Chapter 10

Rigid body mechanics

In the previous chapters we have only considered objects to be point masses. Now we will consider solid (having a certain extent) and rigid (no internal degrees of freedom; undeformable) objects. A rigid solid may undergo translational motion but also rotational motion!

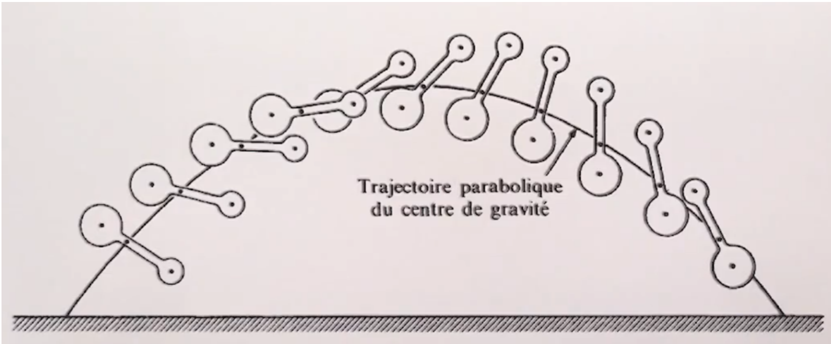


Figure 10.1: The center of mass follows a parabolic trajectory, and the solid rotates around the center of mass.

1 Center of Mass and Newton's Laws

1.1 Formation of the Laws for solids

We previously defined the center of mass (COM) for a collection of point masses and the associated Newton's laws:

$$\overrightarrow{OG} = \frac{\sum_\alpha m_\alpha \vec{r}_\alpha}{\sum_\alpha m_\alpha} = \frac{\sum_\alpha m_\alpha \vec{r}_\alpha}{M} \tag{10.1}$$

$$\sum \vec{F}^{\text{ext}} = M\vec{a}_G \tag{10.2}$$

$$\vec{P}_{\text{tot}} = M\vec{v}_G \tag{10.3}$$

We can extend these to a rigid body by moving from a discrete description to a continuous one:

$$\overrightarrow{OG} = \frac{\int_{vol} \vec{r} dm}{M} = \frac{\int_{vol} \vec{r}\rho(\vec{r})dV}{M} = \frac{\int_x \int_y \int_z \vec{r}(x;y;z)\rho(x;y;z)dV}{M} \tag{10.4}$$

And similarly for angular momentum and its theorem, which for a system of point masses is written:

$$\vec{L}_O^{\text{tot}} = \sum_\alpha \vec{L}_O^\alpha = \sum_\alpha \vec{r}_\alpha \wedge m_\alpha \vec{v}_\alpha \tag{10.5}$$

$$\sum \vec{M}_O^{\text{ext}} = \frac{d\vec{L}_O^{\text{tot}}}{dt} \tag{10.6}$$

And can be extended to the solid by writing:

$$\vec{L}_O^{\text{solid}} = \int_{vol} d\vec{L}_O = \int_{vol} \vec{r} \wedge dm\vec{v}(\vec{r}) \tag{10.7}$$

$$\sum \vec{M}_O^{\text{ext}} = \frac{d\vec{L}_O^{\text{solid}}}{dt} \tag{10.8}$$

Important: The gravitational force will always act at the center of mass G.

It is possible to determine the center of mass of an object without performing the triple integration as long as it possesses one or more axes of symmetry:

1. If the solid has **one** axis of symmetry, the center of mass lies on that axis.
2. If the object has **several** axes of symmetry, the center of mass will be at the intersection of those axes.

1.2 Specific solids

For two-dimensional objects (disks, plates), one can write $M = \rho_S * S$ with ρ_S the surface mass density (in kg/m^2).

For one-dimensional objects (a bar), one can write $M = \rho_l * l$ with ρ_l the linear mass density (in kg/m).

It is also possible, due to the geometry of certain objects, for the center of mass to lie outside of the object itself, for example for a ring (or a hollow cylinder). Its center of mass is located at the geometric center of the object (where there is no material), at the intersection of its axes of symmetry.

1.3 Composite objects

If two solids are superimposed or otherwise combined, one finds the center of mass of the "total" object using the centers of mass and masses of the two individual objects, treating them as point masses. Thus:

$$\overrightarrow{OG} = \frac{m_1\overrightarrow{OG}_1 + m_2\overrightarrow{OG}_2}{m_1 + m_2} \tag{10.9}$$

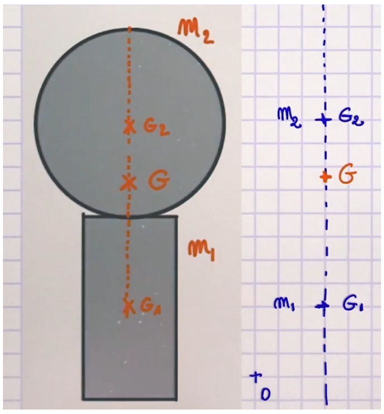


Figure 10.2: Centre of Mass of a Superposition of two solids

1.4 Solid with a Hole

To calculate the COM of a solid with a hole, we can proceed by taking two superimposed objects:

- The plug of mass m_1 with COM denoted $\overrightarrow{OG}_1$, which would fill the hole
- The perforated object of interest, of mass M and with COM denoted $\overrightarrow{OG}$, which we want to determine

By then using the formula for the COM of the total superposed object (the non-perforated base object, called 2, with mass m_2 and COM denoted $\overrightarrow{OG}_2$), we have:

$$\overrightarrow{OG}_2 = \frac{M\overrightarrow{OG} + m_1\overrightarrow{OG}_1}{M + m_1} = \frac{M\overrightarrow{OG} + m_1\overrightarrow{OG}_1}{m_2} \tag{10.10}$$

