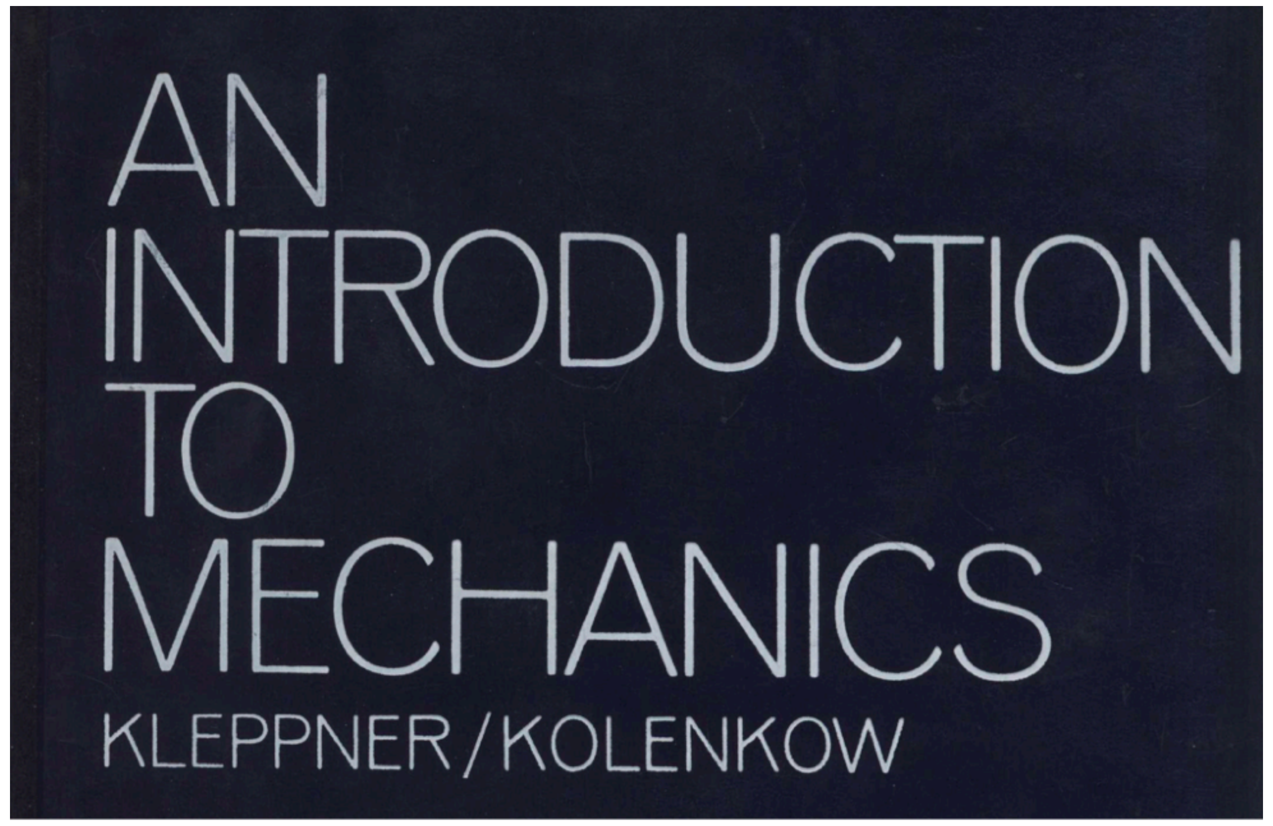
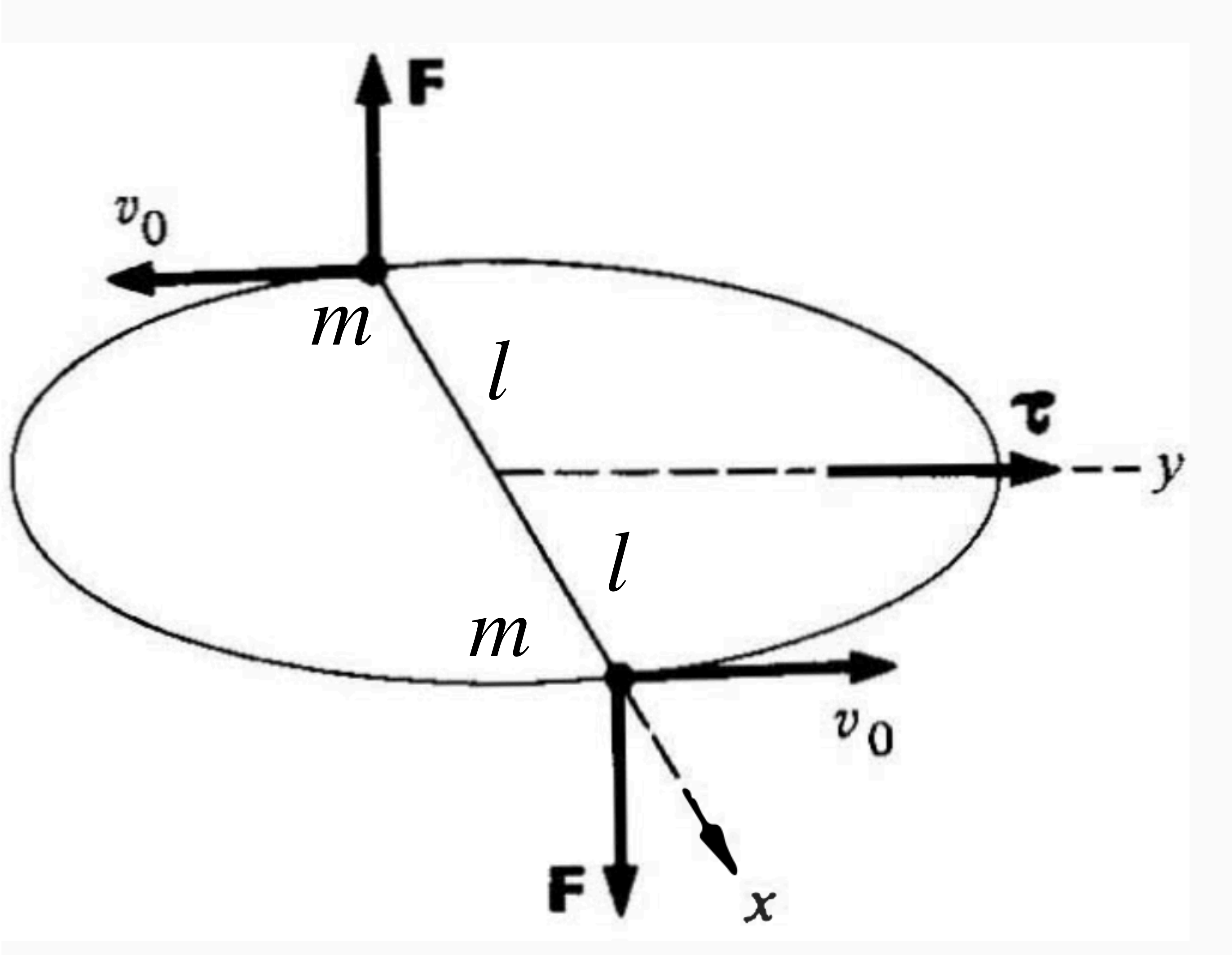
WHY DOES PRECESSION HAPPEN?

