PHYS-101 WEEK 12

EPFL

Physique générale : mécanique (classe inversée en anglais)
Prof. Emma Tolley, 24 November 2025

ANGULAR MOMENTUM & TORQUE

Angular momentum

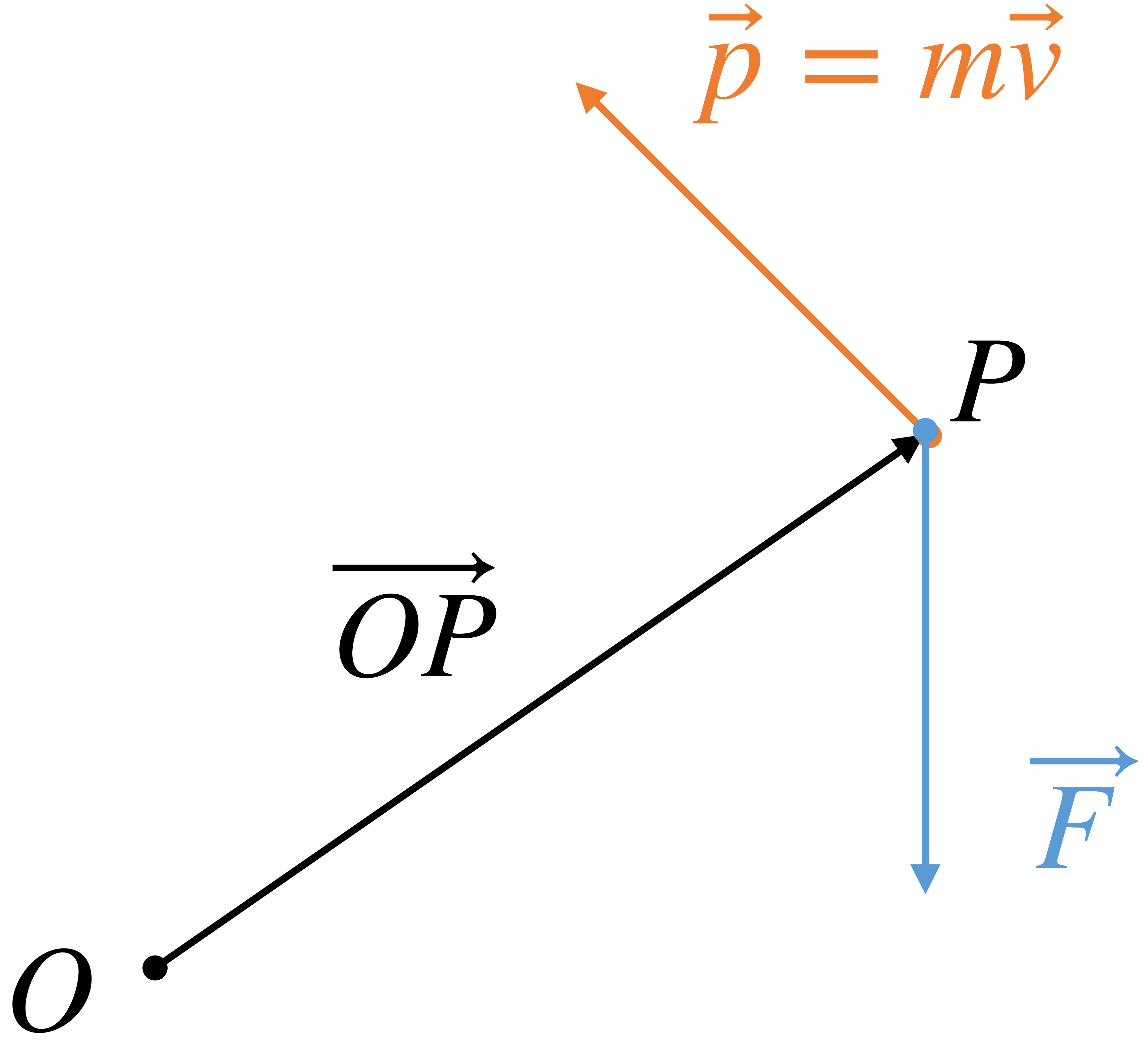
$$\overrightarrow{L_O} = \overrightarrow{OP} \wedge \vec{p}$$

Torque/ moment of a force

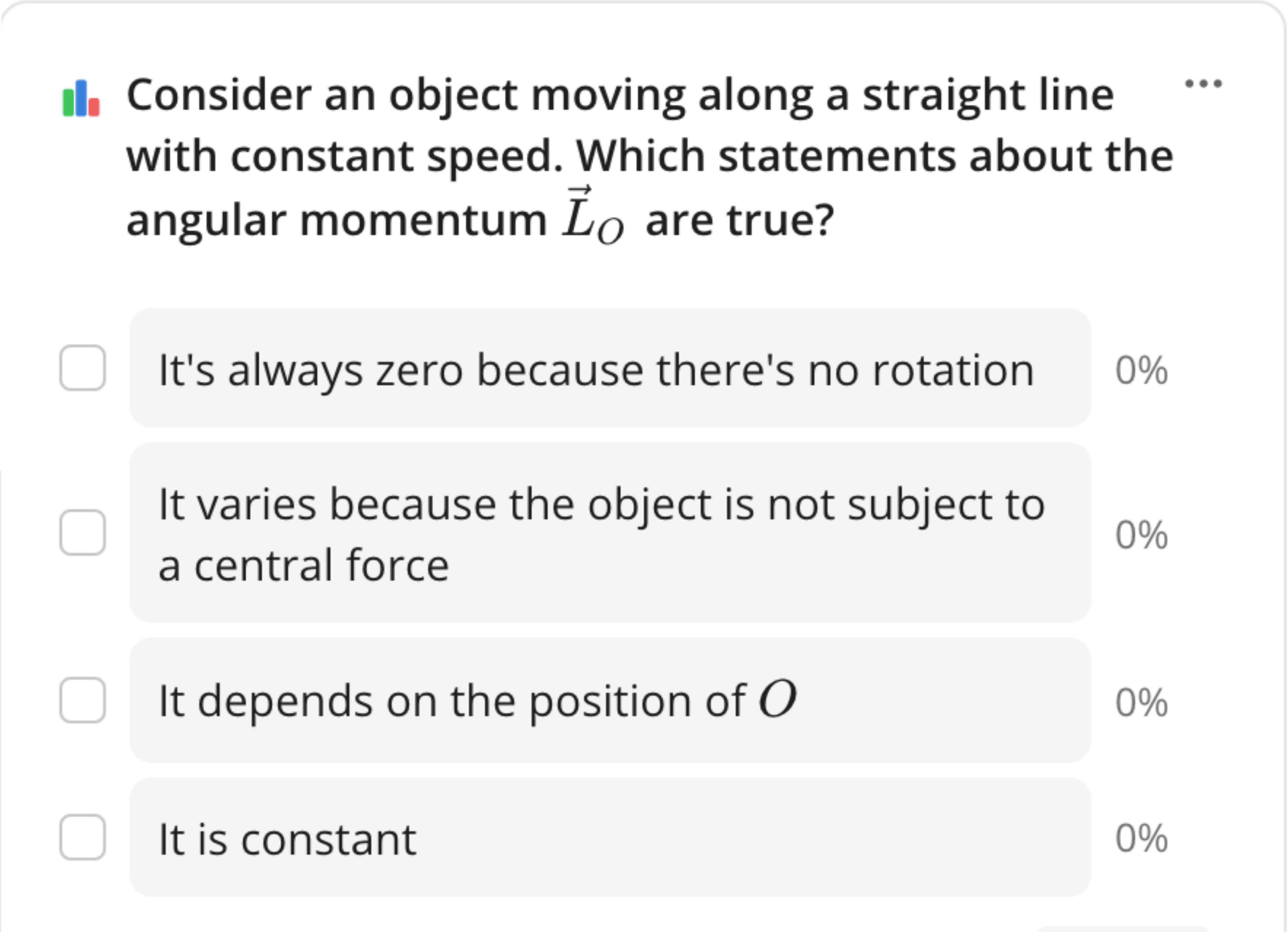
$$\overrightarrow{M_O} = \overrightarrow{OP} \wedge \overrightarrow{F}$$