We can then deduce the desired COM from this expression by isolating $\overrightarrow{OG}$:

$$\overrightarrow{OG} = \frac{m_2\overrightarrow{OG}_2 - m_1\overrightarrow{OG}_1}{M} \tag{10.11}$$

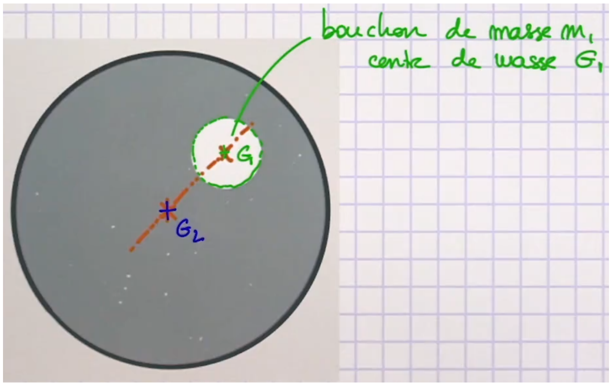


Figure 10.3: Center of Mass of an Object with a Hole

We note that finding the center of mass of an object with a hole amounts to superposing the non-perforated object and a hole of “negative mass.”

2 Statics

In statics we analyze a stationary solid, and we seek the conditions under which it remains stationary. Two conditions must then be satisfied (no translation and no rotation of the object):

$$\vec{F}^{\text{ext}} = \vec{0} \tag{10.12}$$

$$\vec{M}_0^{\text{ext}} = \vec{0} \tag{10.13}$$

With O a fixed point of the reference frame.

Example: Beam (non-homogeneous) on 2 supports

Let a non-homogeneous beam (COM not centered) rest on 2 supports A and B. We know the mass, the position of the COM, and we define the distances to the supports $AG = x_1$ and $GB = x_2$. The objective is to determine the forces at the supports F_A and F_B.

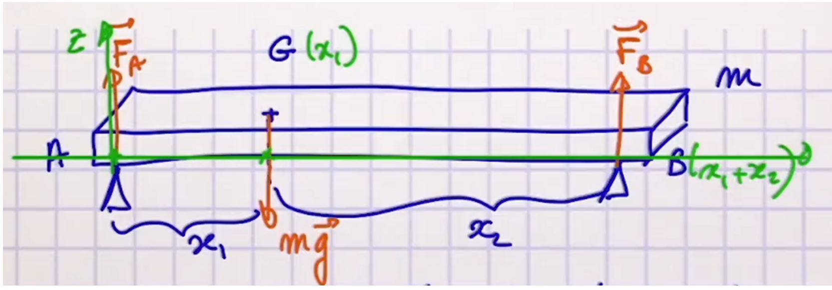


Figure 10.4: Non-homogeneous beam on 2 supports

By applying the two conditions cited above, and taking point A for the sum of moments (so as to use the known distances and have only one unknown), the system is:

$$\sum F^{\text{ext}} = \vec{F}_A + \vec{F}_B + m\vec{g} = (F_A + F_B - mg)\vec{e}_z = \vec{0} \tag{10.14}$$

$$\sum \vec{M}_A = \overrightarrow{AG} \wedge m\vec{g} + \overrightarrow{AB} \wedge \overrightarrow{F_B} = (mgx_1 - (x_1 + x_2)F_B)\vec{e}_y = \vec{0} \tag{10.15}$$

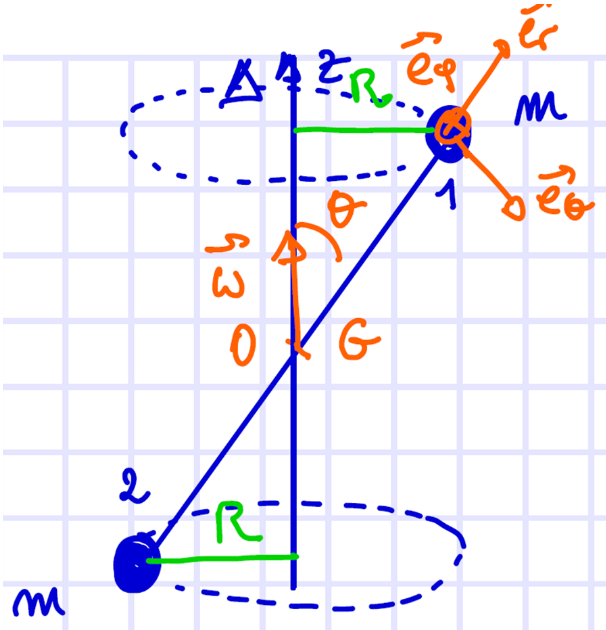
By projecting the sum of forces on the axis z (vertical) and the sum of moments on the axis y, one can isolate the supporting forces:

$$F_A = mg\frac{x_2}{x_1 + x_2} \tag{10.16}$$

$$F_B = mg\frac{x_1}{x_1 + x_2} \tag{10.17}$$

3 (Kinetic) Energy of Rotation

Let us first consider the following system: A rigid rod connects two equal masses each with mass m (its center of mass is therefore at the midpoint of the rod), and it rotates around an axis of rotation (Gz) at constant speed. The masses then each describe uniform circular trajectories.