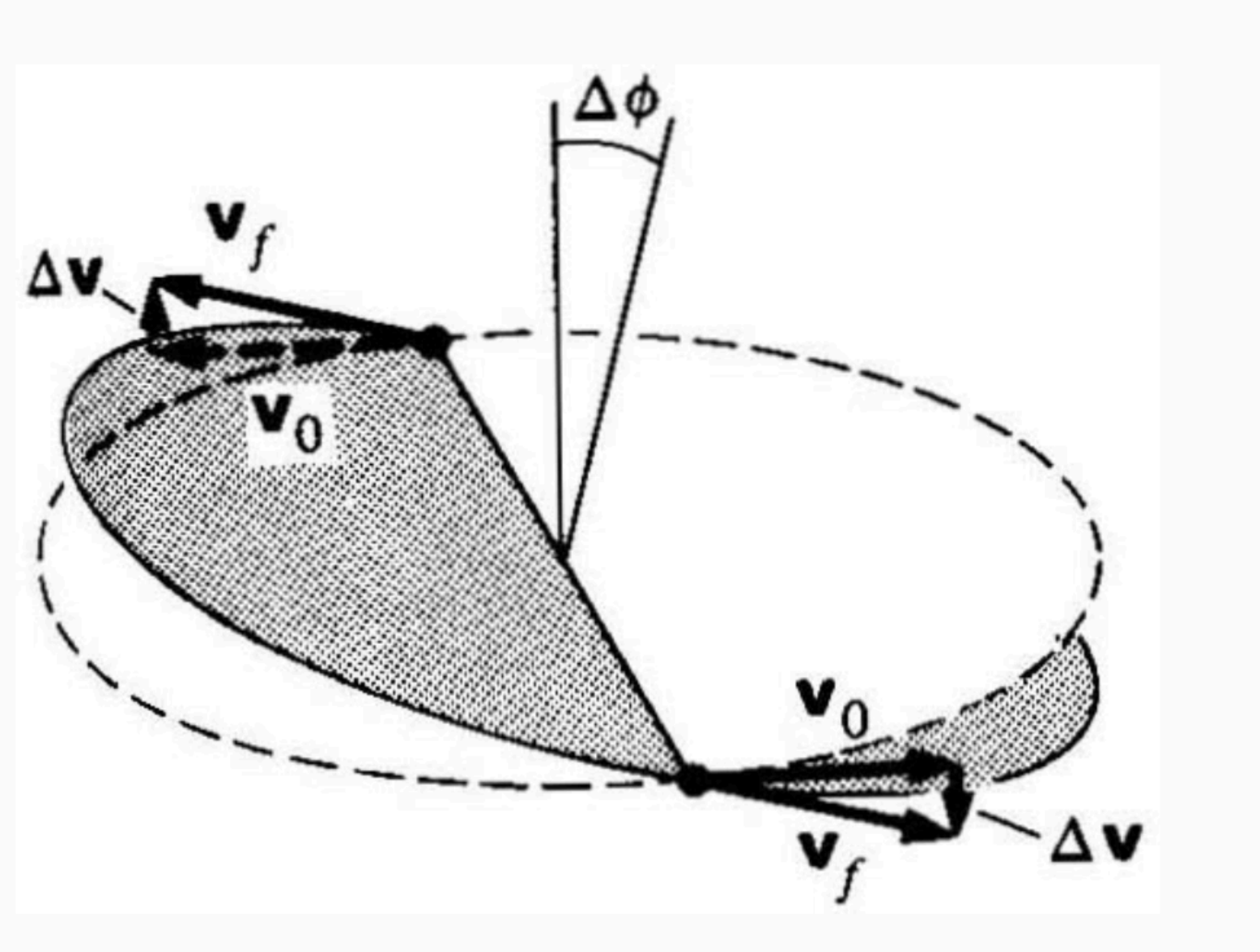

AN
INTRODUCTION
TO
MECHANICS
KLEPPNER/KOLENKOW



Consider a rigid body consisting of two particles of mass m at either end of a rigid massless rod of length $2l$. Suppose that the rod is rotating in free space with its angular momentum $\vec{L}_o = 2ml^2\omega\vec{e}_z$. The speed of each mass is $v_o = \omega l$

Suppose that a torque is applied only during a short time Δt while the rod is instantaneously oriented along the x-axis. We assume that the torque is due to two equal and opposite forces F, as shown. The total force is zero, and the center of mass remains at rest. The momentum of each mass changes by:

$$\Delta \mathbf{p} = m\,\Delta \mathbf{v} = \mathbf{F}\Delta t.$$



The velocity of each mass will change direction, as shown, and the rotation vector will tilt by:

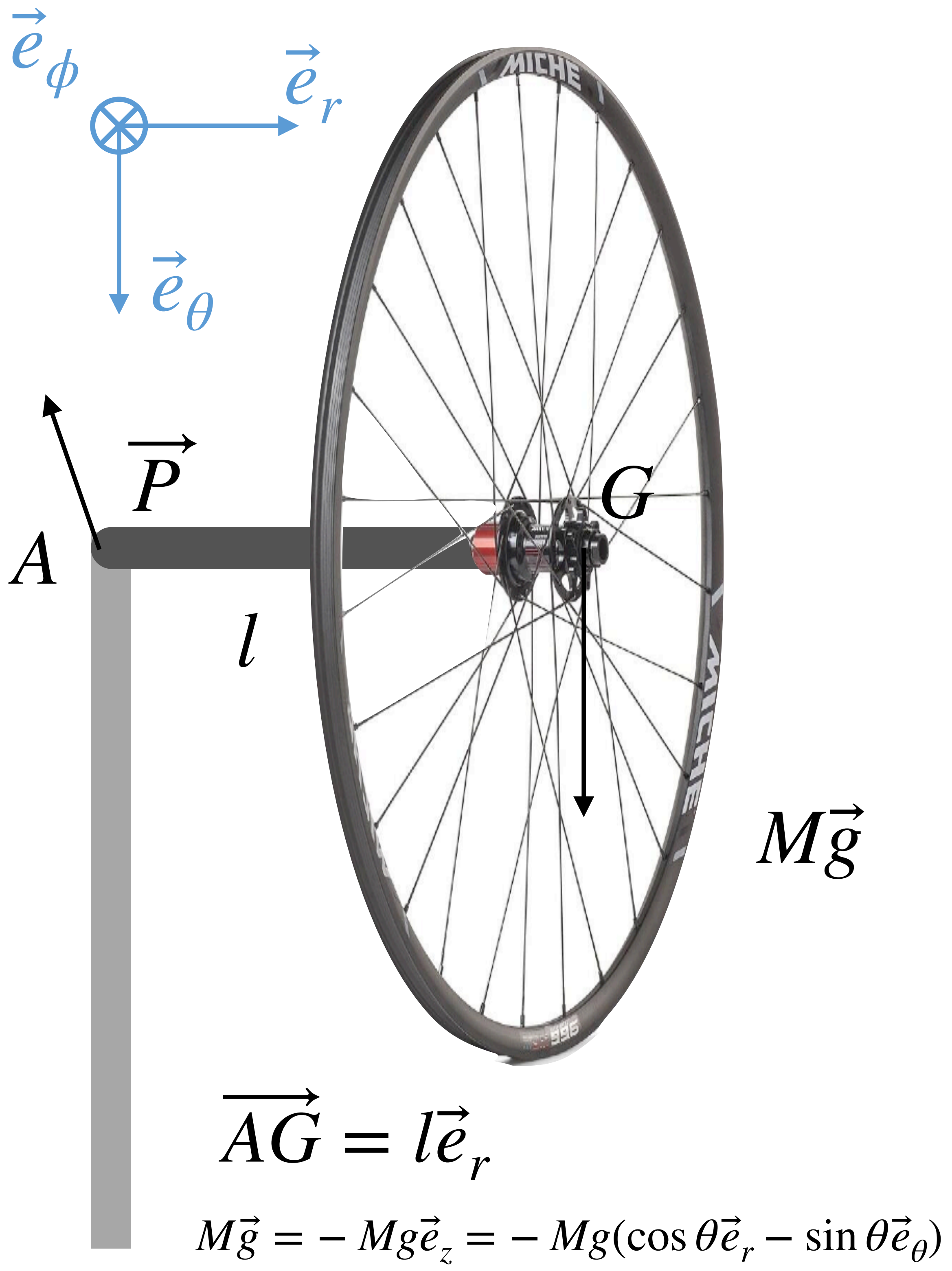
$$\Delta\phi \approx \frac{\Delta v}{v_0} = \frac{F\ \Delta t}{mv_0}.$$

The torque acting on the system is $\vec{M}_o = 2Fl\vec{e}_y$, and the angular momentum is $\vec{L}_o = 2ml^2\vec{w}$.