Theory of angular momentum

$$\sum \overrightarrow{M_O} = \frac{d\overrightarrow{L_O}}{dt}$$

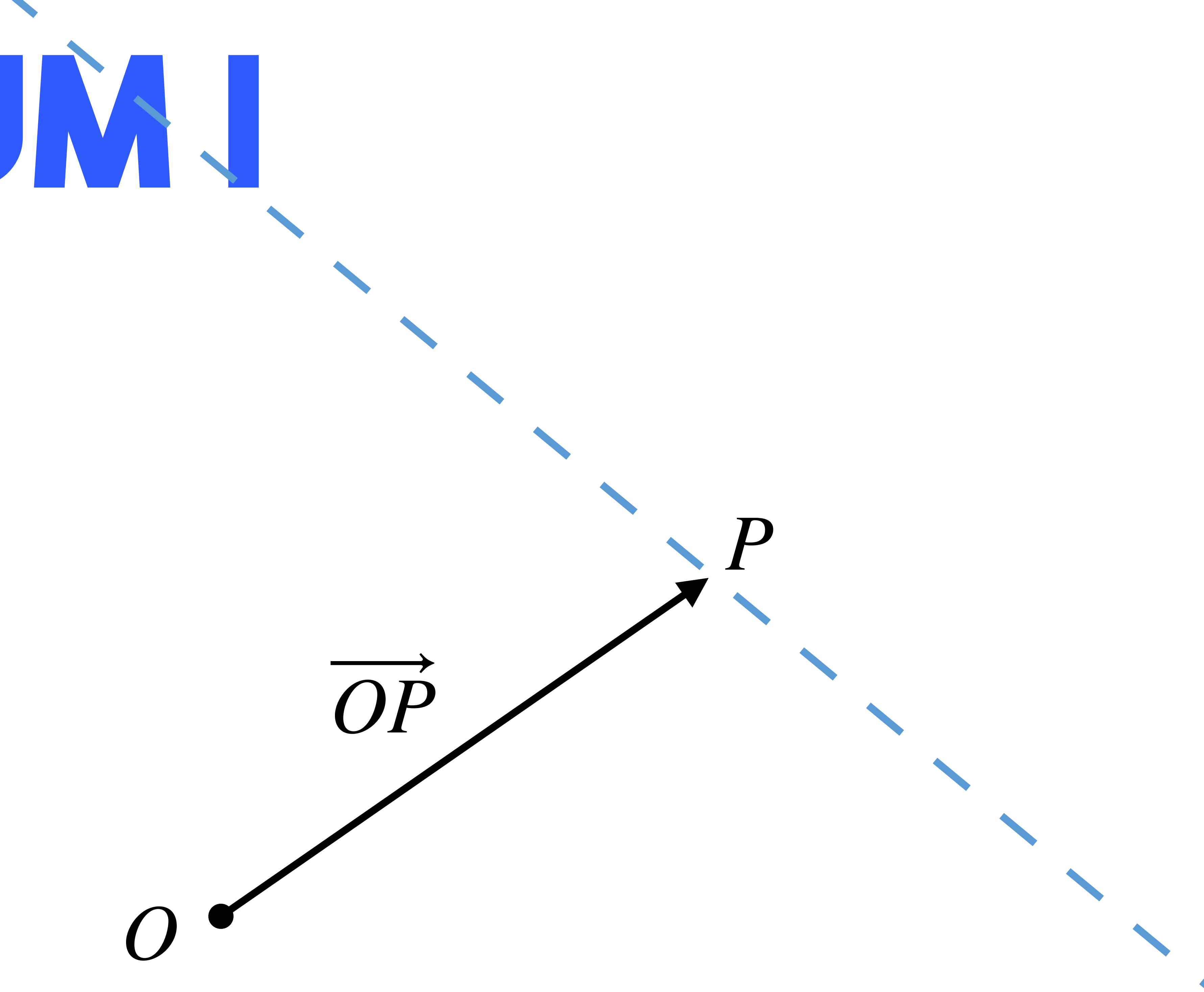


QUIZ: ANGULAR MOMENTUM I



Consider an object moving along a straight line with constant speed. Which statements about the angular momentum $\vec{L}_O$ are true?

- ☐ It's always zero because there's no rotation — 0%
- ☐ It varies because the object is not subject to a central force — 0%
- ☐ It depends on the position of O — 0%
- ☐ It is constant — 0%