Using spherical coordinates, we define:

$$\vec{\omega} = \omega \vec{e}_z \quad (10.18)$$

$$\overrightarrow{OP}_1 = r\vec{e}_r \quad (10.19)$$

$$r = l/2 \quad (10.20)$$

$$R = r\sin\theta \quad (10.21)$$

We also deduce the expressions for velocity and thus for kinetic energy (which are equal) for the respective masses, as well as for the entire system:

$$\text{Velocity of a mass: } \vec{v} = R\omega\vec{e}_\varphi \quad (10.22)$$

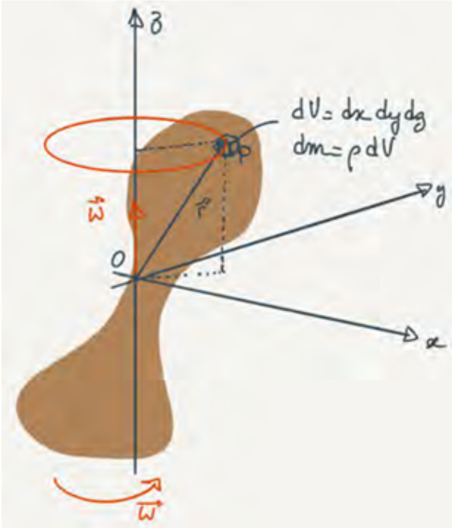
$$\text{Kinetic energy of a mass } E_{c,m} = \frac{1}{2}mv^2 = \frac{1}{2}mR^2\omega^2 \quad (10.23)$$

$$\text{Kinetic energy of the total system:} E_{c,rot} = 2 * \frac{1}{2}mv^2 = \frac{1}{2}(2m)R^2\omega^2 \quad (10.24)$$

We note that the total kinetic energy can be divided into two parts:

1. an expression that depends simply on the geometry of the object: $\frac{1}{2}(2m)R^2$
2. an expression that depends on the rotational speed: ω^2

Let us now generalize this to an arbitrary solid:



An arbitrary point P in the solid behaves like the point mass in the previous case. It is occupying an infinitesimal volume dV with mass dm. This element dm has a rotational energy from which, using the previous case, we deduce the rotational energy of the solid:

$$\text{Mass centered at P:} dm = \rho * dV \tag{10.25}$$

$$\text{Energy: } E_{c,dm} = \frac{1}{2} dm (R\omega)^2 = \frac{1}{2} \rho(\vec{r}) R^2 \omega^2 dV \tag{10.26}$$

$$\text{Energy of the system: } E_{c,rot} = \int_{vol} \frac{1}{2} \rho(\vec{r}) R^2 \omega^2 dV = \frac{1}{2} \omega^2 \int_{vol} \rho(\vec{r}) R^2 dV \tag{10.27}$$

We introduce the notion of **Moment of Inertia** of the solid with respect to the Oz axis as:

$$I_z = \int_{vol} \rho(\vec{r}) R^2 dV \tag{10.28}$$

This allows us to rewrite **the (kinetic) rotational energy** as:

$$E_{c,rot} = \frac{1}{2} I_z \omega^2 \tag{10.29}$$

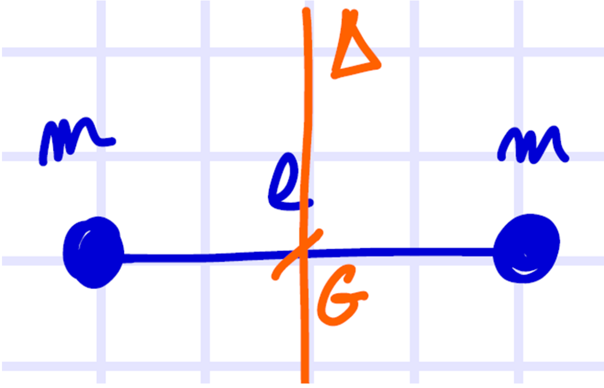
We notice the structure of this expression, which is reminiscent of translational kinetic energy: I plays the role of mass, while ω plays the role of velocity.

4 Moment of Inertia of a Solid with Respect to an Axis

The moment of inertia depends on both the geometry of the solid and the specific axis of rotation.

4.1 Simple rod with two masses

Consider a massless rod, of length l, with 2 masses m at each end. By calculating its moment of inertia with respect to a rotation axis perpendicular to the rod, at the center of mass, we obtain:

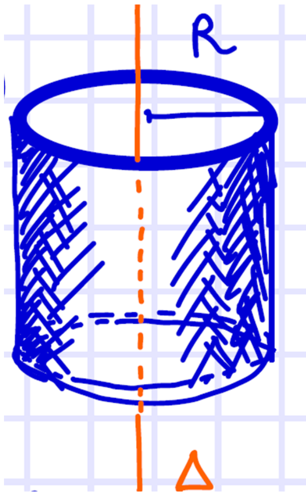


$$I_\Delta = m\left(\frac{l}{2}\right)^2 + m\left(\frac{l}{2}\right)^2 = \frac{1}{2}ml^2 \tag{10.30}$$

4.2 Thin hollow cylinder

The hollow cylinder has a radius R and a total mass M. Its moment of inertia with respect to its axis of symmetry is:

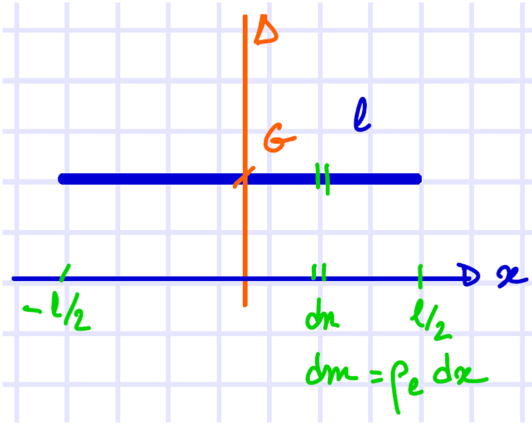
$$I_\Delta = MR^2 \tag{10.31}$$



4.3 Rod: Linear Mass Density

Let us now study a thin rod of mass M distributed along its length L. Its linear mass density is defined as $\rho_L = M/L$. For a moment of inertia around rotation axis Δ perpendicular to the rod at its center of mass, we define an x-axis with the origin at the center of mass G, and then compute:

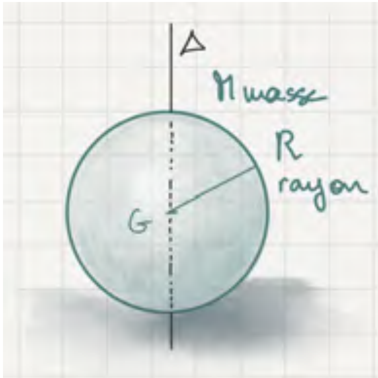
$$I_\Delta = \int_{-l/2}^{l/2} \rho_L * x^2 dx = \frac{1}{12}ML^2 \tag{10.32}$$


Δ
ℓ
G
x
-ℓ/2
dx
ℓ/2
dm = ρℓ dx

4.4 Common Homogeneous Solids

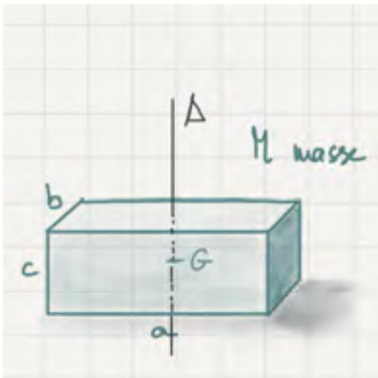
Sphere

$$I_\Delta = \frac{2}{5} M R_S^2 \tag{10.33}$$



Rectangular Parallelepiped

$$I_\Delta = \frac{1}{12} M (a^2 + b^2) \tag{10.34}$$



Remark: For a thin rectangular plate, the expression above can be reused because for this axis, the moment of inertia does not depend on c.

Cylinder

Depending on the chosen axis:

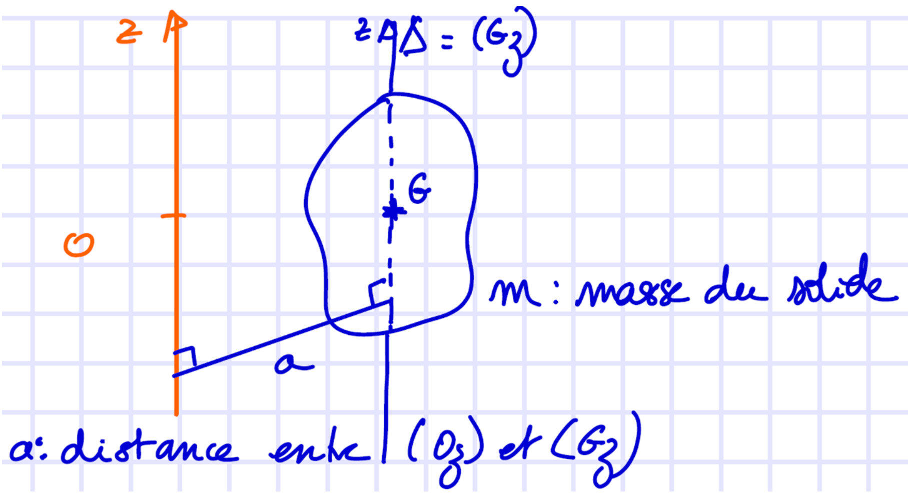
$$I_\Delta = \frac{1}{2} M R^2 \tag{10.35}$$

$$I_{\Delta'} = \frac{1}{4} M (R^2 + \frac{h^2}{3}) \tag{10.36}$$

4.5 Steiner's Theorem

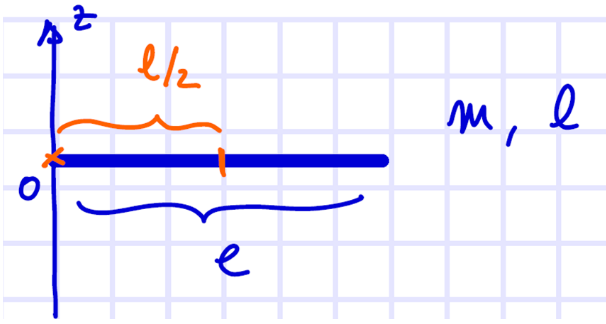
Let two parallel axes (Oz) and (Gz) be separated by a distance a, with G bring the center of mass of the object. If I_{Gz} is the moment of inertia with respect to (Gz), then I_{Oz}, the moment of inertia with respect to (Oz), is:

$$I_{Oz} = I_{Gz} + ma^2 \tag{10.37}$$



Example: Homogeneous Rod Rotating Around One End

The moment of inertia of a homogeneous rod (uniformly distributed linear mass) around an axis perpendicular to its center of mass is known. By applying Steiner's theorem, we can calculate the moment of rotation around one end.