Therefore $\Delta\phi = M_o\Delta t/L_o$, and the rate of precession is $\Omega = \Delta\phi/\Delta t = M_o/L_o$

PRECESSION IN DETAIL

Using spherical coordinates
starting at $\theta = \pi/2$, $\phi = 0$

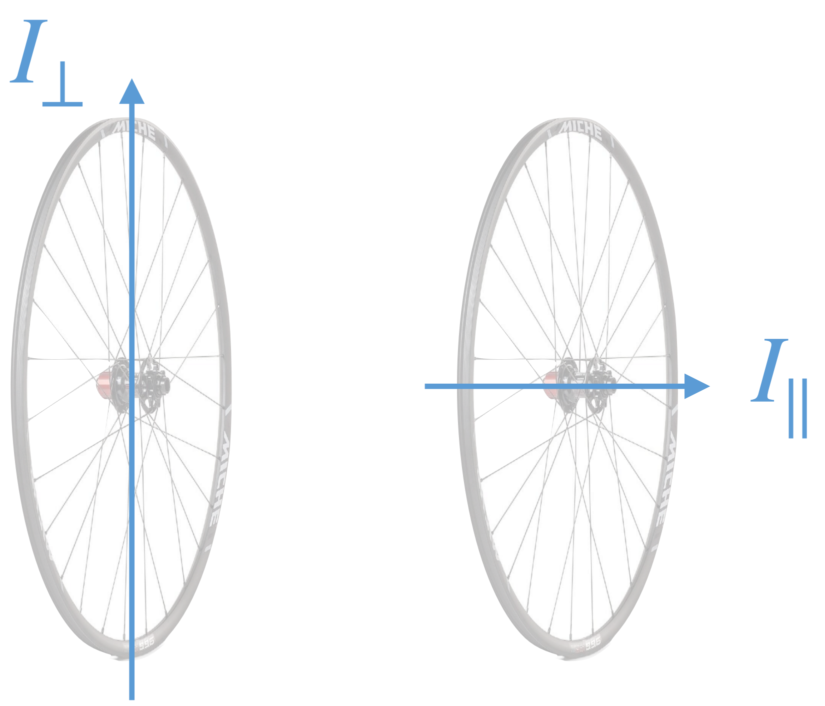


$\overrightarrow{AG} = l\vec{e}_r$

$M\vec{g} = -Mg\vec{e}_z = -Mg(\cos\theta\vec{e}_r - \sin\theta\vec{e}_\theta)$

Consider a bicycle wheel with mass M attached to a fixed pivot A via an axel with length l.

This wheel has the following moments of inertial with respect to the depicted axes passing through the CoM G:



The system has 3 possible modes of rotation:

- Rotating around its own axis $\vec{e}_r$ with some angular velocity ω_s
  - $\vec{L}_r = I_\parallel \omega_s \vec{e}_r$
- rotating around the pivot (rotating around $\vec{e}_z/\vec{e}_\theta$ if $\theta = \pi/2$) with angular velocity $\dot{\phi}$
  - $\vec{L}_\theta = (I_\perp + Ml^2)\dot{\phi}\vec{e}_z$
- or falling down /pointing down (rotating around $\vec{e}_\phi$) with angular velocity $\dot{\theta}$
  - $\vec{L}_\phi = (I_\perp + Ml^2)\dot{\theta}\vec{e}_\phi$

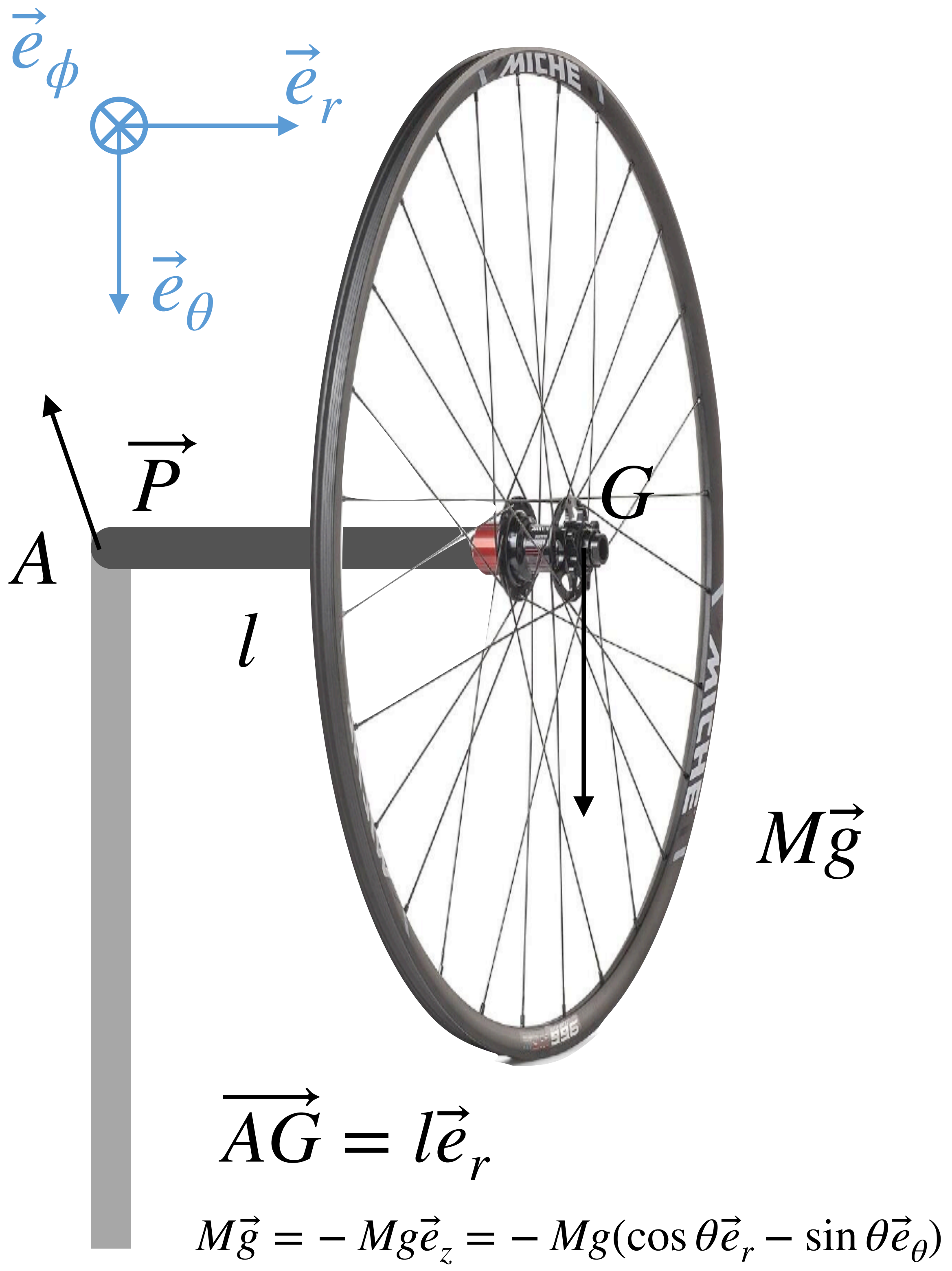
Gravity applies a torque at the center of mass:

$$\begin{aligned}\vec{M}_A = \overrightarrow{AG} \wedge M\vec{g} &= (l\vec{e}_r) \wedge (-Mg(\cos\theta\vec{e}_r - \sin\theta\vec{e}_\theta)) \\ &= -Mgl(-\sin\theta)\vec{e}_r \wedge e_\theta \\ &= Mgl\sin\theta\vec{e}_\phi\end{aligned}$$