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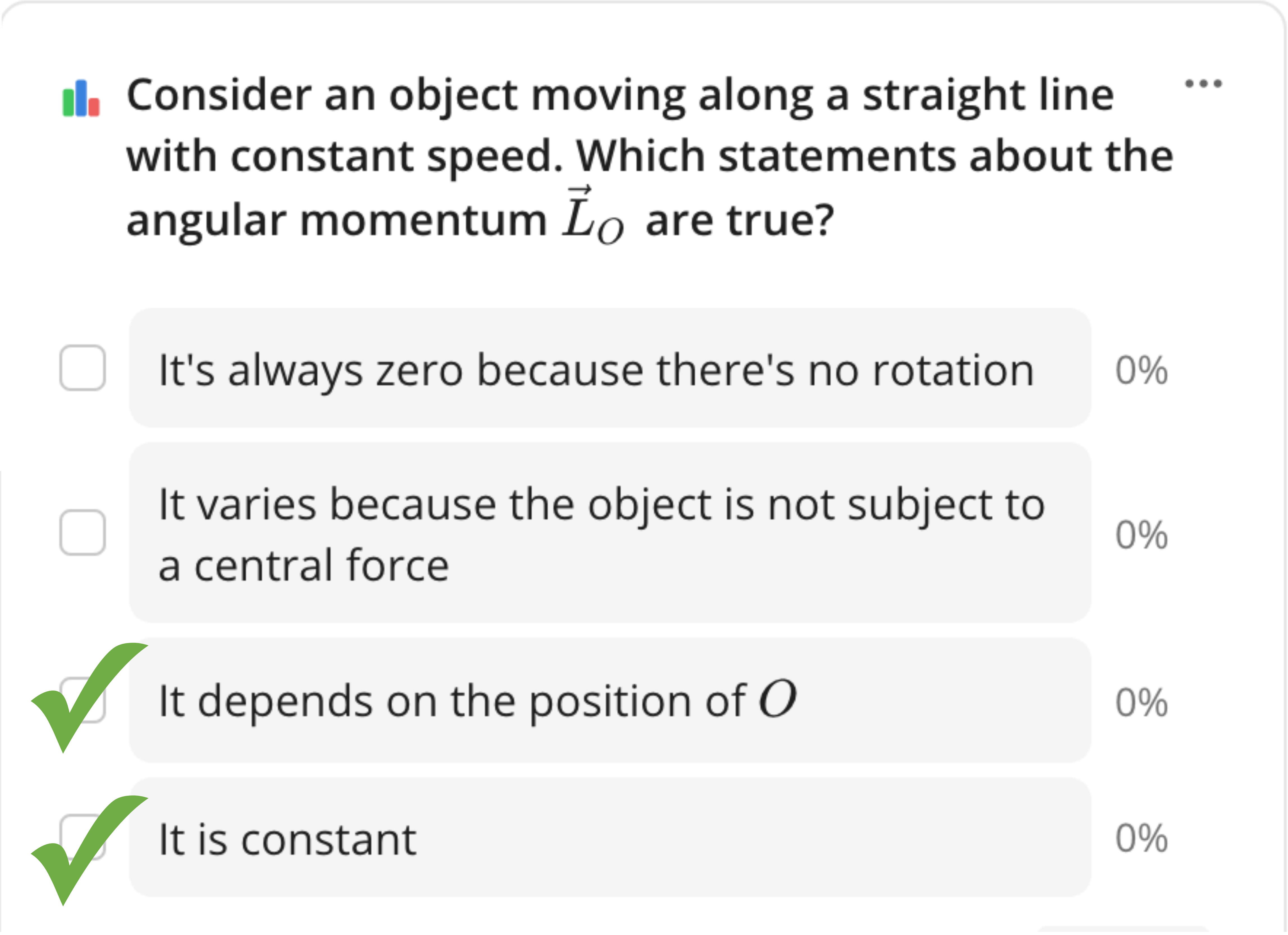

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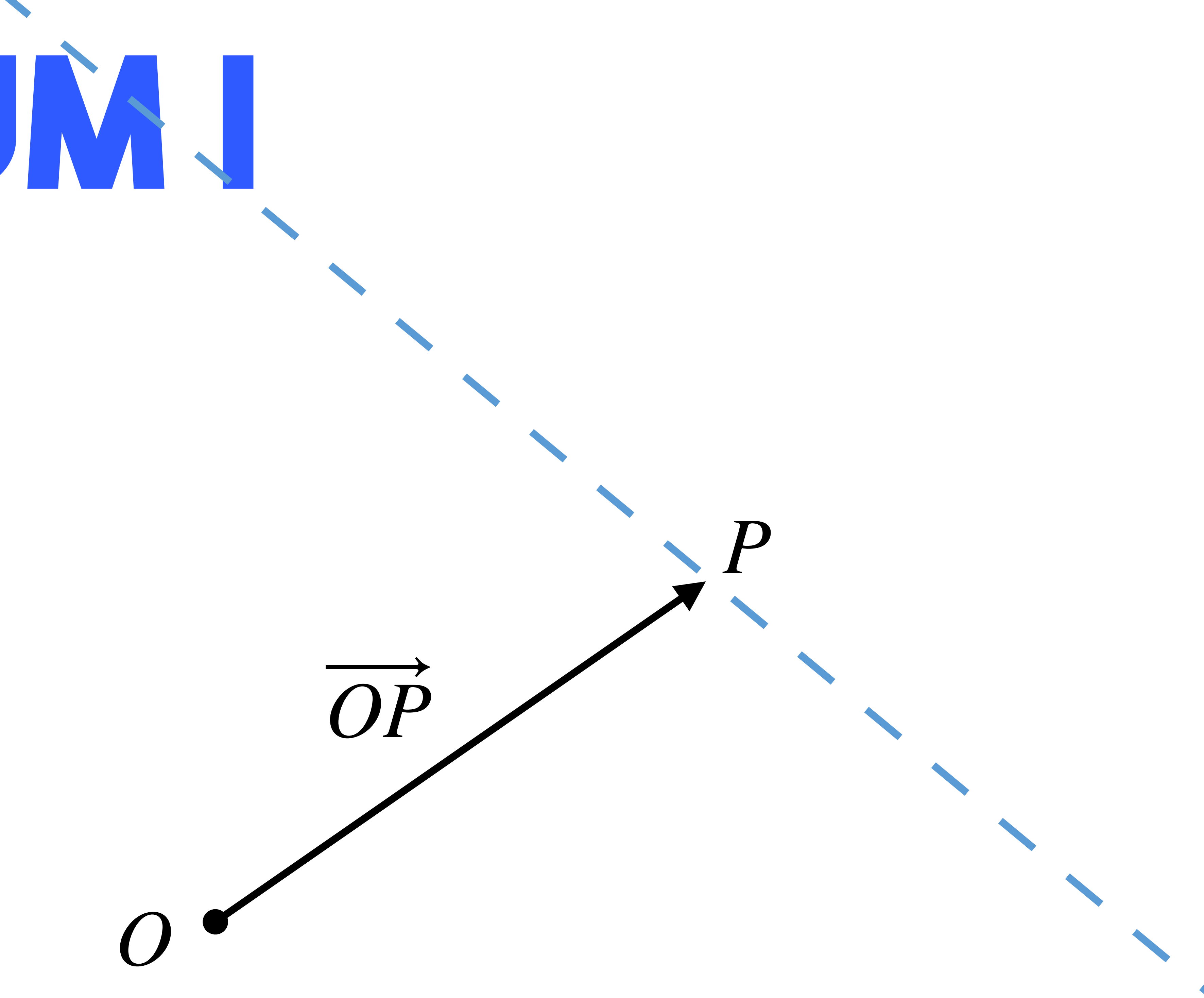