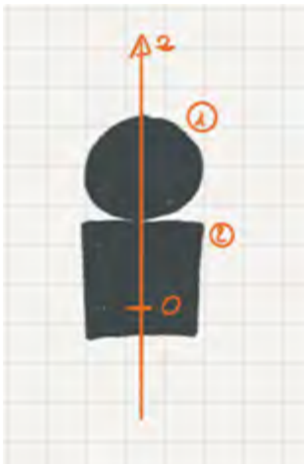
$$I_{Oz} = \frac{1}{12}ml^2 + m\left(\frac{l}{2}\right)^2 \tag{10.38}$$

$$I_{Oz} = ml^2\left(\frac{1}{12} + \frac{1}{4}\right) = \frac{1}{3}ml^2 \tag{10.39}$$

4.6 Composite Solids and Solids with Holes

The moment of inertia is calculated by integration. Since the integral is additive, it is possible to compute the moment of inertia of composite solids by adding the moments of inertia of the solids that compose them:

$$I_{\text{Oz}}^{\text{tot}} = I_{\text{Oz}}^{(1)} + I_{\text{Oz}}^{(2)} \tag{10.40}$$



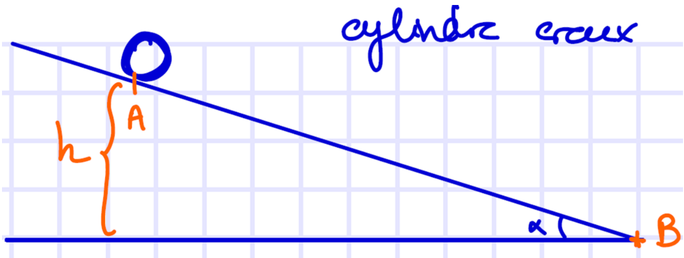
Similarly, the moment of inertia of a solid with a hole is obtained by subtracting from the non-perforated solid the moment of inertia of the plug corresponding to the hole:

$$I_{\text{Oz}}^{\text{bouché}} = I_{\text{Oz}}^{rel} + I_{\text{Oz}}^{bouchon} \tag{10.41}$$

$$I_{\text{Oz}}^{rel} = I_{\text{Oz}}^{\text{bouché}} - I_{\text{Oz}}^{bouchon} \tag{10.42}$$

5 Application

A hollow cylinder of mass M and radius R rolls without slipping down an inclined plane. It is released with no initial velocity at point A at height h, and we seek its velocity at point B.



It is subject to the weight due to gravity, the normal force, and the friction force at the surface. However, only gravity does work because the velocity at the point of contact is zero (rolling without slipping), and the normal force acts perpendicular to the trajectory: $W = W_P + W_R + W_f = W_P$. The mechanical energy of the center of mass G at a given point is written:

$$E_m(G) = E_p + E_{\text{c,rot}} + E_{\text{c,t}} = E_p + \frac{1}{2}I_G\omega^2 + \frac{1}{2}Mv_G^2 \tag{10.43}$$

$$I_G = MR^2 \tag{10.44}$$

$$v_G = R\omega \Leftrightarrow \omega = \frac{v_G}{R} \tag{10.45}$$

Let us take as reference for the potential energy the base of the inclined plane, so that $E_p(A) = Mgh$ and $E_p(B) = 0$. By conservation of mechanical energy between A and B (because the weight is a conservative force):

$$E_m(A) = E_m(B) \Leftrightarrow Mgh = \frac{1}{2}I_G\omega_B^2 + \frac{1}{2}Mv_B^2 = \frac{1}{2}MR^2 * \frac{v_B^2}{R^2} + \frac{1}{2}Mv_B^2 \tag{10.46}$$

$$v_B = \sqrt{gh} \tag{10.47}$$

The expression resembles that obtained for an object that slides without rolling: $v_B = \sqrt{2gh}$. The velocity here is smaller because for the same initial potential energy, part of it is used to make the object rotate and give it rotational energy.

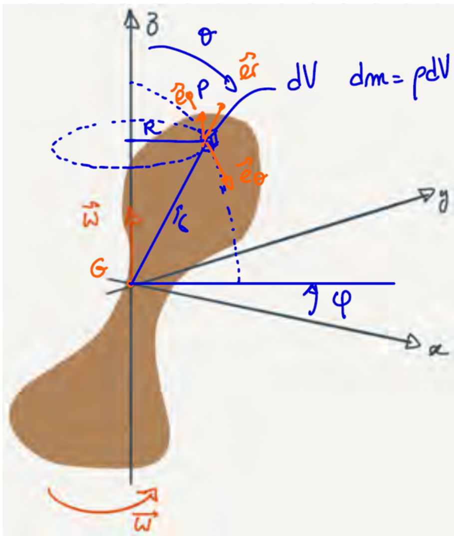
6 Angular Momentum of a Solid

Let us now study the angular momentum of a general solid and no longer of a simple particle. The quantity was defined by equation (9.1), and the angular momentum theorem by (9.3).

General Case: arbitrary solid in rotation

Let a solid be rotating around an axis passing through G. We choose the axis (Gz) so that it is the axis of rotation. The goal will be to compute the angular momentum of this solid associated with the rotation.

Consider an infinitesimal element of the solid centered at point P, with volume dV and mass $dm = \rho dV$. Let us study the problem in spherical coordinates.