Produces a torque pointing into the screen

PRECESSION IN DETAIL

Using spherical coordinates starting at $\theta = \pi/2,\ \phi = 0$



$\overrightarrow{AG} = l\vec{e}_r$

$M\vec{g} = -Mg\vec{e}_z = -Mg(\cos\theta\vec{e}_r - \sin\theta\vec{e}_\theta)$

Total angular momentum is:

$$\vec{L}_A = \vec{L}_r + \vec{L}_\theta + \vec{L}_\phi$$

$$= I_\parallel \omega_s \vec{e}_r + (I_\perp + Ml^2)\dot{\phi}\vec{e}_z + (I_\perp + Ml^2)\dot{\theta}\vec{e}_\phi$$

Time derivative is:

$$\dot{\vec{L}}_A = \dot{\vec{L}}_r + \dot{\vec{L}}_\theta + \dot{\vec{L}}_\phi$$

$$\dot{\vec{L}}_r = I_\parallel \dot{\omega}_s \vec{e}_r + I_\parallel \omega_s \dot{\vec{e}}_r$$

$$\dot{\vec{L}}_\theta = (I_\perp + Ml^2)\ddot{\phi}\vec{e}_z$$

$$\dot{\vec{L}}_\phi = (I_\perp + Ml^2)(\ddot{\theta}\vec{e}_\phi + \dot{\theta}\dot{\vec{e}}_\phi)$$

$$\dot{\vec{L}}_A = I_\parallel \dot{\omega}_s \vec{e}_r + I_\parallel \omega_s \dot{\vec{e}}_r + (I_\perp + Ml^2)(\ddot{\phi}\vec{e}_z + \ddot{\theta}\vec{e}_\phi + \dot{\theta}\dot{\vec{e}}_\phi)$$

PRECESSION IN DETAIL

Using spherical coordinates

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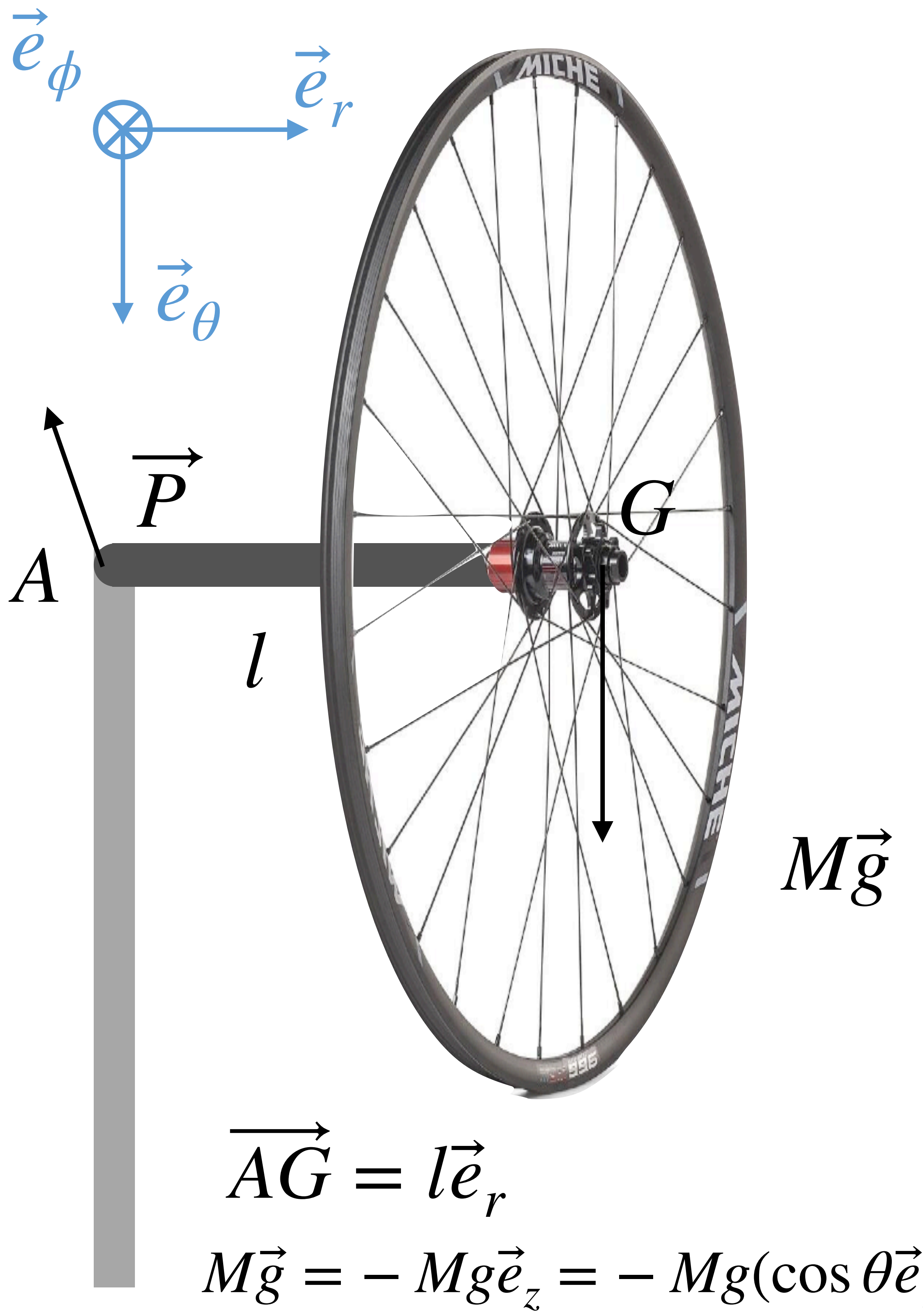


$\overrightarrow{AG} = l\vec{e}_r$

$M\vec{g} = -Mg\vec{e}_z = -Mg(\cos\theta\vec{e}_r - \sin\theta\vec{e}_\theta)$

Consider the case when we start with no spin $\omega_s = 0$.

At t=0, $\vec{L}_A = 0$

$$\dot{\vec{L}}_A = (I_\perp + Ml^2)(\ddot{\phi}\vec{e}_z + \ddot{\theta}\vec{e}_\phi + \dot{\theta}\dot{\vec{e}}_\phi)$$

$$\dot{\vec{L}}_A = \vec{M}_A = Mgl\sin\theta\vec{e}_\phi$$

Because we start with no initial rotation

On z: $(I_\perp + Ml^2)\ddot{\phi} = 0 \implies \dot{\phi} = 0, \phi(t) = 0$

On phi: $(I_\perp + Ml^2)(\ddot{\theta}\vec{e}_\phi + \dot{\theta}\dot{\vec{e}}_\phi) = Mgl\sin\theta\vec{e}_\phi$