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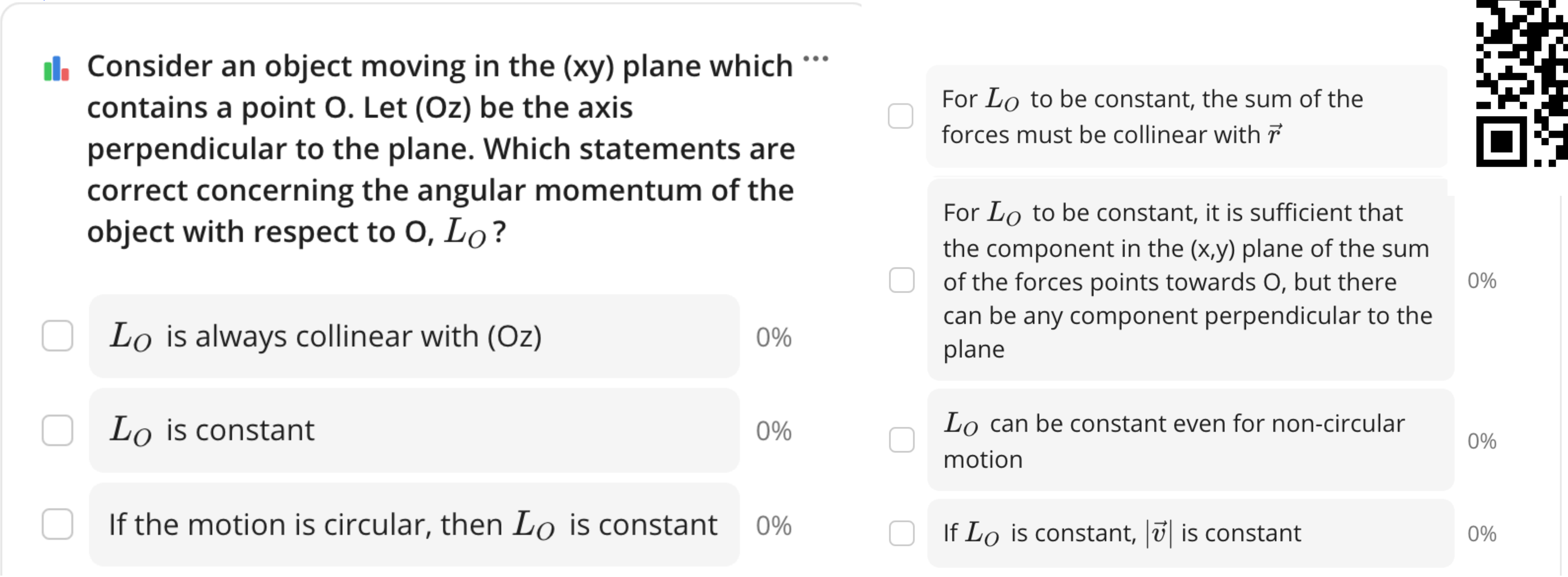

Consider an object moving in the (xy) plane which contains a point O. Let (Oz) be the axis perpendicular to the plane. Which statements are correct concerning the angular momentum of the object with respect to O, L_O?

- ☐ L_O is always collinear with (Oz) 0%
- ☐ L_O is constant 0%
- ☐ If the motion is circular, then L_O is constant 0%
- ☐ For L_O to be constant, the sum of the forces must be collinear with $\vec{r}$
- ☐ For L_O to be constant, it is sufficient that the component in the (x,y) plane of the sum of the forces points towards O, but there can be any component perpendicular to the plane 0%
- ☐ L_O can be constant even for non-circular motion 0%
- ☐ If L_O is constant, $|\vec{v}|$ is constant 0%

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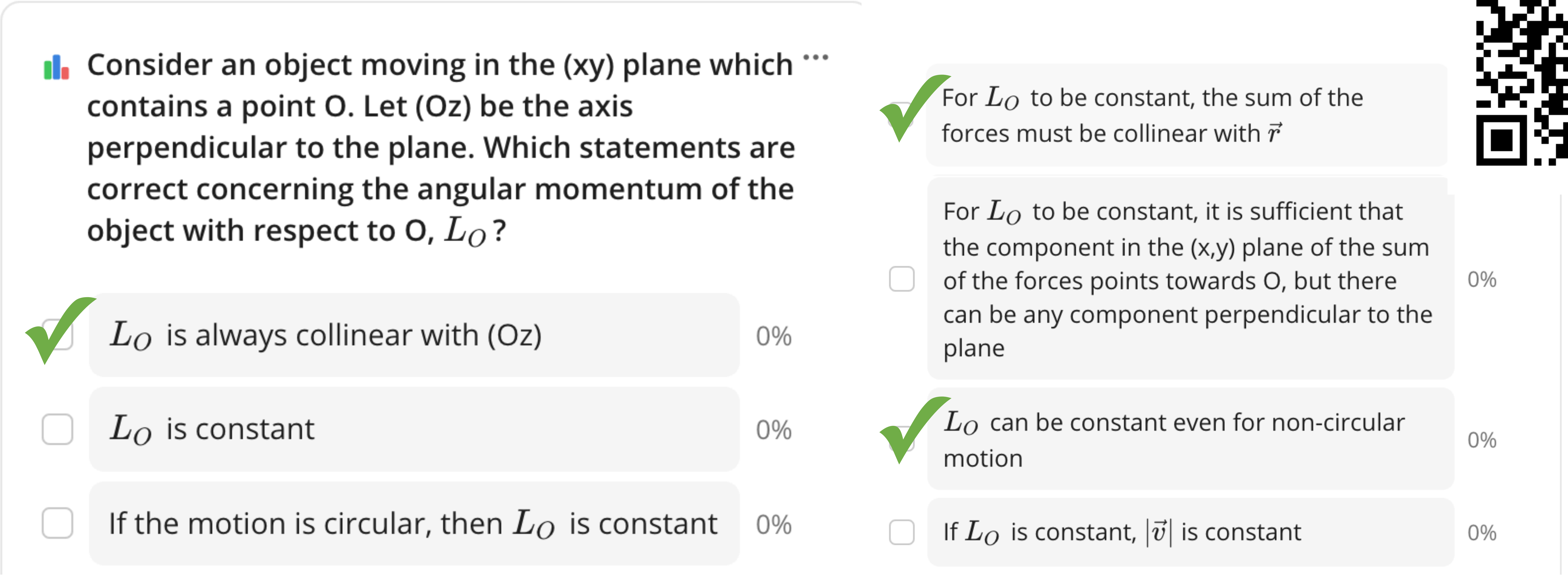