The infinitesimal angular momentum in the neighborhood of the chosen point P is written:

$$d\vec{L}_G = \vec{r} \wedge d\vec{p} = \vec{r} \wedge dm * \vec{v}(P) \tag{10.48}$$

$$\vec{v}(P) = R\omega\vec{e}_\varphi = r \sin\theta \; \omega\vec{e}_\varphi \tag{10.49}$$

$$\rightarrow d\vec{L}_G = r\vec{e}_r \wedge dm * r \sin\theta \; \omega\vec{e}_\varphi = r^2 dm \sin\theta\omega(-\vec{e}_\theta) \tag{10.50}$$

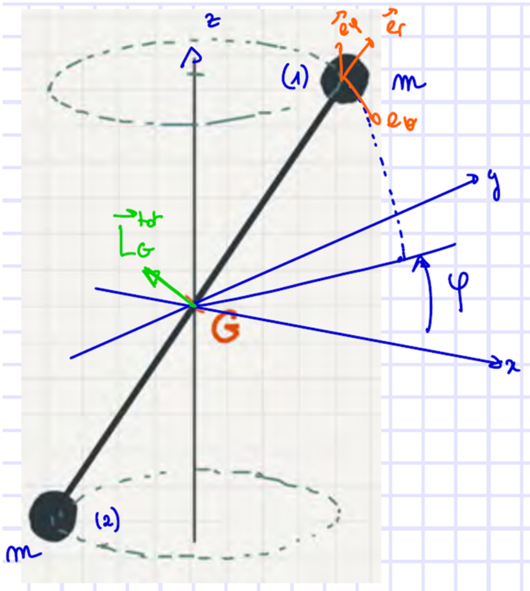
For the entire solid, we then write:

$$\vec{L}_G = \int_{\text{vol}} r^2 dm \sin\theta \; \omega(-\vec{e}_\theta) \tag{10.51}$$

For each point P of the solid, the resulting infinitesimal angular momentum will each have a different direction. Therefore this computation is an integration of many vector elements with different directions, and it is far too complex to intuitively obtain $\vec{L}_G$ in the general case.

Simple Case: dumbbell in rotation

Let us simplify the problem by studying a dumbbell (massless rod with small mass attached at each end).



The total angular momentum will be the sum of the angular momentum of each mass. We can write:

$$\vec{L}_G^{(1)} = \vec{r} \wedge m\vec{v} = r\vec{e}_r \wedge m(r\omega \sin\theta \vec{e}_\varphi) = -mr^2\omega \sin\theta \vec{e}_\theta \quad (10.52)$$

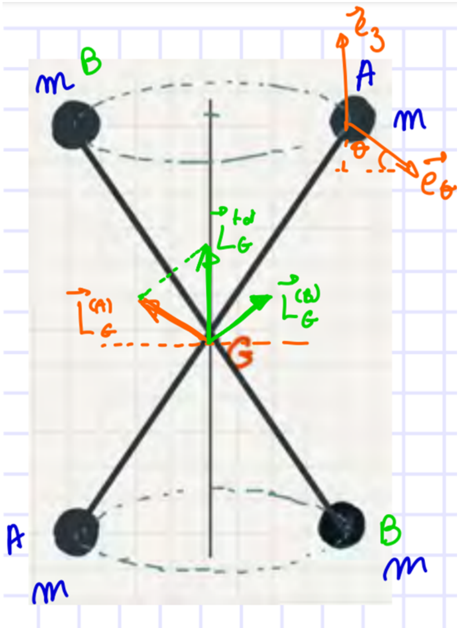
$$\vec{L}_G^{(2)} = -\vec{r} \wedge m(-\vec{v}) = \vec{L}_G^{(1)} \quad (10.53)$$

$$\vec{L}_G^{\text{sol}} = \vec{L}_G^{(1)} + \vec{L}_G^{(2)} = -(2m)r^2\omega \sin\theta \vec{e}_\theta = -Mr^2\omega \sin\theta \vec{e}_\theta \quad (10.54)$$

The obtained angular momentum is perpendicular to the axis of the rod, passing through the center of mass G. While rotating, the solid traces a cone, and **the direction of the angular-momentum vector also changes, describing a cone** with a different angle relative to the horizontal (angle θ).
If the angular momentum changes, its derivative is nonzero, and therefore it is necessary to apply a torque to maintain the rotation (consistent with the angular momentum theorem seen previously).
In general the angular momentum is not parallel to the axis of rotation of the solid, unless some symmetry exists in the mass distribution. Maintaining such an axis of rotation requires applying a torque.

Examples of Symmetric Cases

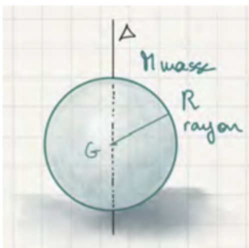
Let us now double dumbbell. We have the first dumbbell with two masses (A), and have added a second identical dumbbell (B), symmetric with respect to the axis of rotation.

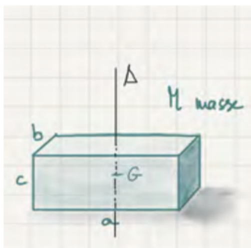


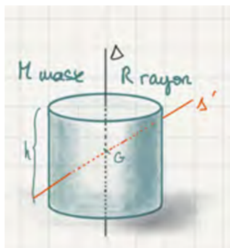
The total angular momentum of the solid will be $\vec{L}_G^{\text{tot}} = \vec{L}_G^A + \vec{L}_G^B$, collinear with $\vec{e}_z$ due to the symmetry of the problem. Using the expression for solid (A) projected onto the z-axis by a dot product and $M = 4m$ the total mass, we then have:

$$\vec{L}_G^{\text{tot}} = |L_G^{\text{tot}}|\vec{e}_z = (2\vec{L}_G^A \cdot \vec{e}_z)\vec{e}_z = 4m(r^2 \sin^2\theta)\omega\vec{e}_z = MR^2\omega\vec{e}_z = I_G\vec{\omega} \tag{10.55}$$

Here the angular momentum is parallel to the vector $\vec{\omega}$ because the problem is symmetric! It is possible to generalize this expression to standard solids:





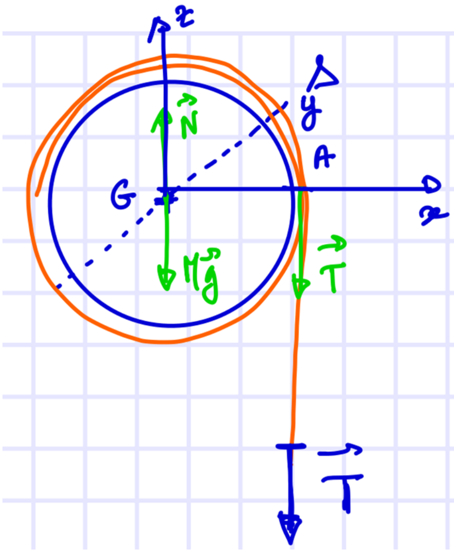


In all three cases, the solid can be cut along two planes that allow us, by gathering the four quadrants into four infinitesimal masses, to recover the previous situation (double symmetric dumbbell).

There therefore exist symmetry axes such that for a rotation around such an axis, $\vec{L}_G = I_{Gz}\vec{\omega}$: these are called **principal axes of inertia. Every solid has at least 3 principal axes of inertia**.

Example: Pulley and Rope

Let us consider a pulley modeled as a homogeneous solid disk fixed to the axis Δ passing through G, of mass M and radius R. A massless rope winds around it. During a time t_0, a tension $\vec{T}$ is applied to the rope, and the system is initially at rest: we want to determine the length of rope unwound after time t_0.



Taking into account all the forces present (weight of the pulley, tension of the rope, reaction at the axis) and paying attention to their respective points of application (at G or A respectively). We then apply the angular momentum theorem at the center of mass G:

$$\sum \vec{M}_G^{\text{ext}} = RT\vec{e}_y = \frac{d\vec{L}_G}{dt} \tag{10.56}$$

Since the axis Δ is a principal axis of inertia, the angular momentum can be written as $\vec{L}_G = I_{Gy}\omega\vec{e}_y = \frac{1}{2}MR^2\omega\vec{e}_y$. In this expression, only ω depends on time, the other terms are constant. Returning to the angular momentum theorem and projecting onto the y-axis, we can then isolate and integrate twice:

$$\frac{1}{2}MR^2\dot{\omega} = RT \tag{10.57}$$

$$\dot{\omega} = \frac{2T}{MR} = cste \tag{10.58}$$

$$\omega = \frac{2T}{MR}t + \omega(t=0) = \frac{2T}{MR}t \tag{10.59}$$

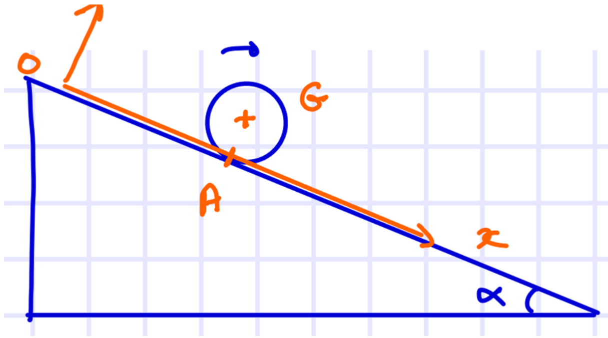
$$\theta = \frac{T}{MR}t^2 \tag{10.60}$$

The length of rope unwound is then written:

$$L = \theta_f * R = \frac{T}{M}t_0^2 \tag{10.61}$$

7 Rolling Solid

Let us now study a cylinder rolling without slipping along an inclined plane with angle α.



We have seen so far that it is possible to apply the angular momentum theorem to solve the problem at a fixed point. The 3 points of interest here are O, the origin of the coordinate system, A, the point of contact between the solid and the plane, and G, the center of mass of the cylinder. O is fixed; A and G are not!

Furthermore, we must be able to compute the angular momentum at the chosen point of application. This is possible for the center of mass G or for a point of the solid with zero velocity:

- for G: $\vec{L}_G = I_{Gz}\vec{\omega}$ if Gz is a principal axis of inertia
- for A: $\vec{L}_A = I_{Az}\vec{\omega}$ if A is a point of the solid with zero velocity and Az is a principal axis of inertia
- for O: neither a point of the solid with zero velocity nor the center of mass. We cannot write $\vec{L}_O = I_{Oz}\vec{\omega}$

It is therefore a problem to apply the angular momentum theorem in this example: **the fixed-point constraint in the reference frame is too restrictive.**

Let us study its application at an arbitrary non-fixed point A in the reference frame $\mathcal{R}$. Returning to the integral expression of angular momentum, and using the known relations ($\vec{p}^{\text{tot}} = M\vec{v}_{\mathcal{R}}(G)$; $\sum \vec{M}_O = \sum \overrightarrow{OP}\text{app} \wedge \vec{F}^{\text{ext}}$; $\vec{L}_O = \int \text{vol}\overrightarrow{OP} \wedge dm, \vec{v}_{\mathcal{R}}(P)$) :

$$\vec{L}_A = \int_{\text{vol}} \overrightarrow{AP} \wedge dm\vec{v}_{\mathcal{R}}(P) = \int_{\text{vol}} (\overrightarrow{AO} + \overrightarrow{OP}) \wedge dm\vec{v}_{\mathcal{R}}(P) \tag{10.62}$$

