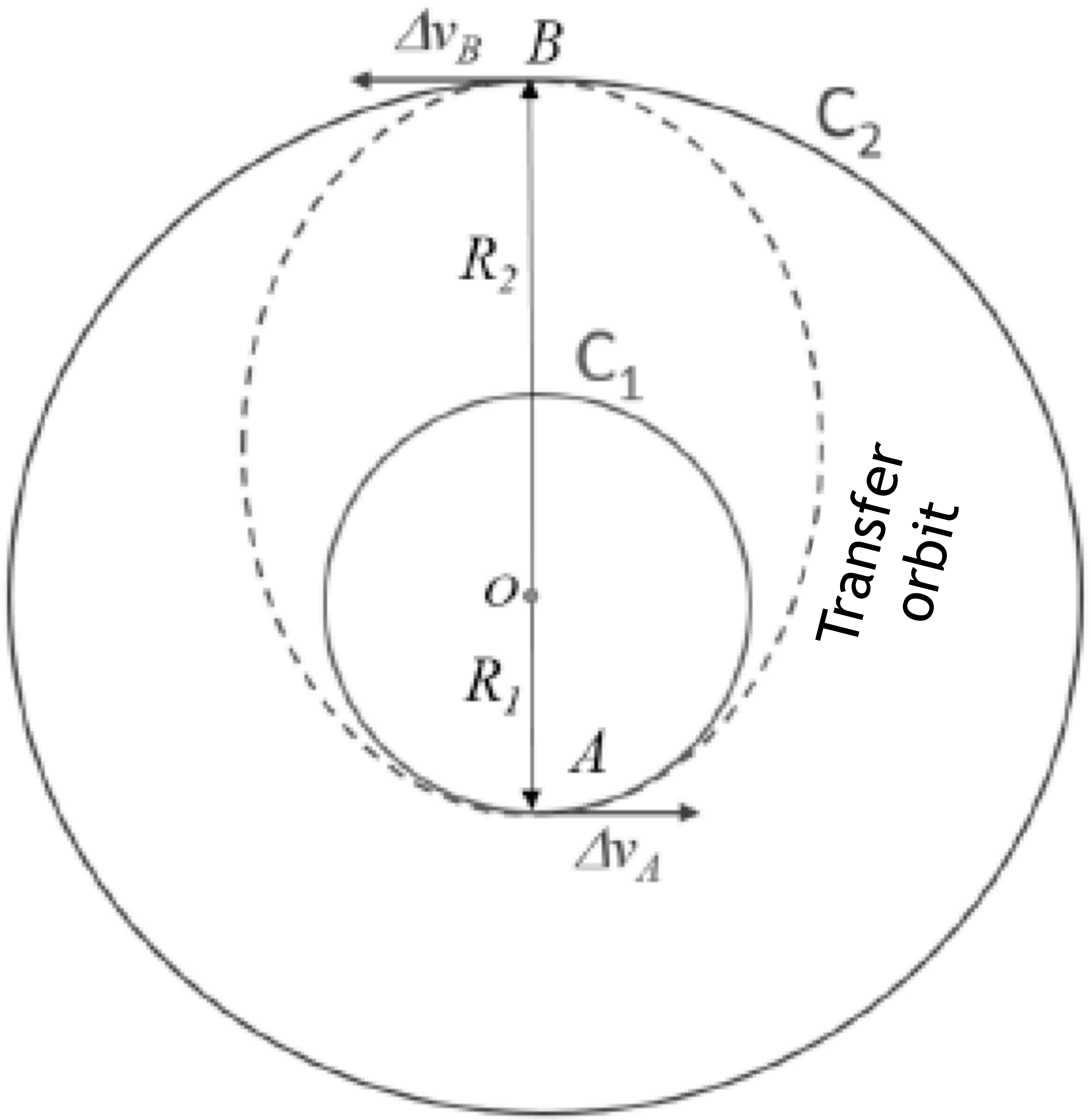
EXERCISE: TRANSFER ORBIT

(Question from 2018 exam)



The International Space Station is a satellite orbiting the Earth. Astronauts are periodically resupplied by a shuttle launched by a rocket. Let G be the universal gravitational constant and M the mass of the Earth. After being released by the rocket, the shuttle of mass m is placed in a circular orbit C_1 of radius R_1, which is smaller than the circular orbit C_2 of radius R_2 of the space station.