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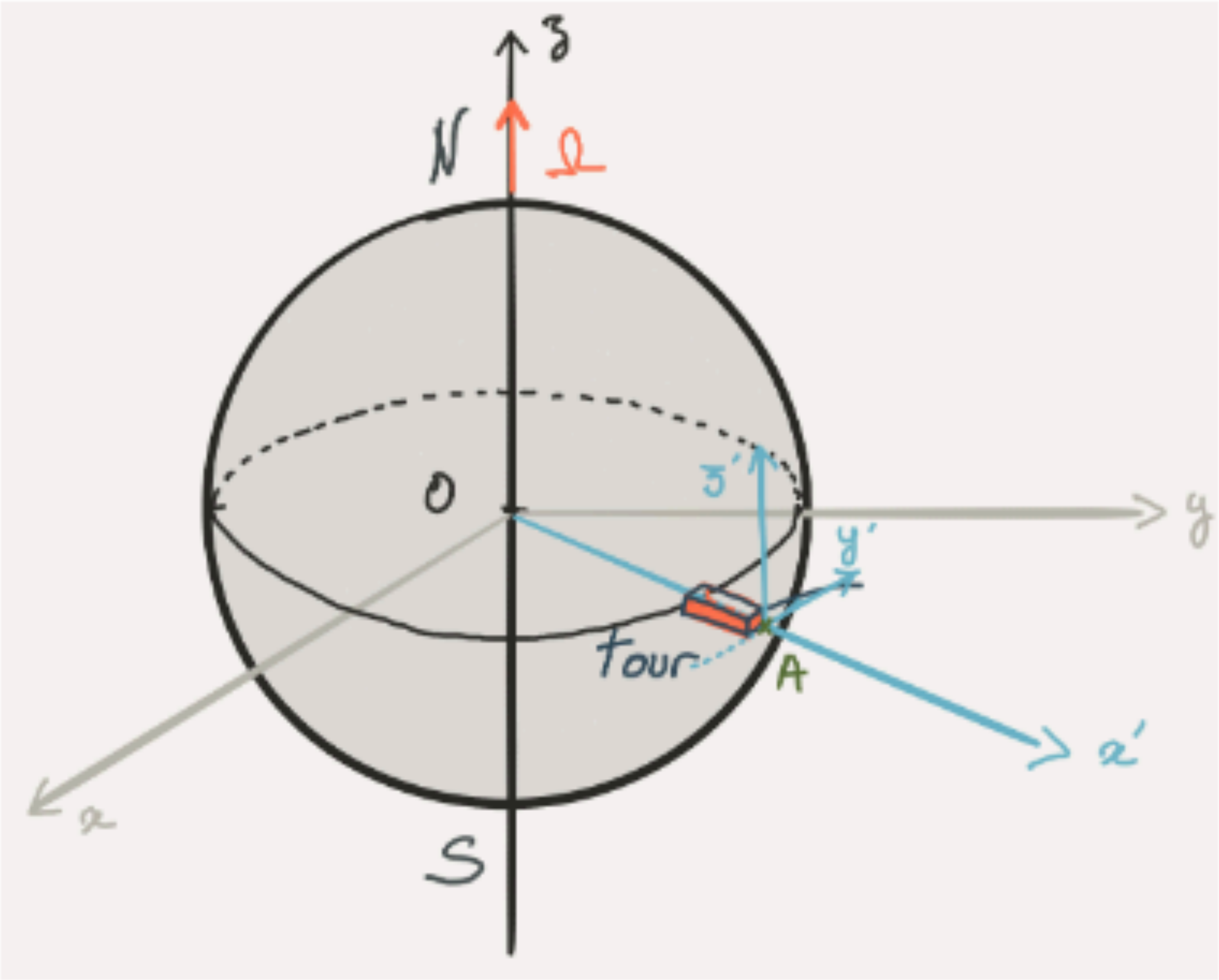
REVISITING THE TOWER FROM CH4

We consider dropping a stone from a height h from a tower located at the equator. By how much and in which direction is the stone deflected?



"Intuitive" calculation in the Galilean reference frame $\mathscr{R}$ using the fact that the top of the tower is moving faster than the ground due to the Earth's rotation

$$\vec{D} = \vec{D}_{\text{stone}} - \vec{D}_{\text{ground}} = h\,\Omega\sqrt{\frac{2h}{g}}\,\vec{e}_y$$

Calculation in the rotating reference frame $\mathscr{R}'$ using the Coriolis force

$$\vec{D} = y'(t_f)\,\vec{e}_y = \frac{2}{3}\,\Omega\,h\sqrt{\frac{2h}{g_{\text{eff}}}}\,\vec{e}_y$$

REVISITING THE TOWER: INTUITIVE CALCULATION

What went wrong?

We assume that the stone is given an initial constant velocity due to the Earth's rotation (along $\vec{e}_y$):

$$\text{stone}: \quad \vec{v}_i = (R+h)\,\Omega\,\vec{e}_y \tag{4.20}$$

$$\text{ground}: \quad \vec{v}_s = R\,\Omega\,\vec{e}_y \tag{4.21}$$

$$t_{\text{fall}} = \sqrt{\frac{2h}{g}} \tag{4.22}$$

The displacement vectors (during the fall time) $\vec{D}_{\text{stone}}$ and $\vec{D}_{\text{ground}}$ are along $\vec{e}_y$:

$$\vec{D}_{\text{stone}} = \vec{v}_i\,t_{\text{fall}} = (R+h)\,\Omega\sqrt{\frac{2h}{g}}\,\vec{e}_y \tag{4.23}$$

$$\vec{D}_{\text{ground}} = \vec{v}_s\,t_{\text{fall}} = R\,\Omega\sqrt{\frac{2h}{g}}\,\vec{e}_y \tag{4.24}$$

The deviation is therefore given by:

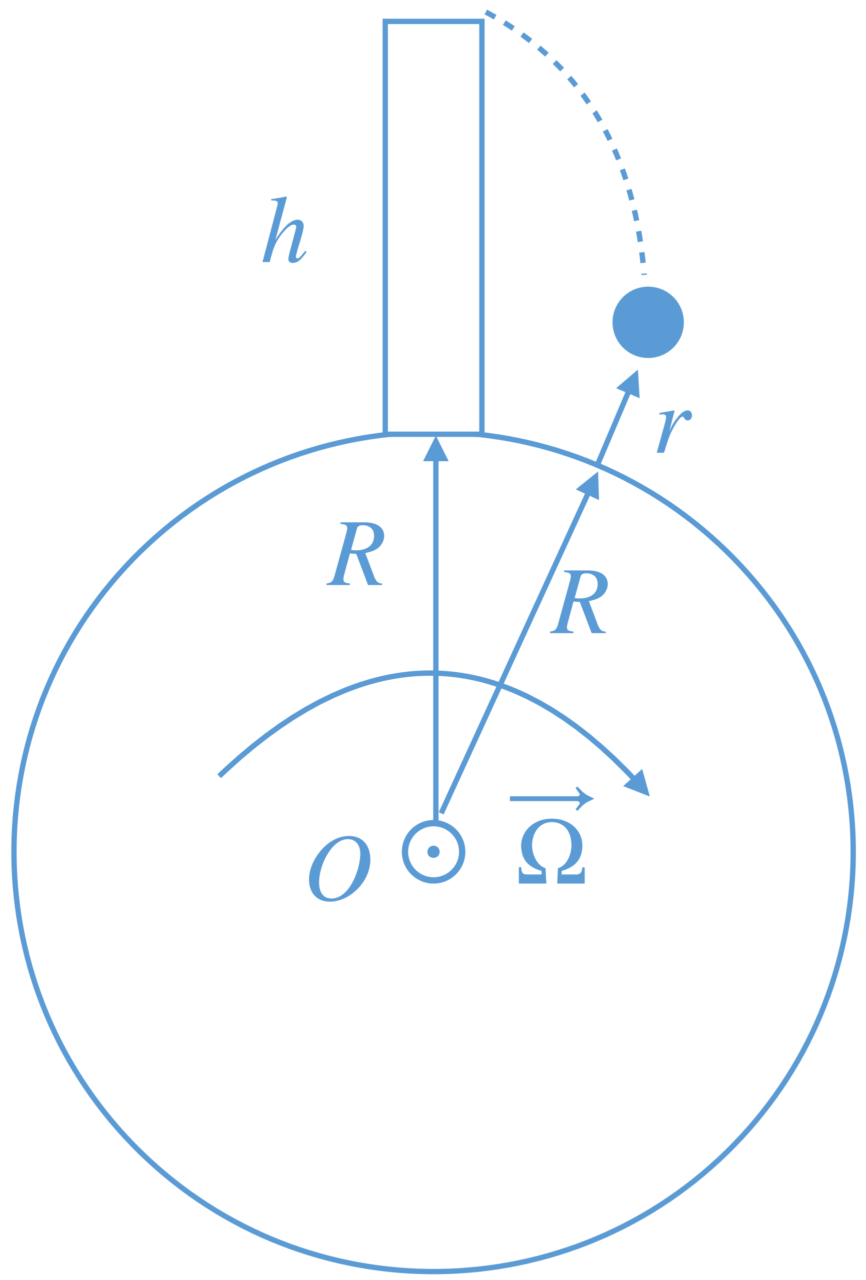
$$\vec{D} = \vec{D}_{\text{stone}} - \vec{D}_{\text{ground}} = h\,\Omega\sqrt{\frac{2h}{g}}\,\vec{e}_y \tag{4.25}$$