$$\vec{L}_A = \overrightarrow{AO} \wedge \int_{\text{vol}} dm\vec{v}_{\mathcal{R}}(P) + \int_{\text{vol}} \overrightarrow{OP} \wedge dm\vec{v}_{\mathcal{R}}(P) = -\overrightarrow{OA} \wedge \vec{p}^{\text{tot}} + \vec{L}_O \tag{10.63}$$

$$\frac{d\vec{L}_A}{dt} = -\vec{v}_{\mathcal{R}}(A) \wedge \vec{p}^{\text{tot}} - \overrightarrow{OA} \wedge \sum \vec{F}^{\text{ext}} + \sum \vec{M}_O \tag{10.64}$$

$$\frac{d\vec{L}_A}{dt} = -M\vec{v}_{\mathcal{R}}(A) \wedge \vec{v}_{\mathcal{R}}(G) + \sum (\overrightarrow{AO} + \overrightarrow{OP}_{\text{app}}) \wedge \vec{F}^{\text{ext}} \tag{10.65}$$

From this we deduce the **general formula for an arbitrary point A**:

$$\frac{d\vec{L}_A}{dt} = \sum \vec{M}_A - M\vec{v}_{\mathcal{R}}(A) \wedge \vec{v}_{\mathcal{R}}(G) \tag{10.66}$$

The second term will be zero in three cases:

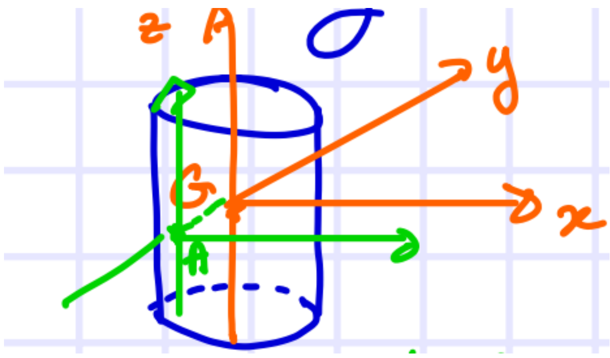
- if A is fixed in $\mathcal{R}$
- if A is the center of mass
- if $\vec{v}_{\mathcal{R}}(A)$ is colinear with $\vec{v}_{\mathcal{R}}(G)$

Returning now to the case of the cylinder and the inclined plane: the reference point A **in the frame $\mathcal{R}$ has a nonzero velocity and is colinear with that of G** (different from the zero velocity of the material point in contact with the plane), because at each instant t the point A changes position. Therefore, we can eliminate the second term and apply $\frac{d\vec{L}_A}{dt} = \sum \vec{M}_A$ directly. This is also possible, as discussed earlier, for the center of mass G, but not for point O (due to the problem of computing $\vec{L}_O$)!

Furthermore, knowing the principal axis of inertia is important in order to compute the angular momentum at the chosen point. Under what condition, then, are the axes (O, x, y, z) **principal axes of inertia** for a point O?

We assume that this is the case if (G, x, y, z) are principal axes of inertia, and if O lies on one of them. In this case, for a rotation around (Oz): $\vec{\omega} = \omega \vec{e}_z$ and $\vec{L}_O = I_{Oz}\vec{\omega}$.

In our example, the point A lies on the principal axis Oy, and therefore the axes (A, x, y, z) are also principal axes.



8 Inertia Tensor (outside the syllabus)

In the general case, when the rotation does not occur around a principal axis of inertia, the angular momentum is written as the product between a tensor (matrix) and a vector:

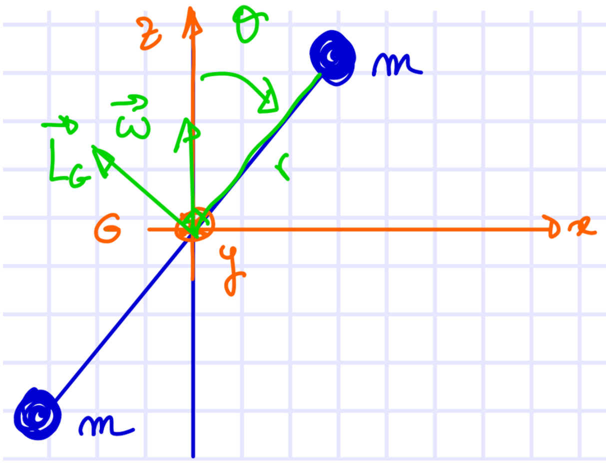
$$\vec{L}_G = \underline{I}_G \vec{\omega} \tag{10.67}$$

With $\underline{I}_G$ the inertia tensor, which depends on the chosen origin and axes. Its expression is:

$$\underline{I}_G = \begin{bmatrix} \int (y^2+z^2)dm & -\int xydm & -\int xzdm \\ -\int xydm & \int (x^2+z^2)dm & -\int yzdm \\ -\int xzdm & -\int yzdm & \int (x^2+y^2)dm \end{bmatrix} \tag{10.68}$$

Example

For the example of the simple non-symmetric dumbbell, the calculation allows us to obtain in (G, x, y, z):



$$\vec{\omega} = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} \tag{10.69}$$

$$\underline{I}_G = \begin{bmatrix} 2m(r\cos\theta)^2 & 0 & -2mr^2\sin\theta\cos\theta \\ 0 & 2mr^2 & 0 \\ -2mr^2\sin\theta\cos\theta & 0 & 2m(r\sin\theta)^2 \end{bmatrix} \tag{10.70}$$

$$\vec{L}_G = \underline{I}_G\vec{\omega} = \begin{pmatrix} -2m\omega r^2\sin\theta\cos\theta \\ 0 \\ 2m\omega r^2\sin^2\theta \end{pmatrix} \tag{10.71}$$