https://phyanim.sciences.univ-nantes.fr/Meca/Planetes/transfert.php

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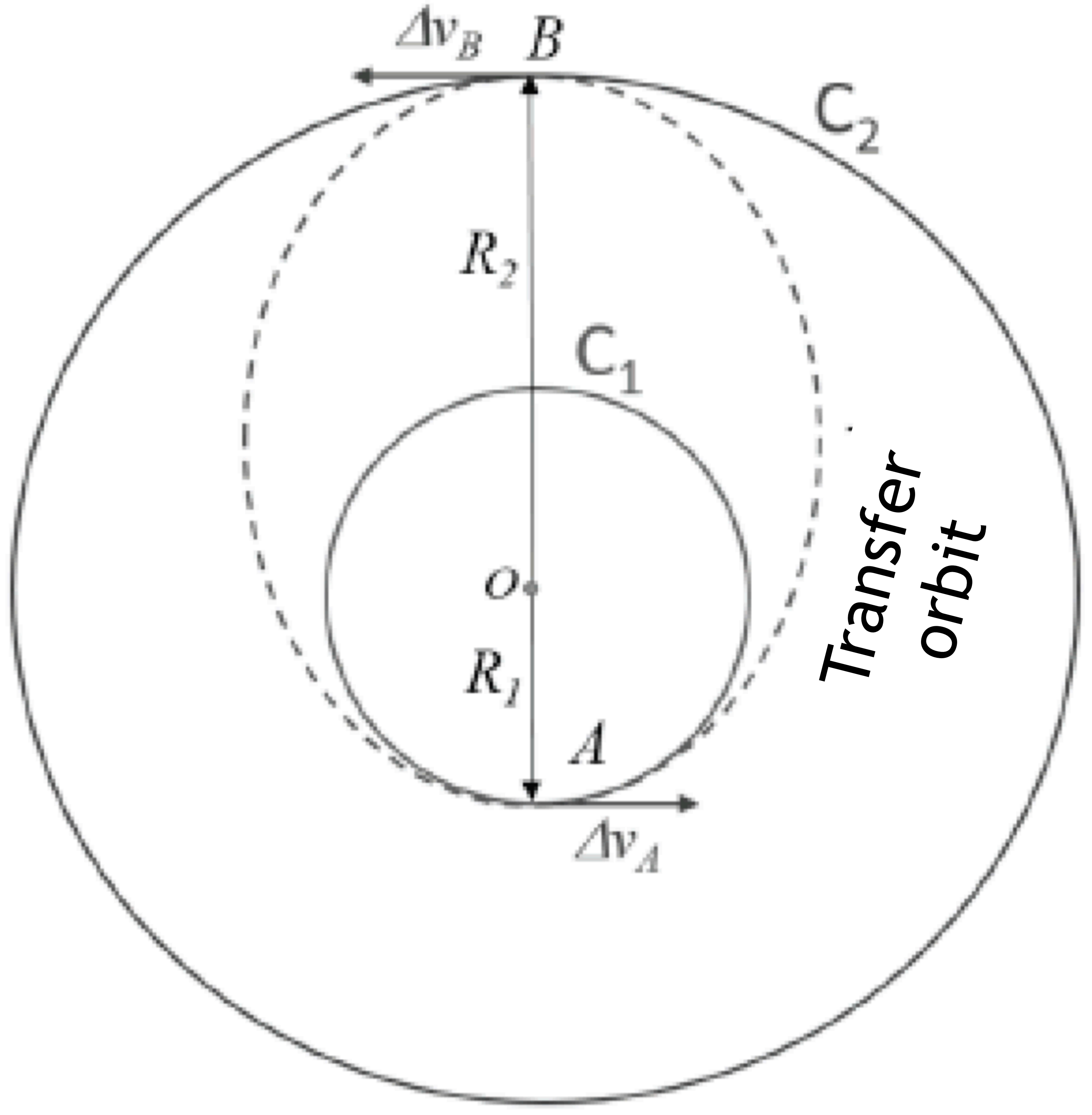

1. Prove that the speed of a satellite in a circular orbit is constant.

2. Express the speed v_1 of the shuttle in the circular orbit C_1 as a function of the given information.

3. Give the expression for the mechanical energy E_1 in orbit C_1 as a function of G, m, M, and R_1.

4. Calculate the work W_{12} done by the gravitational force F acting on the shuttle when it moves from orbit C_1 to orbit C_2. In practice, to reach the circular orbit C_2, it is first necessary to pass through a transfer orbit which is elliptical, as shown by the dotted line in the diagram

5. The shuttle is in the transfer orbit. Express the shuttle's velocity v_B at point B as a function of its velocity v_A at point A.

6. Determine the expression for the mechanical energy E_T in the transfer orbit as a function of G, m, M, R_1, and R_2.

7. Express the velocity $v_A = v_1 + \Delta v_1$ that must be imparted to the shuttle to move from the circular orbit C_1 to the transfer orbit. The result should be expressed as a function of E_T, E_1, and m.

