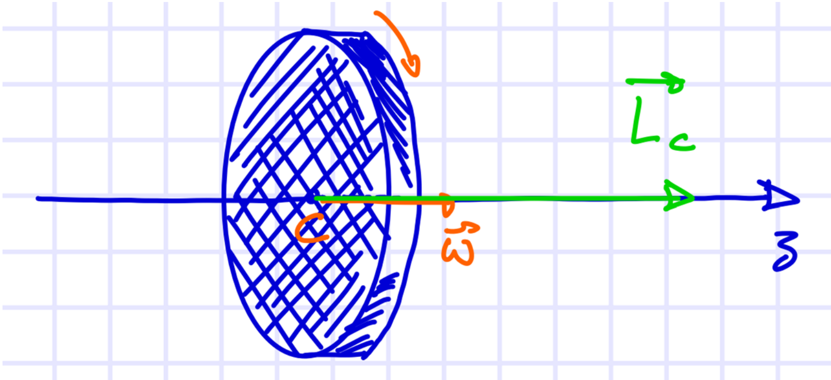
It is then possible to plot graphically the acceleration of the bar as a function of the angle θ for the two particular points:



We notice graphically that **the acceleration at the end exceeds the gravitational acceleration** g **from approximately 50 degrees.** This may seem surprising, but it is due to the fact that the bar is rigid and fixed at a pivot. Each part of the bar contributes to driving the next part, exerting internal forces, which allow $a(A) > g$.

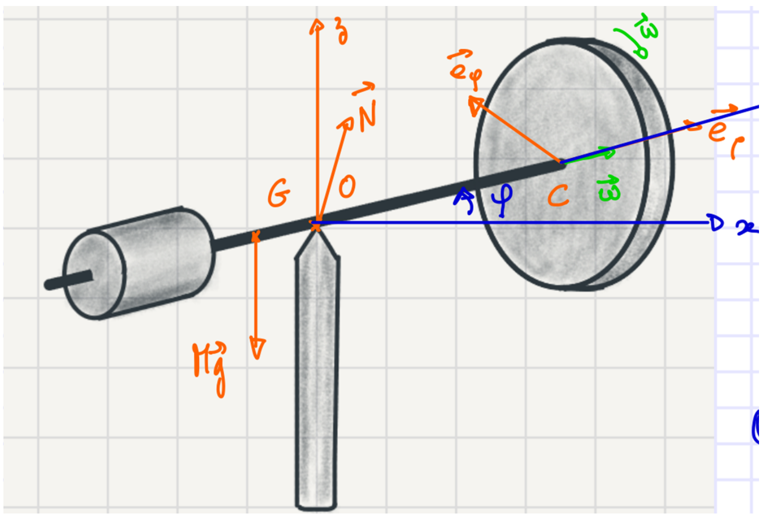
2 Gyroscopic Motion

Let a solid disk rotate around an axis (Cz) passing through its center C.



In this case, $\vec{L}_C = I_{Cz}\vec{\omega}$ and $\sum \vec{M}_C = \frac{d\vec{L}_C}{dt}$. The angular momentum can be changed if a torque is applied.

If we want to rotate the object around a vertical axis, we see that $\vec{L}_C$ moves while remaining in a horizontal plane. Therefore, its derivative is also in this plane. The *moment* of the force is thus also in this horizontal plane, and therefore the force itself is along a vertical axis, since it is perpendicular to the moment! The larger $\vec{L}_C$ is (i.e., the heavier or faster the disk rotates), the greater the torque needed for a given deviation angle α.
Now consider a gyroscope, composed of a spinning disk connected to a counterweight and held by a support:



This system is subjected to the normal force of the support (applied at point O) and the weight (applied at the center of mass G). The disk rotates with angular velocity $\vec{\omega} = \omega\vec{e}_\rho$. We use the angular momentum theorem here at point O because it is a fixed point, in order to eliminate the reaction force from the equations:

$$\sum \vec{M}_O = \overrightarrow{OO} \wedge \vec{N} + \overrightarrow{OG} \wedge M\vec{g} = \frac{d\vec{L}_O}{dt} \tag{11.22}$$

The total angular momentum consists of that due to the rotation of the disk, as well as that due to the rotation of the entire system on the horizontal plane. However, the contribution of the latter is negligible compared to that of the disk, so we consider $\vec{L}_O = \vec{L}_O^{\text{disk}} = I_O^{\text{disk}}\vec{\omega}$. Thus, assuming the disk does not slow down its rotation ($\omega = cste$) and denoting $d_G = OG$, developing each side of the equation and isolating the **angular velocity of precession** (angular velocity of the gyroscope around its support) $\Omega = \dot{\varphi}$, we have:

$$(-OG\vec{e}_\rho) \wedge (-Mg\vec{e}_z) = Mgd_G(-\vec{e}_\varphi) \tag{11.23}$$

$$\frac{d}{dt}(I_O\omega\vec{e}_\rho) = I_O\omega\frac{d}{dt}(\vec{e}_\rho) = I_O\omega\dot{\varphi}\vec{e}_\varphi \tag{11.24}$$

$$\Omega = \dot{\varphi} = \frac{Mgd_G}{I_O\omega} \tag{11.25}$$

We notice that this decreases when ω increases, because if the disk spins faster, its angular momentum increases and it will be more difficult to change it by the same angle φ.
It should be noted, however, that here we neglect the angular momentum due to the rotation of the gyroscope itself, which would lead to a motion not strictly circular in the horizontal plane, but would add vertical oscillations called "nutation".