$\dot{\vec{e}}_\varphi = -\dot{\varphi}\sin\theta\,\vec{e}_r - \dot{\varphi}\cos\theta\,\vec{e}_\theta$ =0 because $\dot{\phi} = 0$

$$(I_\perp + Ml^2)(\ddot{\theta}) = Mgl\sin\theta$$

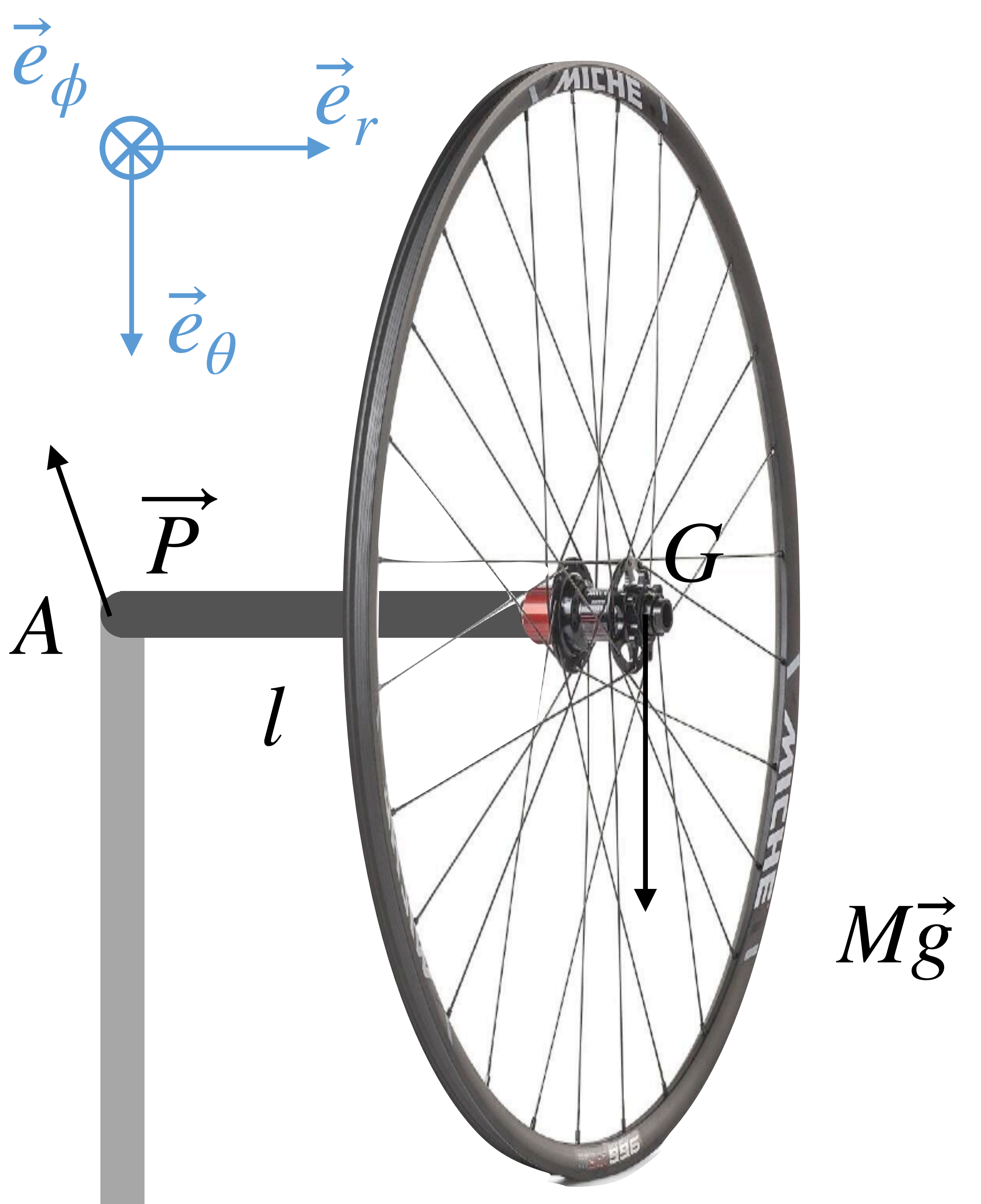
We've recovered similar equation to falling bar problem

Theta changes, the wheel falls. Torque and angular momentum always point along $\vec{e}_\phi$ which is fixed during the motion

PRECESSION IN DETAIL

Using spherical coordinates

starting at $\theta = \pi/2,\ \phi = 0$



$\overrightarrow{AG} = l\vec{e}_r$

$M\vec{g} = -Mg\vec{e}_z = -Mg(\cos\theta\vec{e}_r - \sin\theta\vec{e}_\theta)$

Consider the case when we start with a very large spin ω_s. At t=0, $\vec{L}_A = I_{\|}\omega_s\vec{e}_r$. Assume that ω_s stays constant: $\dot{\omega}_s = 0$, and the $\vec{L}_r$ term dominates such that $\vec{L}_a = \vec{L}_r$

The situation we considered in the lecture video

$$\vec{L}_A = I_{\|}\omega_s\vec{e}_r \qquad \dot{\vec{e}}_r = \dot{\theta}\,\vec{e}_\theta + \dot{\varphi}\sin\theta\,\vec{e}_\varphi$$

$$\dot{\vec{L}}_A = I_{\|}\omega_s[\dot{\theta}\vec{e}_\theta + \dot{\phi}\sin\theta\vec{e}_\phi]$$

$$\dot{\vec{L}}_A = \vec{M}_A = Mgl\sin\theta\vec{e}_\phi$$

On theta: $I_{\|}\omega_s\dot{\theta} = 0$

Theta is constant, so stays at $\theta = \pi/2$

On phi: $I_{\|}\omega_s\dot{\phi}\sin\theta = Mgl\sin\theta$

$$\implies \dot{\phi} = \frac{Mgl}{I_{\|}\omega_s}$$

Constant precession velocity

PRECESSION IN DETAIL

Using spherical coordinates

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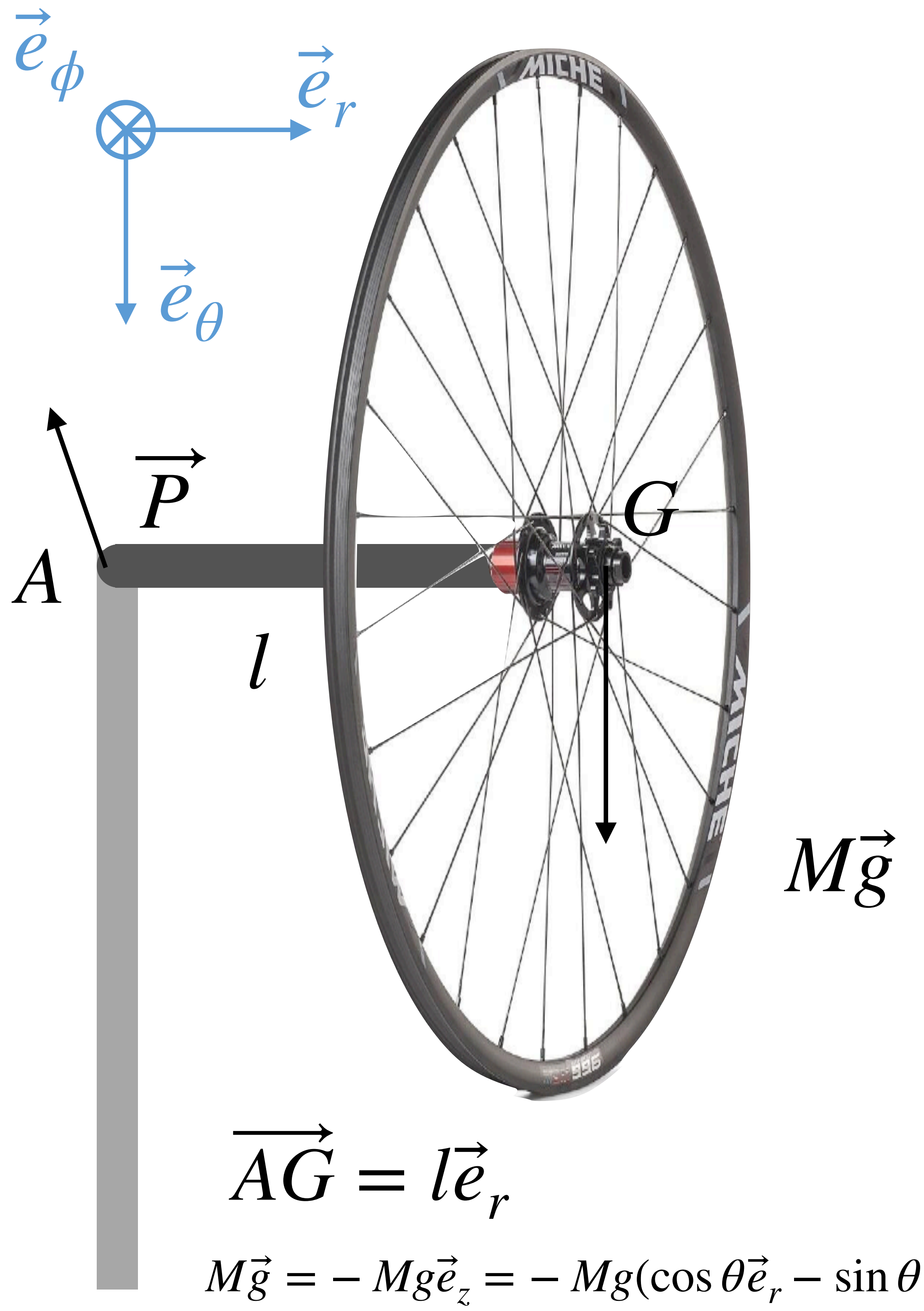