8. Is the change in velocity Δv_2 of the shuttle at B, to go from the transfer orbit to C_2, positive or negative? Justify your answer without calculation

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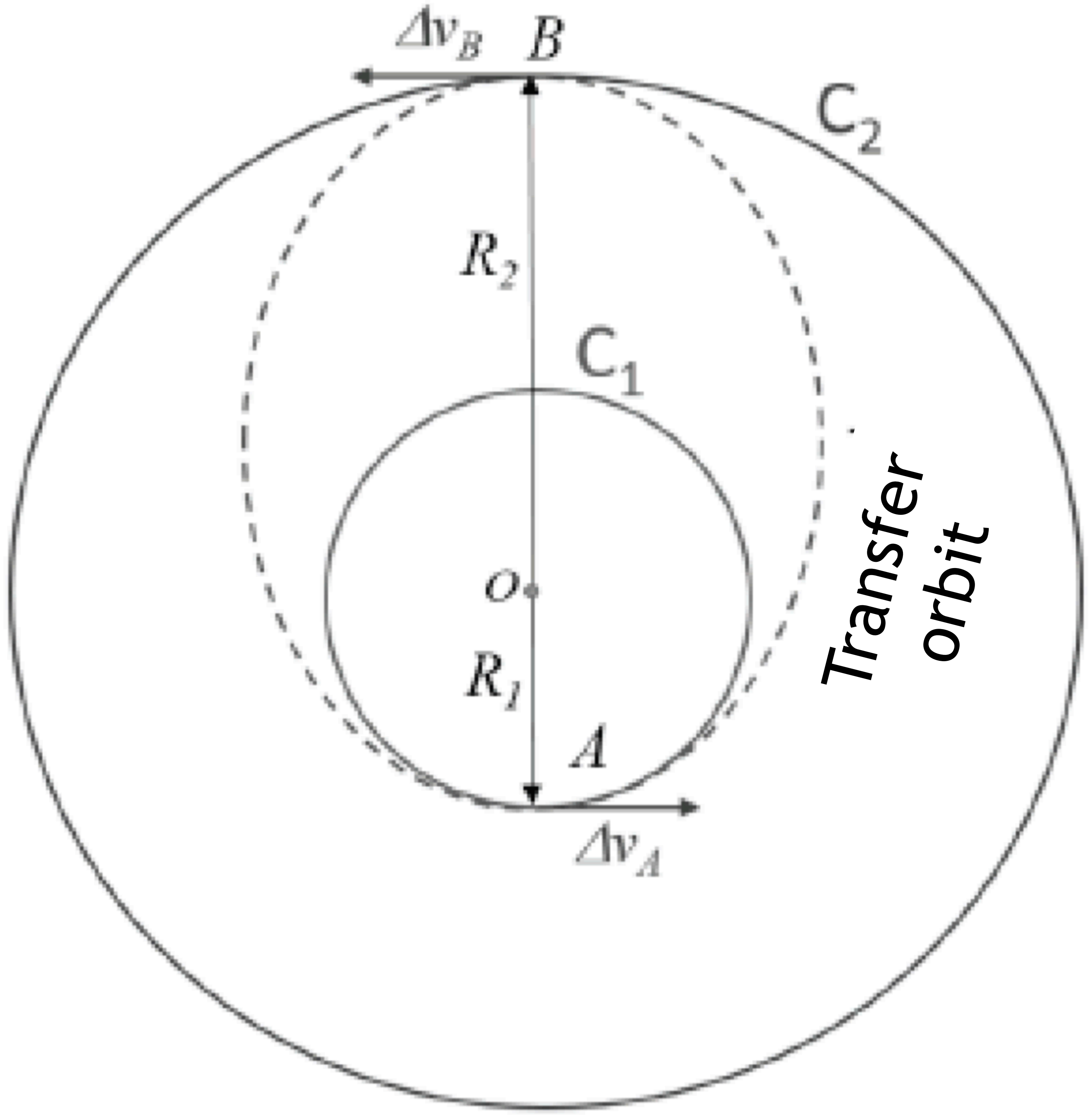

1. Prove that the speed of a satellite in a circular orbit is constant.

$$\vec{r} = R\vec{e}_\rho \qquad \vec{v} = R\dot{\phi}\vec{e}_\phi$$

$$\overrightarrow{L}_o = \vec{r} \wedge \vec{v} = R^2 m\dot{\phi} = cte$$

$$R = cte, \quad m = cte, \quad R^2 m\dot{\phi} = cte \implies \boxed{\dot{\phi} = cte}$$

2. Express the speed v_1 of the shuttle in the circular orbit C_1 as a function of the given information.

Using curvilinear coordinates

$$\left.\begin{aligned} \overrightarrow{F}_G &= \vec{n}\frac{mMG}{R_1^2} \\ \vec{a}_n &= \vec{n}\frac{v_1^2}{R_1} \end{aligned}\right\} \quad \overrightarrow{F}_G = m\vec{a} \implies \frac{mMG}{R_1^2} = m\frac{v_1^2}{R_1} \implies \boxed{v_1 = \sqrt{\frac{MG}{R_1}}}$$