REVISITING THE TOWER: ANGULAR MOMENTUM

Initial angular momentum $\overrightarrow{L}_o^i = m(R+h)^2\overrightarrow{\Omega} = m(R+h)^2\Omega E\vec{e}_z$

Angular momentum as a function of height r $\overrightarrow{L}_o = m(R+r)^2\omega(r)\vec{e}_z$

When the stone is dropped, it will only move under gravity which is a central force. Therefore angular **momentum is conserved**



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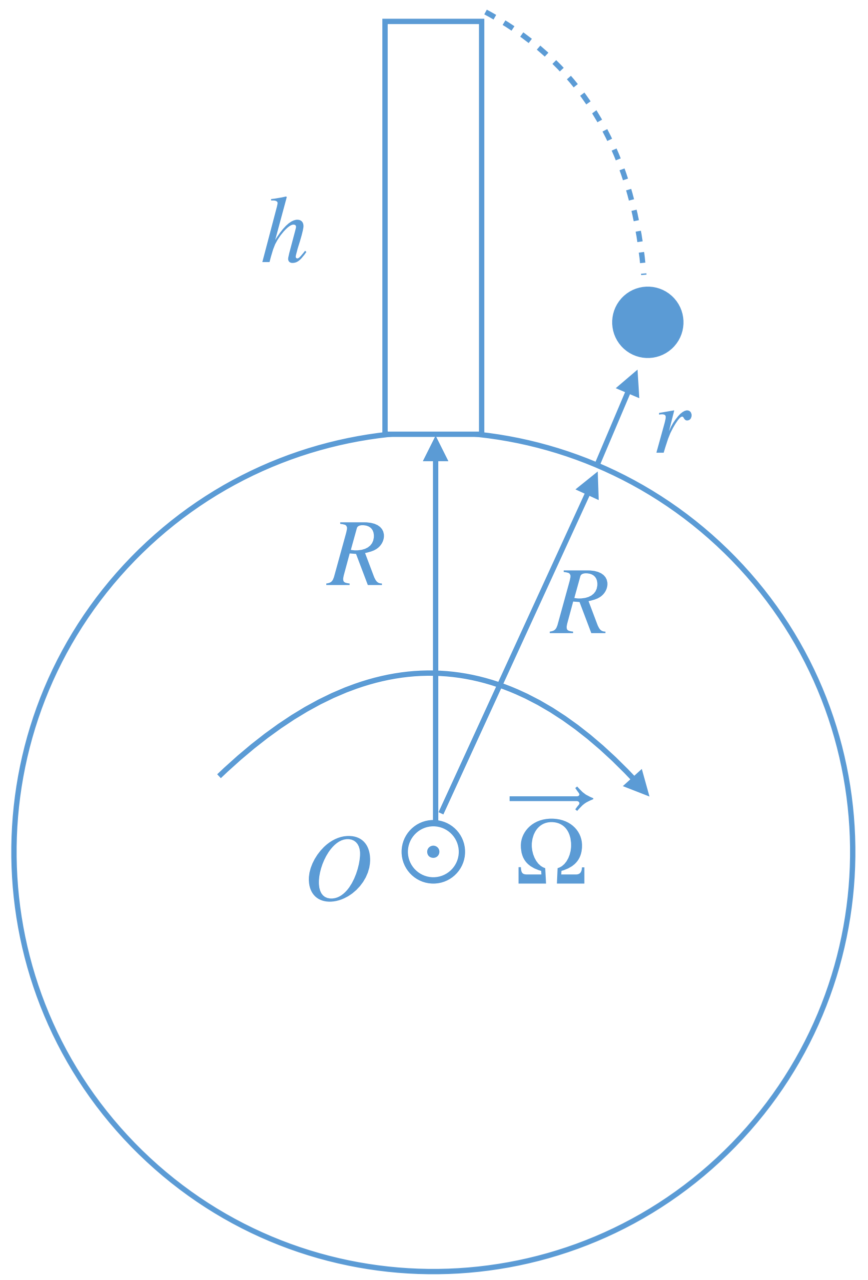
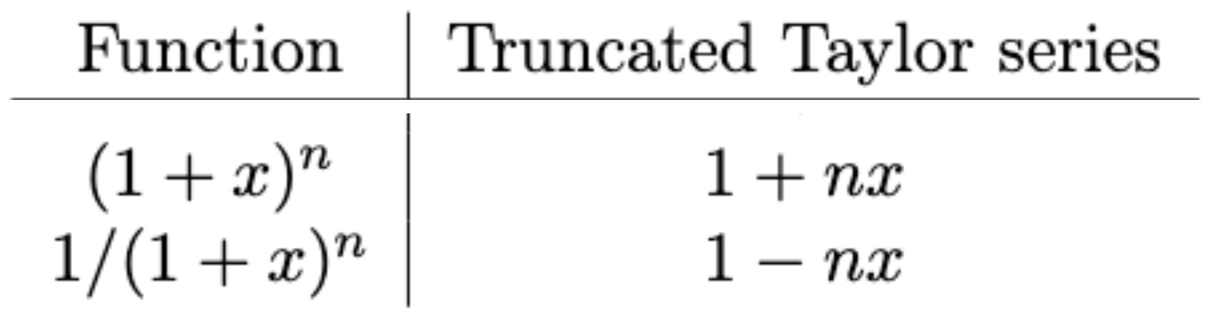


$$m(R+r)^2\omega(r)\vec{e}_z = m(R+h)^2\Omega\vec{e}_z$$

$$\omega(r) = \frac{(R+h)^2}{(R+r)^2}\Omega = \frac{(1+h/R)^2}{(1+r/R)^2}\Omega$$



Function	Truncated Taylor series
$(1+x)^n$	$1+nx$
$1/(1+x)^n$	$1-nx$

$$\omega(r) \approx (1+2h/R)(1-2r/R)\Omega \approx \Omega\left(1+\frac{2}{R}(h-r)\right)$$