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Consider the case when we start with a very large spin ω_s. At t=0, $\vec{L}_A = I_{\parallel}\omega_s\vec{e}_r$. Assume that ω_s stays constant: $\dot{\omega}_s = 0$

An intermediate situation. Let's compare the evolution of the theta coordinate

$$\dot{\vec{L}}_A = I_{\parallel}\omega_s\dot{\vec{e}}_r + (I_\perp + Ml^2)(\ddot{\phi}\vec{e}_z + \ddot{\theta}\vec{e}_\phi + \dot{\theta}\dot{\vec{e}}_\phi)$$

$$\dot{\vec{e}}_r = \dot{\theta}\,\vec{e}_\theta + \dot{\varphi}\sin\theta\,\vec{e}_\varphi \qquad \dot{\vec{e}}_\varphi = -\dot{\varphi}\sin\theta\,\vec{e}_r - \dot{\varphi}\cos\theta\,\vec{e}_\theta \qquad \vec{e}_z = \cos\theta\vec{e}_r - \sin\theta\vec{e}_\theta$$

$$\dot{\vec{L}}_A = I_{\parallel}\omega_s[\dot{\theta}\vec{e}_\theta + \dot{\phi}\sin\theta\vec{e}_\phi]$$
$$+(I_\perp + Ml^2)(\ddot{\phi}[\cos\theta\vec{e}_r - \sin\theta\vec{e}_\theta] + \ddot{\theta}\vec{e}_\phi + \dot{\theta}\dot{\phi}[\sin\theta\vec{e}_r - \cos\theta\vec{e}_\theta])$$

$$\dot{\vec{L}}_A = \vec{M}_A = Mgl\sin\theta\vec{e}_\phi$$

On r: $(I_\perp + Ml^2)(\ddot{\phi}\cos\theta + \dot{\theta}\dot{\phi}\sin\theta) = 0 \implies \ddot{\phi}\cos\theta + \dot{\theta}\dot{\phi}\sin\theta = 0$ ①

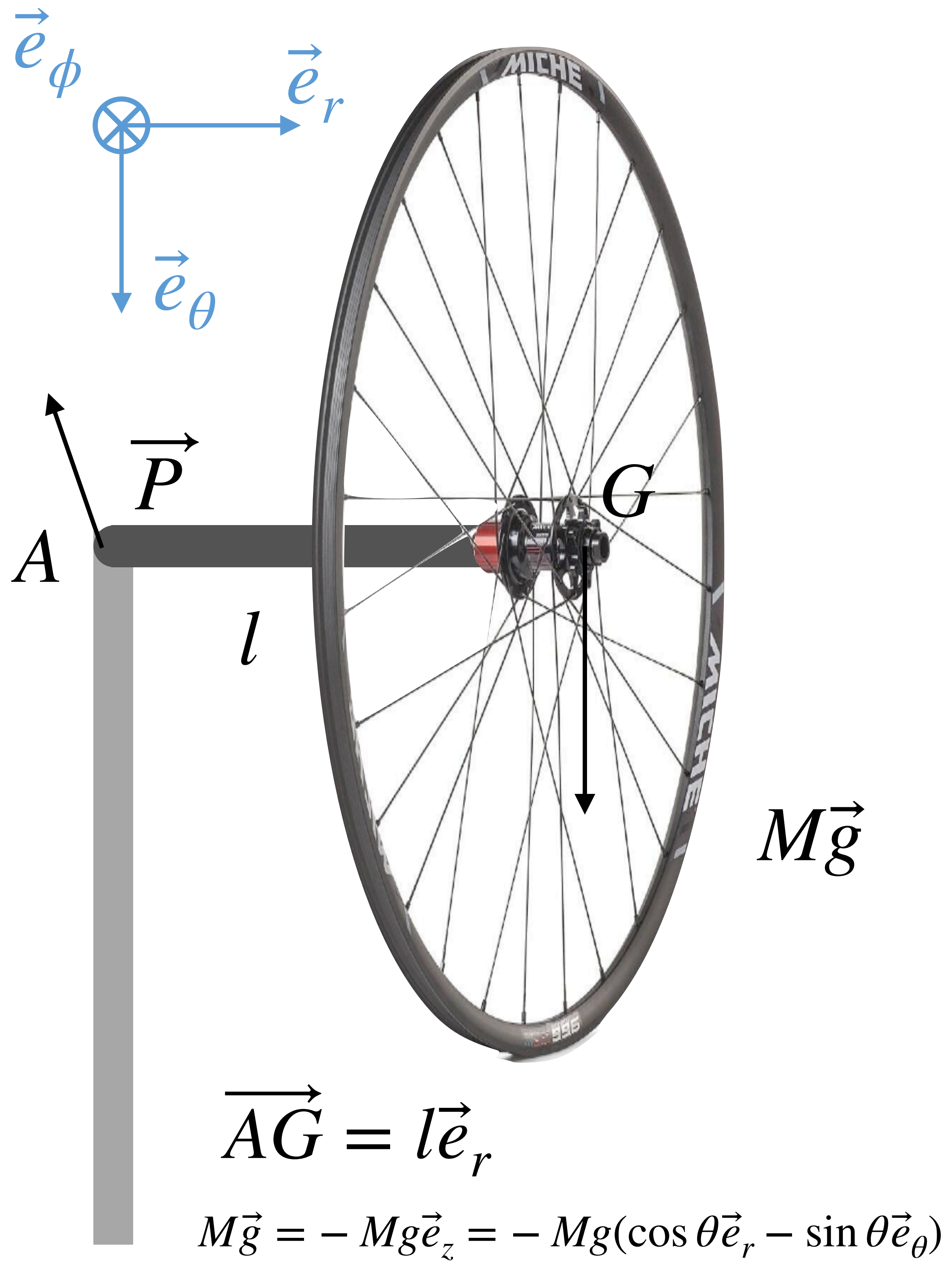
On theta: $I_{\parallel}\omega_s\dot{\theta} - (I_\perp + Ml^2)(\ddot{\phi}\sin\theta + \dot{\theta}\dot{\phi}\cos\theta) = 0$ ②

On phi: $I_{\parallel}\omega_s\dot{\phi}\sin\theta + (I_\perp + Ml^2)\ddot{\theta} = Mgl\sin\theta \implies (I_\perp + Ml^2)\ddot{\theta} = \sin\theta(Mgl - I_{\parallel}\omega_s\dot{\phi})$ ③