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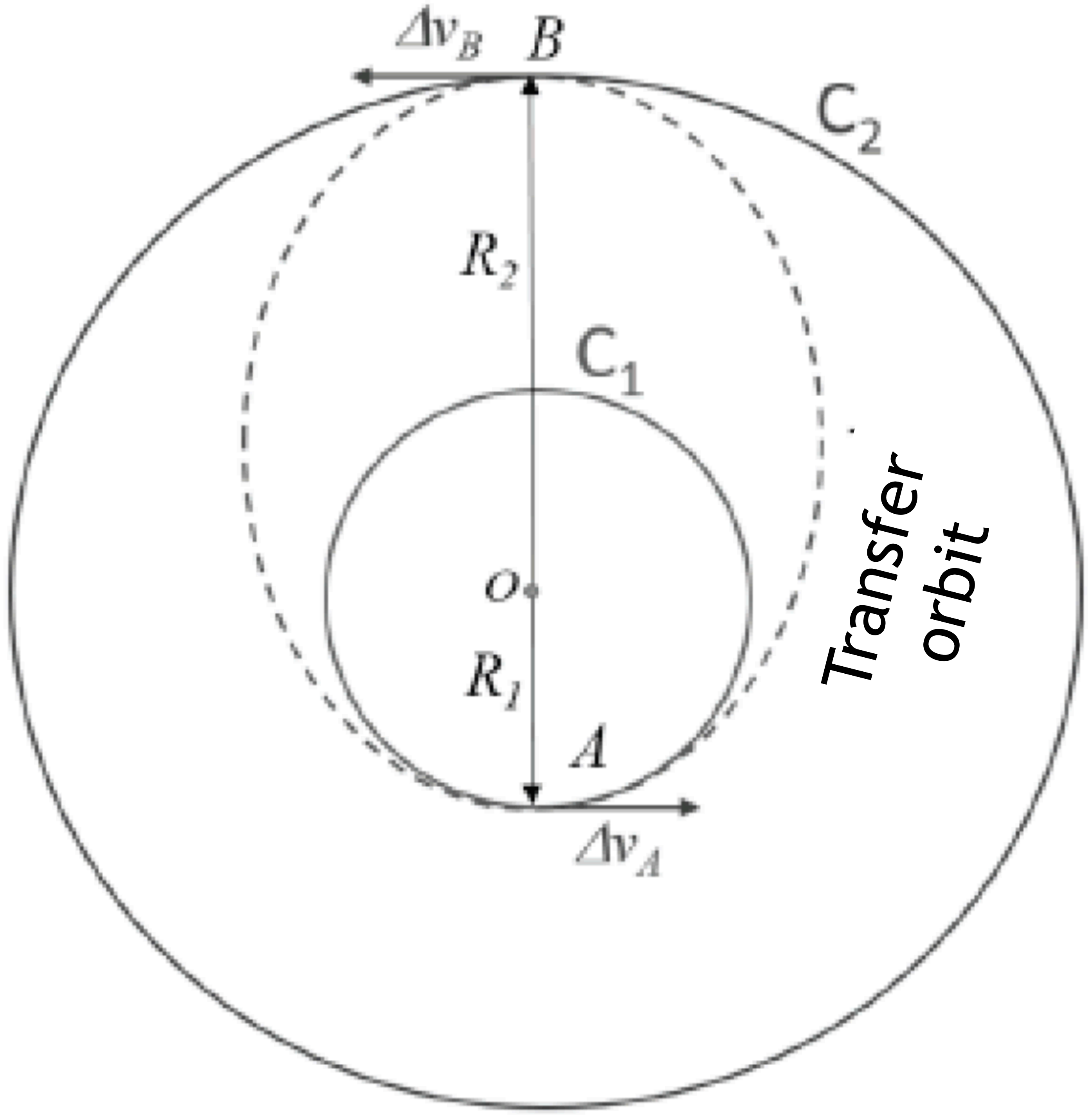

3. Give the expression for the mechanical energy E_1 in orbit C_1 as a function of G, m, M, and R_1.

$$E_1 = E_c + E_p = \frac{1}{2}mv_1^2 - \frac{mMG}{R_1} = \frac{1}{2}m\frac{MG}{R_1} - \frac{mMG}{R_1}$$

$$\boxed{E_1 = -\frac{1}{2}\frac{mMG}{R_1}}$$

4. Calculate the work W_{12} done by the gravitational force F acting on the shuttle when it moves from orbit C_1 to orbit C_2. In practice, to reach the circular orbit C_2, it is first necessary to pass through a transfer orbit which is elliptical, as shown by the dotted line in the diagram

Gravity is conservative, so work done by gravity is $W_{12}^G = E_p(R_1) - E_p(R_2) = \boxed{GMm\left(\frac{1}{R_2} - \frac{1}{R_1}\right)}$

However $W_{12}^G \neq E_c(R1) - E_c(R2)$ because $W_{12}^G \neq W_{12}^{total}$ The rocket had exert a force to move to orbit C2