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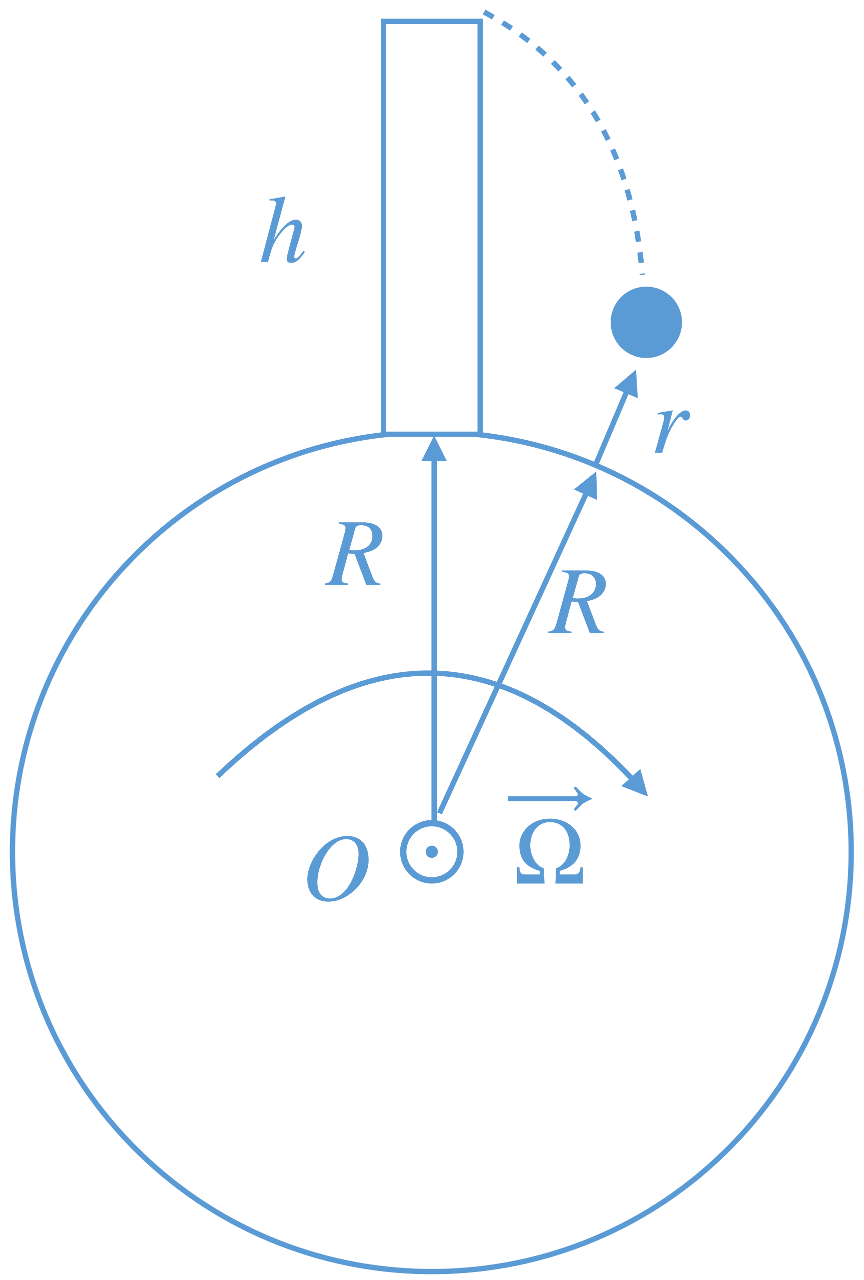

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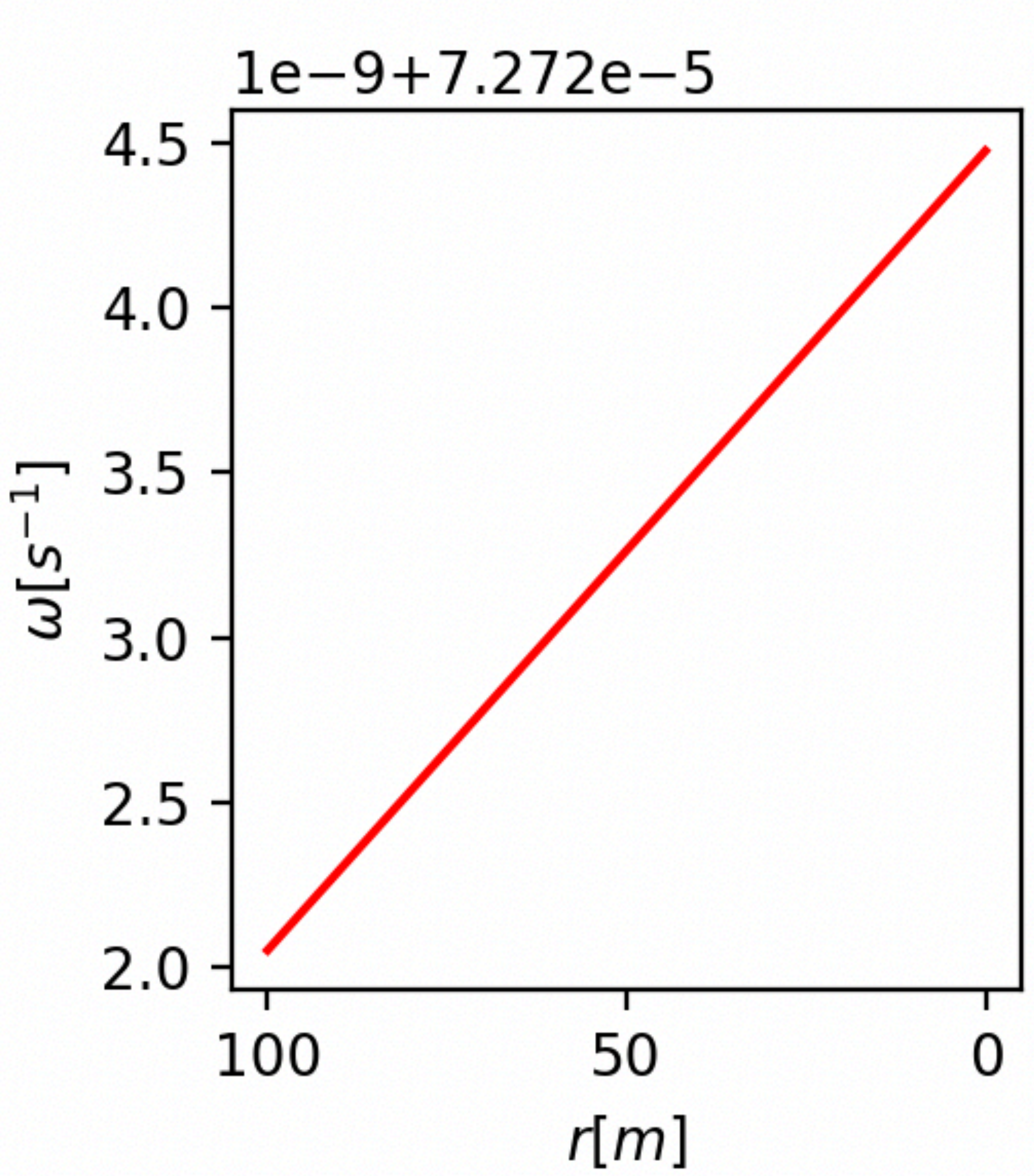
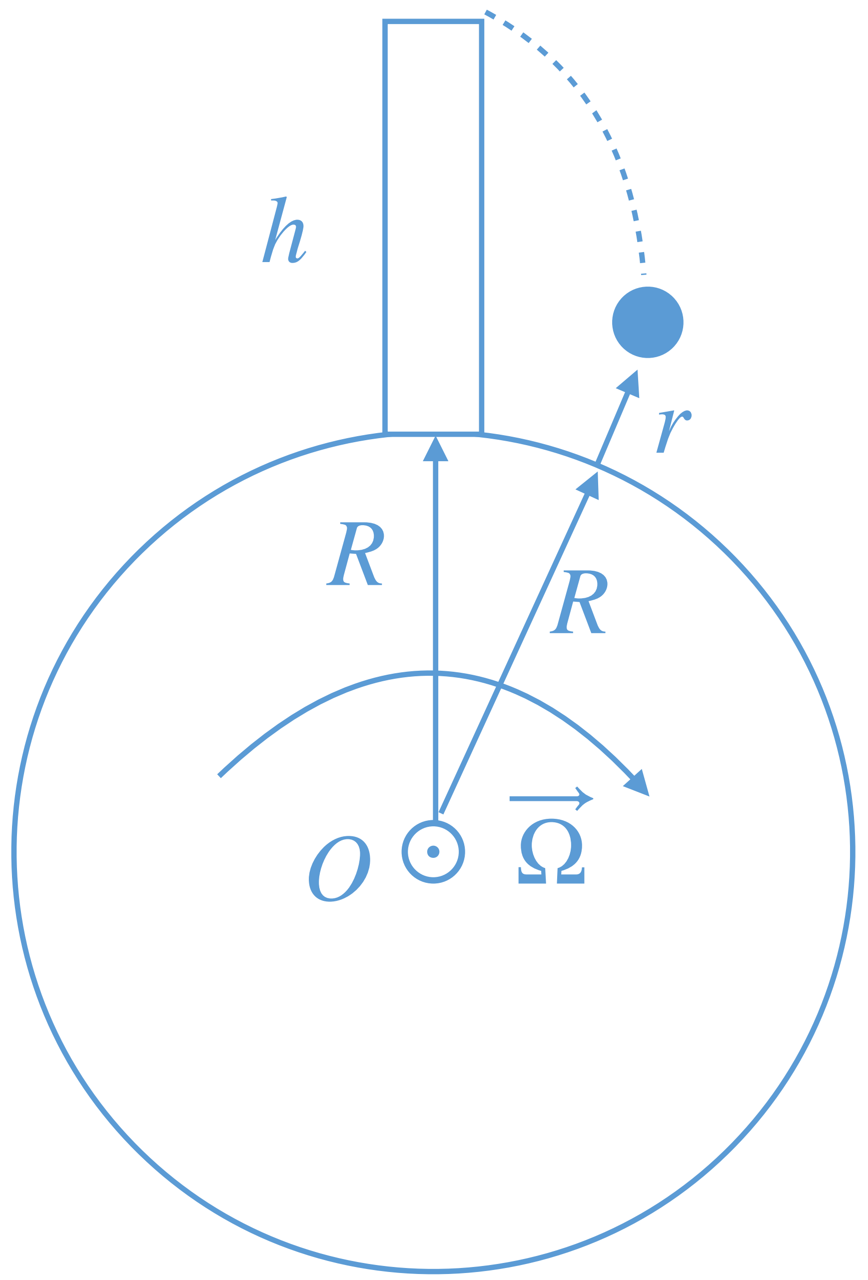
Angular velocity increases as the stone falls