PRECESSION IN DETAIL

Using spherical coordinates

starting at $\theta = \pi/2$, $\phi = 0$



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$M\vec{g} = -Mg\vec{e}_z = -Mg(\cos\theta\vec{e}_r - \sin\theta\vec{e}_\theta)$

Consider the case when we start with a very large spin ω_s. At t=0, $\overrightarrow{L}_A = I_{\|}\omega_s\vec{e}_r$. Assume that ω_s stays constant: $\dot{\omega}_s = 0$

① $\ddot{\phi}\cos\theta + \dot{\theta}\dot{\phi}\sin\theta = 0$

$$\ddot{\phi} = -\dot{\theta}\dot{\phi}\sin\theta/\cos\theta$$

② $I_{\|}\omega_s\dot{\theta} - (I_\perp + Ml^2)(\ddot{\phi}\sin\theta + \dot{\theta}\dot{\phi}\cos\theta) = 0$

$$\implies I_{\|}\omega_s\dot{\theta} - (I_\perp + Ml^2)\big((-\dot{\theta}\dot{\phi}\sin\theta/\cos\theta)\sin\theta + \dot{\theta}\dot{\phi}\cos\theta\big) = 0$$

$$\implies I_{\|}\omega_s - (I_\perp + Ml^2)\big((-\dot{\phi}\sin^2\theta/\cos\theta) + \dot{\phi}\cos^2\theta/\cos\theta\big) = 0$$

$$\implies I_{\|}\omega_s - (I_\perp + Ml^2)\frac{\dot{\phi}}{\cos\theta}\big(-\sin^2\theta + \cos^2\theta\big) = 0 \qquad \cos^2\theta - \sin^2\theta = \cos 2\theta$$

$$\implies I_{\|}\omega_s - (I_\perp + Ml^2)\frac{\dot{\phi}\cos 2\theta}{\cos\theta} = 0$$

$$\implies \dot{\phi} = \frac{I_{\|}\omega_s}{I_\perp + Ml^2}\frac{\cos(\theta)}{\cos(2\theta)}$$

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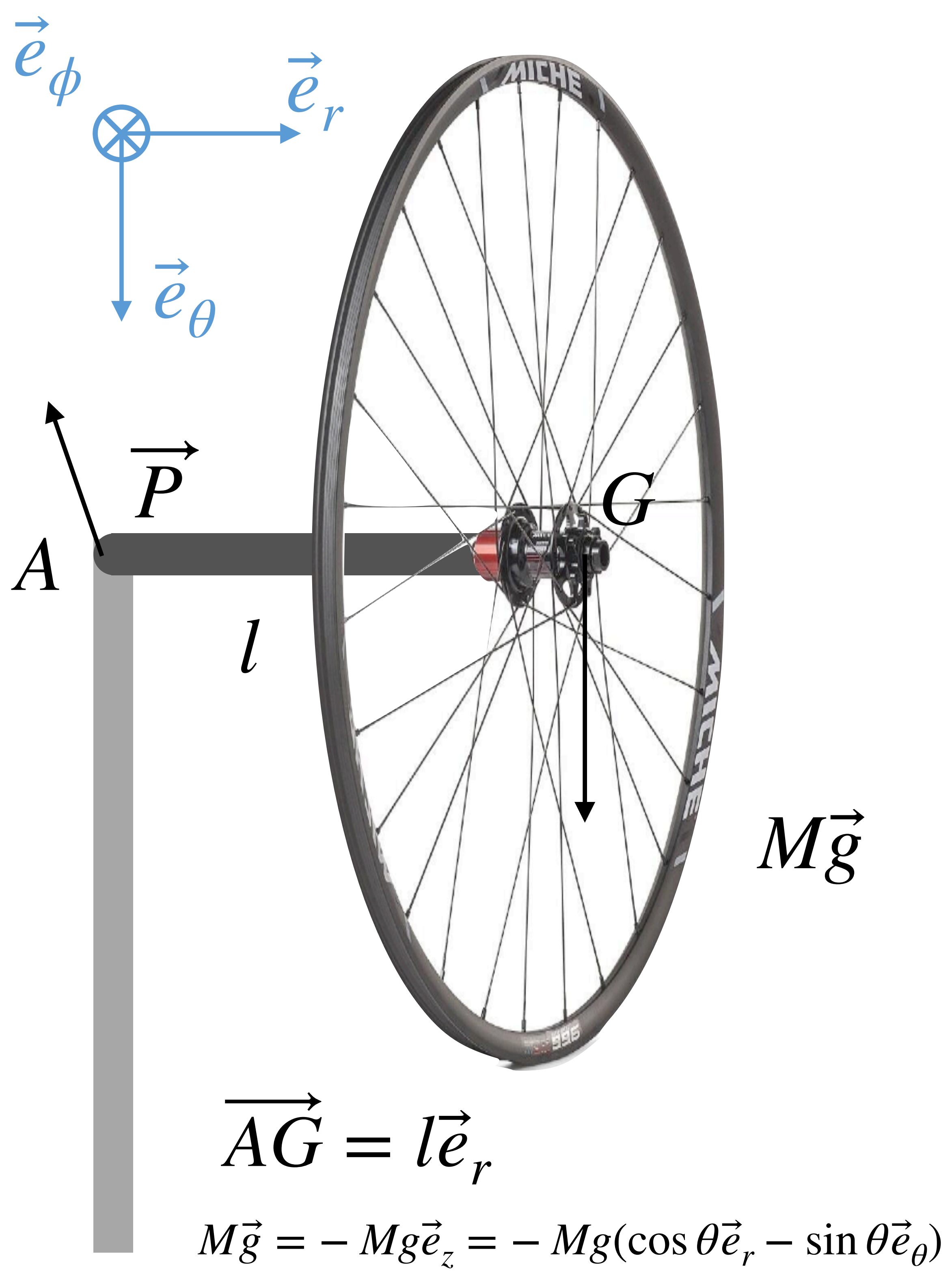

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Consider the case when we start with a very large spin ω_s. At t=0, $\overrightarrow{L}_A = I_\parallel\omega_s\vec{e}_r$. Assume that ω_s stays constant: $\dot{\omega}_s = 0$

$$\implies \dot{\phi} = \frac{I_\parallel\omega_s}{I_\perp + Ml^2}\frac{\cos(\theta)}{\cos(2\theta)}$$

③ $$(I_\perp + Ml^2)\ddot{\theta} = \sin\theta(Mgl - I_\parallel\omega_s\dot{\phi})$$

$$(I_\perp + Ml^2)\ddot{\theta} = \sin\theta\left(Mgl - I_\parallel\omega_s\left(\frac{I_\parallel\omega_s}{I_\perp + Ml^2}\frac{\cos(\theta)}{\cos(2\theta)}\right)\right)$$

$$2\cos\theta\sin\theta = \sin 2\theta$$

$$(I_\perp + Ml^2)\ddot{\theta} = Mgl\sin\theta - I_\parallel\omega_s\left(\frac{I_\parallel\omega_s}{I_\perp + Ml^2}\frac{\sin(2\theta)}{2\cos(2\theta)}\right)$$

$$\boxed{\ddot{\theta} = \frac{Mgl}{I_\perp + Ml^2}\sin\theta - \frac{(I_\parallel\omega_s)^2}{(I_\perp + Ml^2)^2}\frac{\sin(2\theta)}{2\cos(2\theta)}}$$