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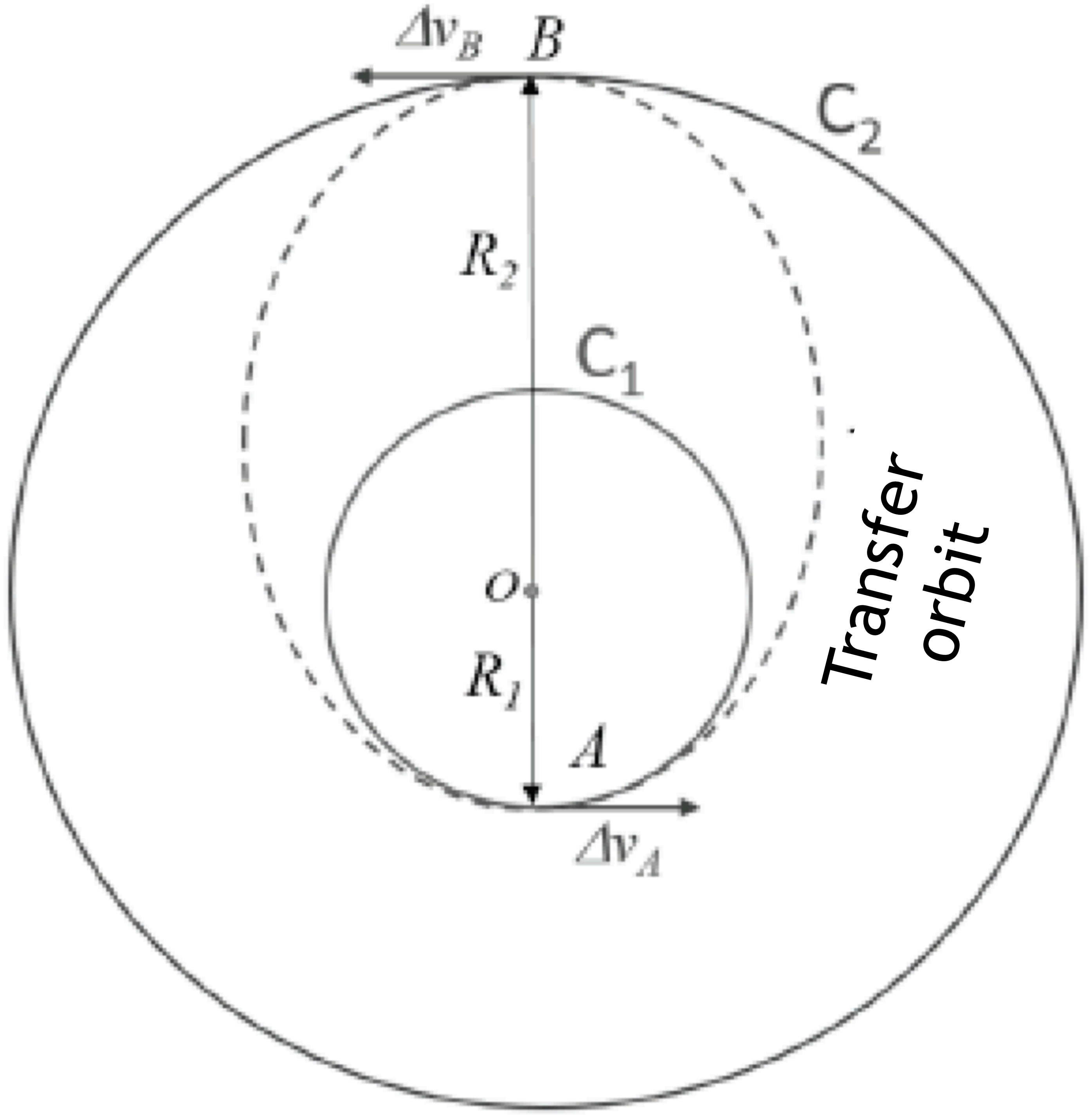

5. The shuttle is in the transfer orbit. Express the shuttle's velocity v_B at point B as a function of its velocity v_A at point A.

Transfer orbit is elliptical, so $\overrightarrow{L}_o = cte, \quad R \neq cte$

At point A: $\vec{r}_1 = R_1 \vec{e}_\rho \qquad \vec{v}_A = v_A \vec{e}_\phi \qquad \overrightarrow{L}_o^A = \vec{r}_1 \wedge m\vec{v}_A = mR_1 v_A \vec{e}_z$

At point B: $\vec{r}_2 = R_2 \vec{e}_\rho \qquad \vec{v}_B = v_B \vec{e}_\phi \qquad \overrightarrow{L}_o^B = \vec{r}_2 \wedge m\vec{v}_B = mR_2 v_B \vec{e}_z$

$$\overrightarrow{L}_o = cte \implies \overrightarrow{L}_o^A = \overrightarrow{L}_o^B \implies R_1 v_A = R_2 v_B \implies \boxed{v_B = \frac{R_1}{R_2} v_A}$$