With spin, the acceleration along theta is changed by this second term

Motion with no spin $\omega_s = 0$

$$\ddot{\theta} = \frac{Mgl}{I_\perp + Ml^2}\sin\theta$$

PRECESSION IN DETAIL

Using spherical coordinates

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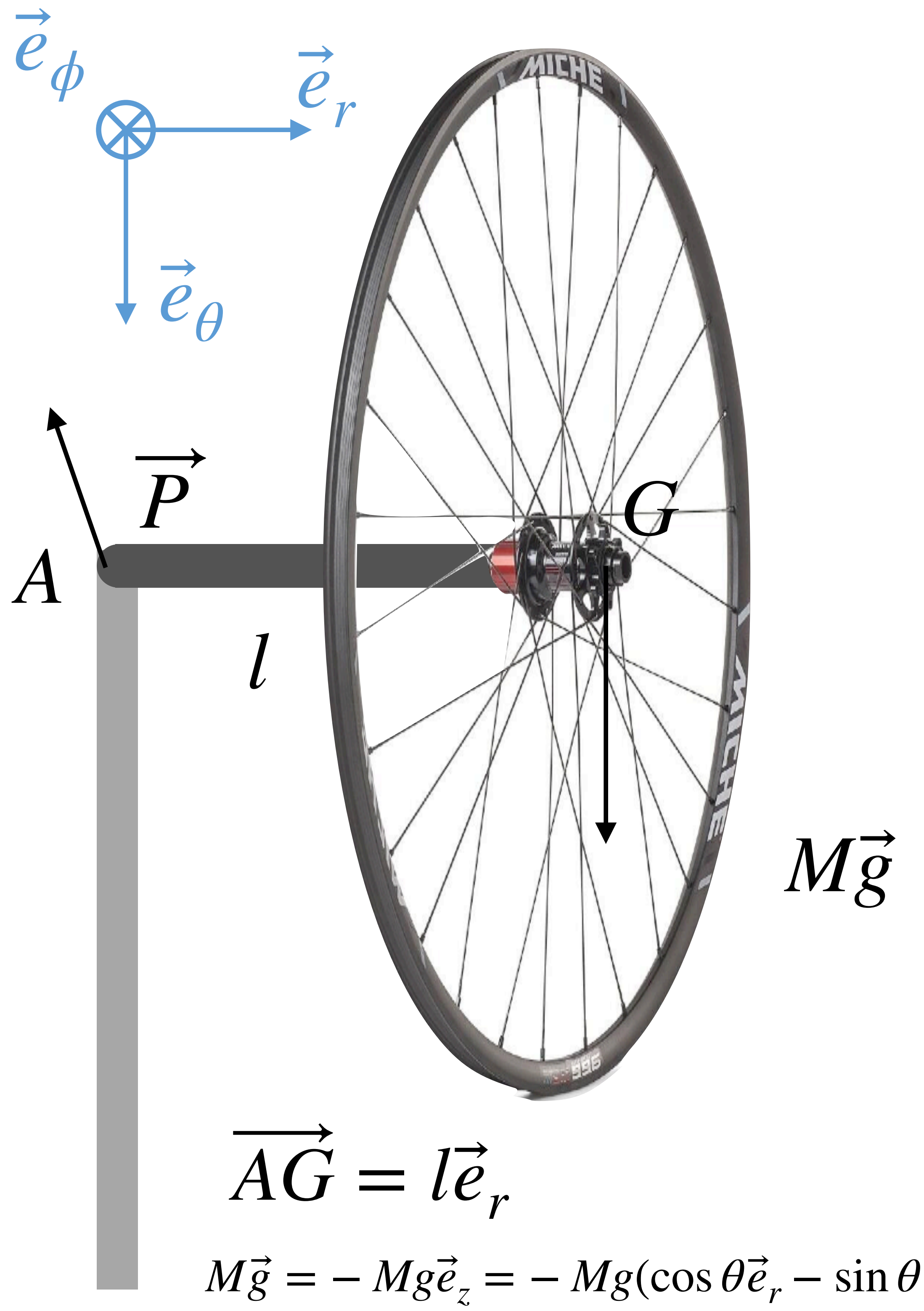

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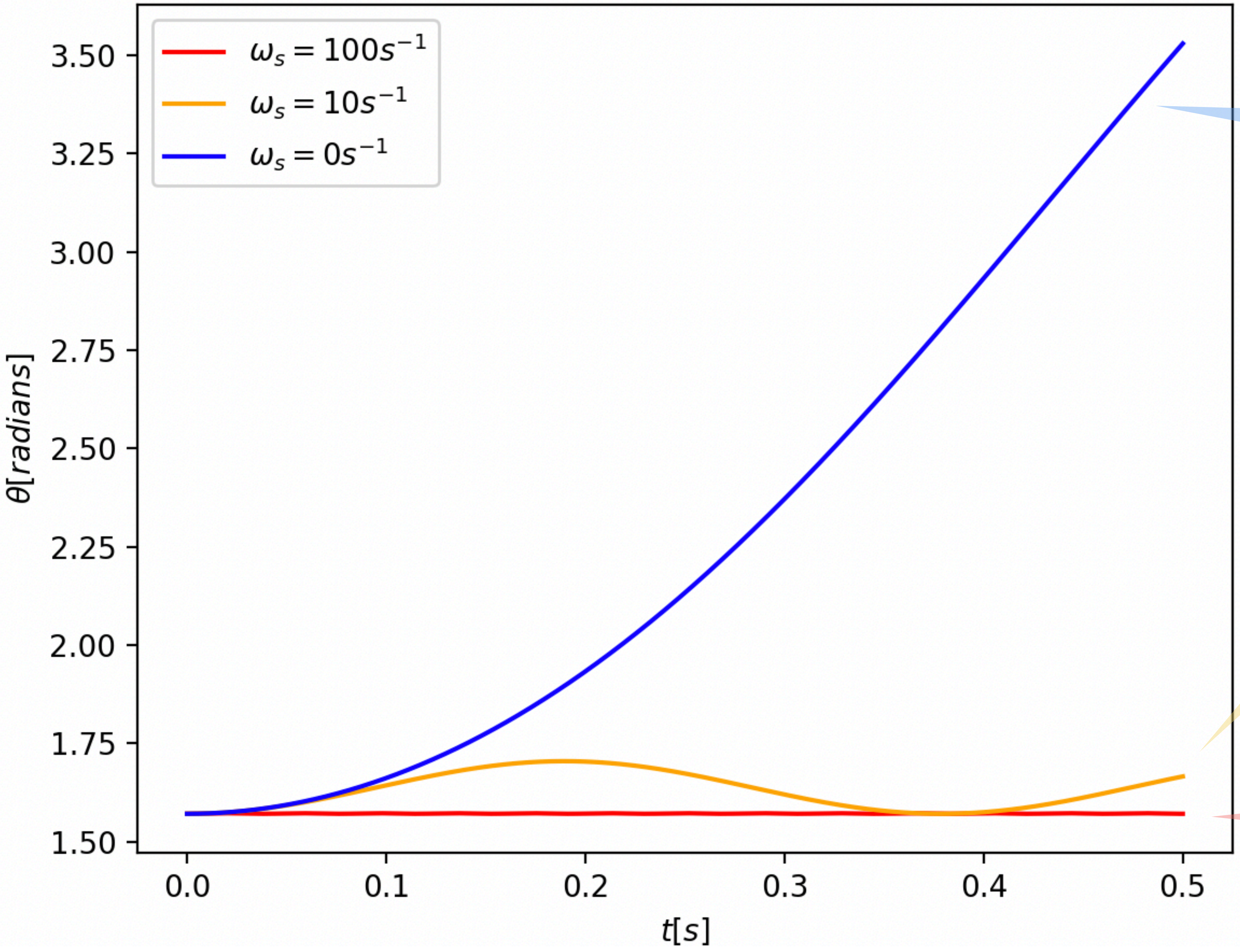
$$\boxed{\ddot{\theta} = \frac{Mgl}{I_\perp + Ml^2}\sin\theta - \frac{(I_\parallel \omega_s)^2}{(I_\perp + Ml^2)^2}\frac{\sin(2\theta)}{2\cos(2\theta)}}$$

Solve for theta numerically



Has no spin and drops

Intermediate regime where theta oscillates

Has very high spin and theta is a constant