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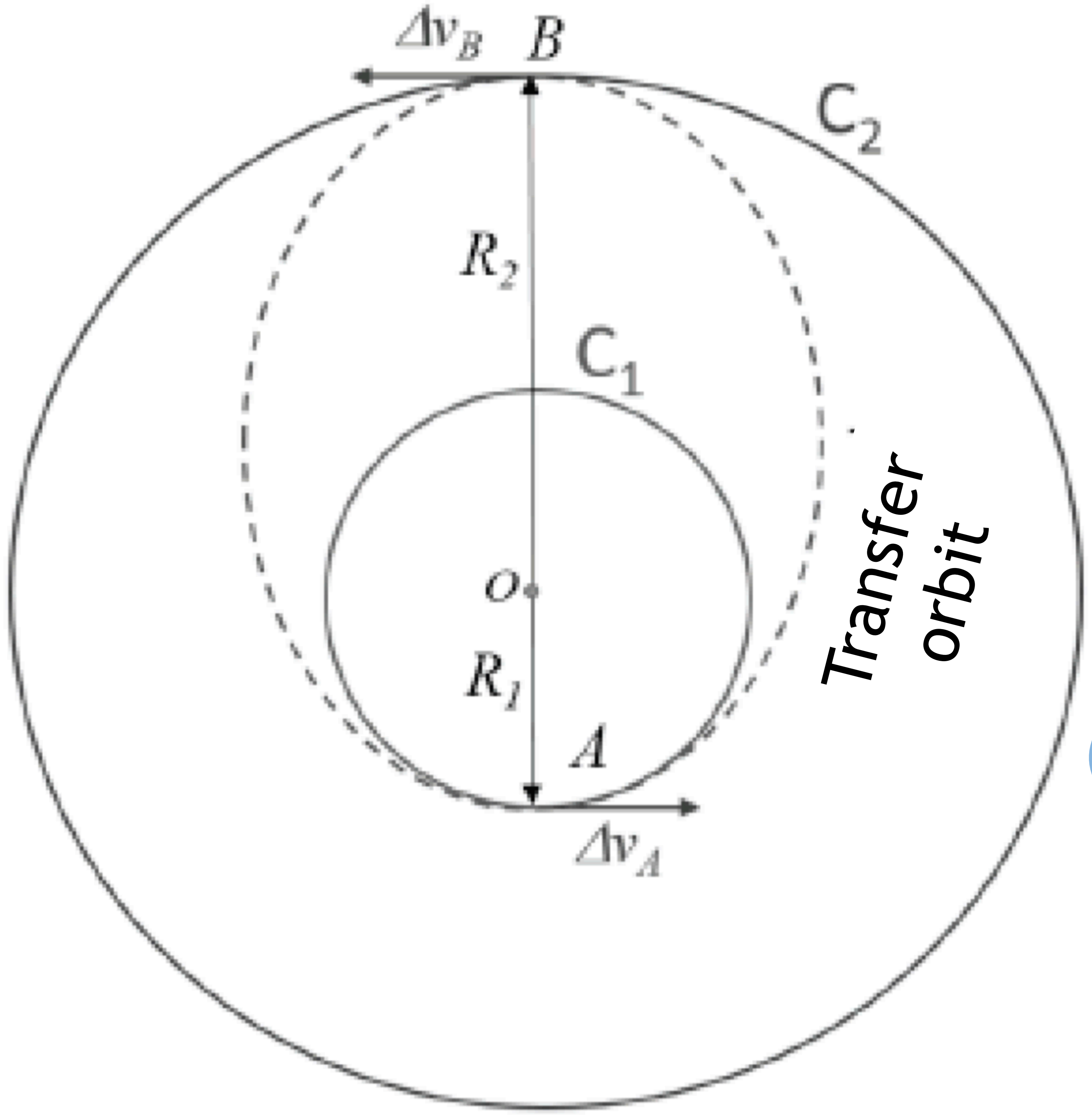

6. Determine the expression for the mechanical energy E_T in the transfer orbit as a function of G, m, M, R_1, and R_2.

Mechanical energy is constant, so $E_T^A = E_T^B$

① $E_T^A = E_T = \frac{1}{2}mv_A^2 - \frac{mMG}{R_1}$

② $E_T^B = E_T = \frac{1}{2}mv_B^2 - \frac{mMG}{R_2}$

$v_B = \frac{R_1}{R_2}v_A$

Need to eliminate v_A, v_B

② $E_T = \frac{1}{2}m\frac{R_1^2}{R_2^2}v_A^2 - \frac{mMG}{R_2} \implies \frac{1}{2}mv_A^2 = \frac{R_2^2}{R_1^2}\left(E_T + \frac{mMG}{R_2}\right)$

① $E_T = \frac{R_2^2}{R_1^2}\left(E_T + \frac{mMG}{R_2}\right) - \frac{mMG}{R_1} \implies E_T\left(1 - \frac{R_2^2}{R_1^2}\right) = \left(\frac{R_2}{R_1^2} - \frac{1}{R_1}\right)mMG \implies E_T\left(R_1^2 - R_2^2\right) = \left(R_2 - R_1\right)mMG$

$\implies E_T = \frac{R_2 - R_1}{R_1^2 - R_2^2}R_1^2 mMG$

$\implies E_T\left(R_1 + R_2\right)\left(R_1 - R_2\right) = \left(R_2 - R_1\right)mMG \implies$ $E_T = -\frac{mMG}{R_1 + R_2}$

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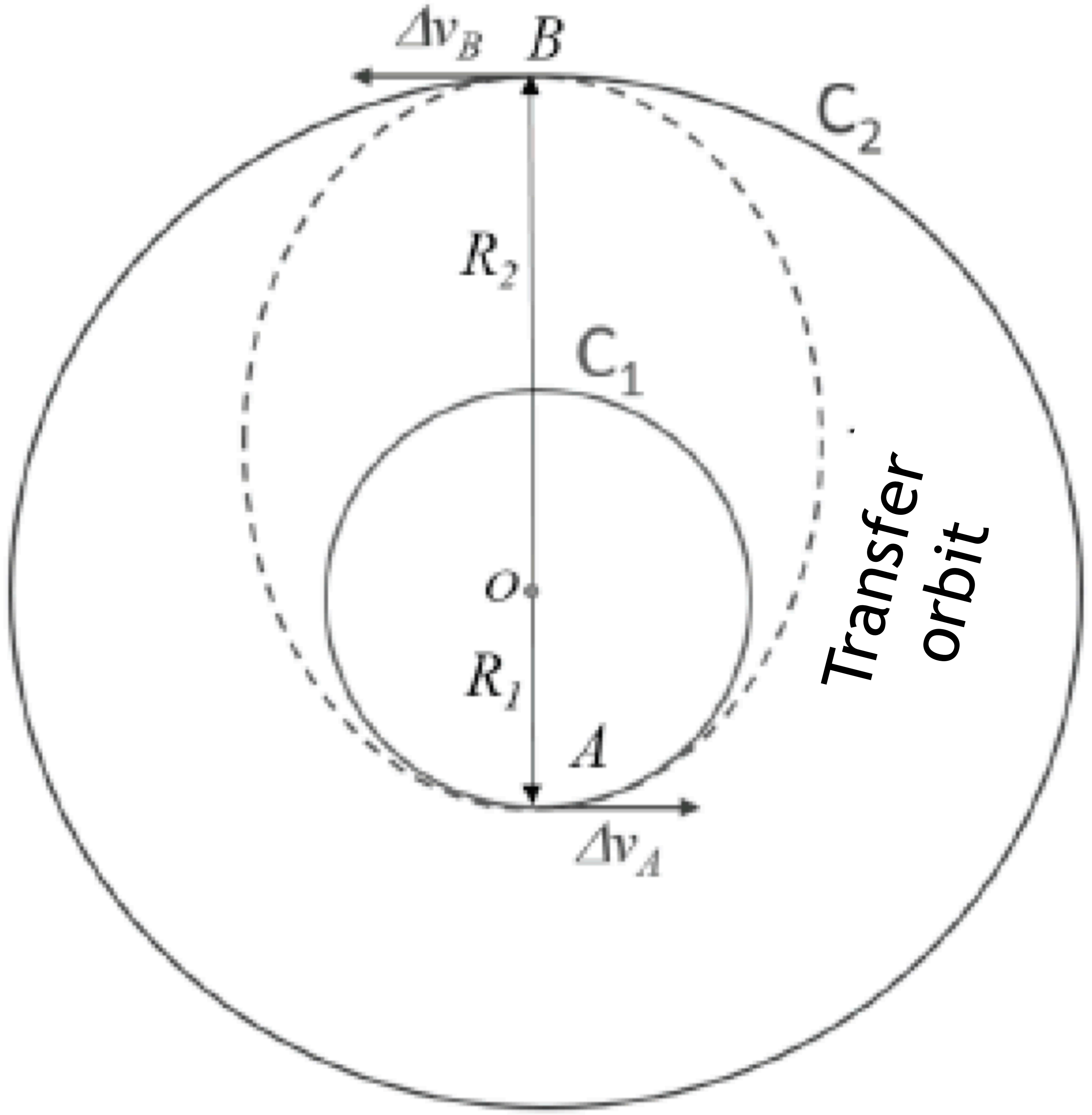

7. Express the velocity $v_A = v_1 + \Delta v_1$ that must be imparted to the shuttle to move from the circular orbit C_1 to the transfer orbit. The result should be expressed as a function of E_T, E_1, and m.

From previous steps

$$v_1 = \sqrt{\frac{MG}{R_1}} \implies v_1^2 = \frac{MG}{R_1}$$

$$E_1 = -\frac{1}{2}\frac{mMG}{R_1} = -\frac{m}{2}v_1^2$$

$$E_T = \frac{1}{2}mv_A^2 - \frac{mMG}{R_1} = \frac{1}{2}mv_A^2 + 2E_1$$

$$\implies E_T - 2E_1 = \frac{1}{2}mv_A^2$$

$$\implies v_A = \sqrt{\frac{2E_T - 4E_1}{m}}$$

$$\implies \Delta v_1 = \sqrt{\frac{2E_T - 4E_1}{m}} - v_1 = \boxed{\sqrt{\frac{2E_T - 4E_1}{m}} - \sqrt{\frac{-2E_1}{m}}}$$

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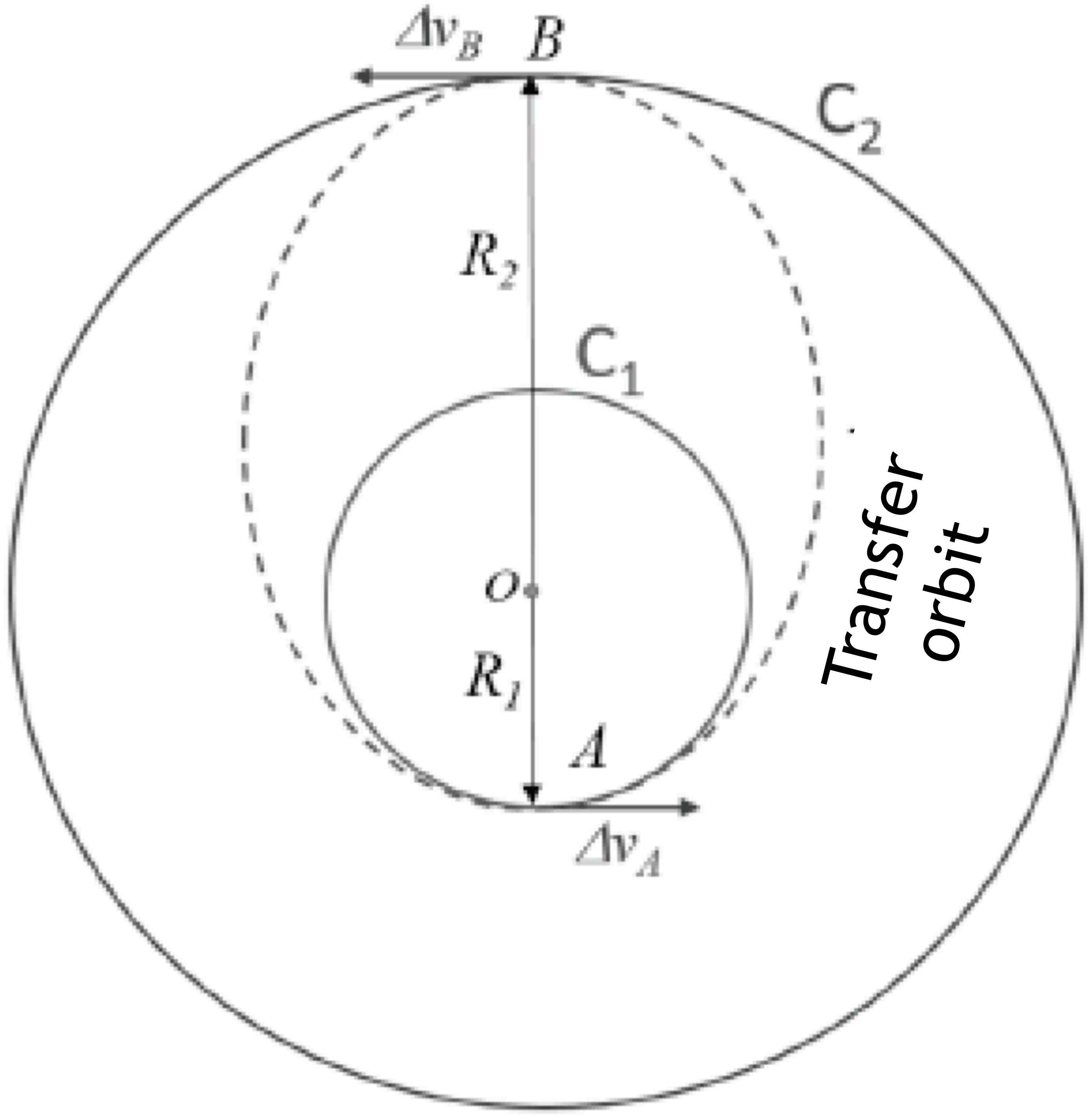

8. Is the change in velocity Δv_2 of the shuttle at B, to go from the transfer orbit to C_2, positive or negative? Justify your answer without calculation

Positive, because at point B the radius of curvature of C2 is larger than the radius of curvature of the transfer orbit. The centripetal force is the same so the velocity must be increased.