REVISITING THE TOWER: ANGULAR MOMENTUM



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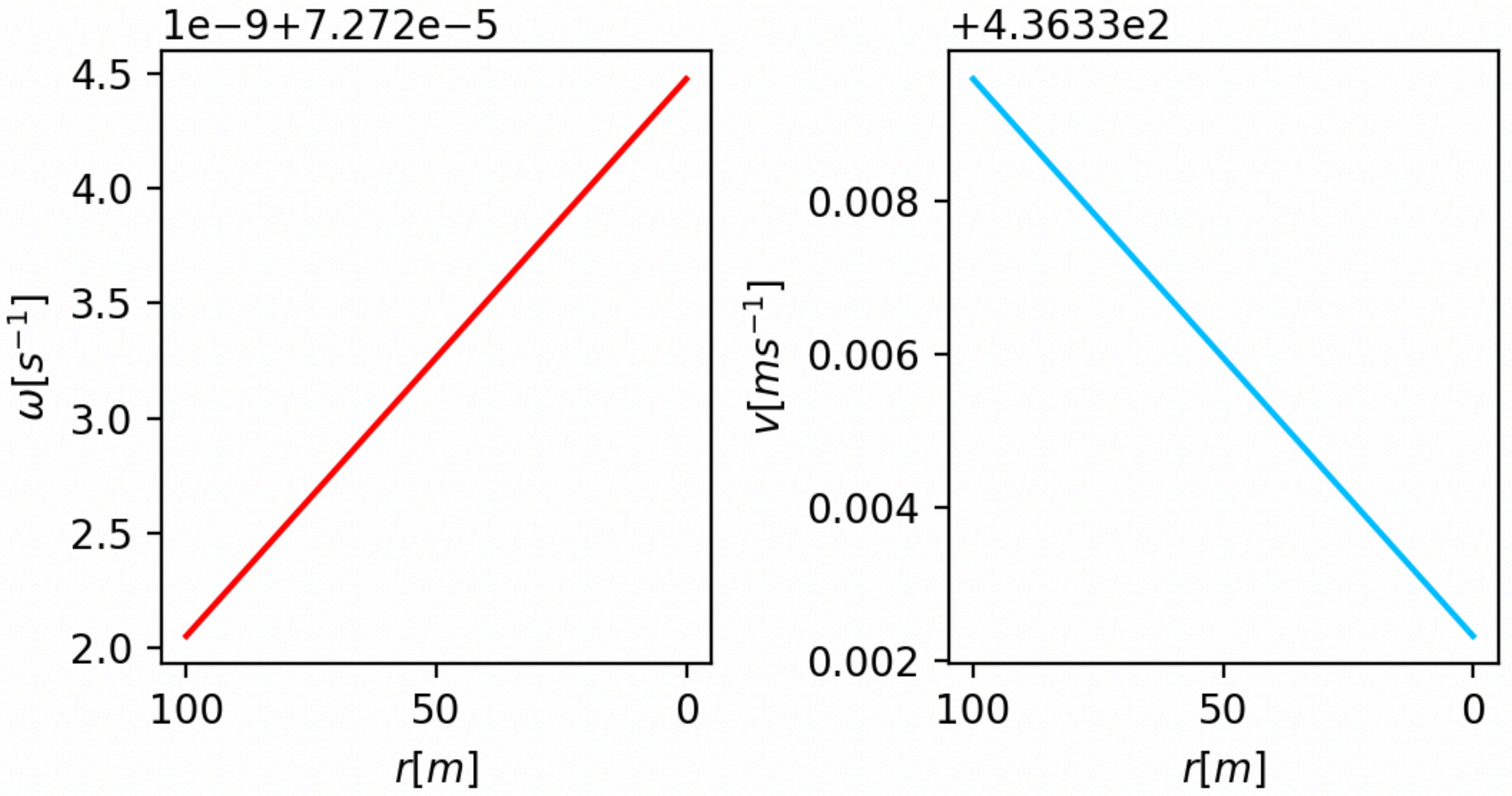
Angular velocity increases as the stone falls

$$v(r) = (R+r)\omega(r) = (R+r)\Omega\left(1+\frac{2}{R}(h-r)\right)$$

$$v(r) = \Omega\left(R-r+2h+2rh/R-2r^2/R\right)$$

Horizontal velocity decreases as the stone falls



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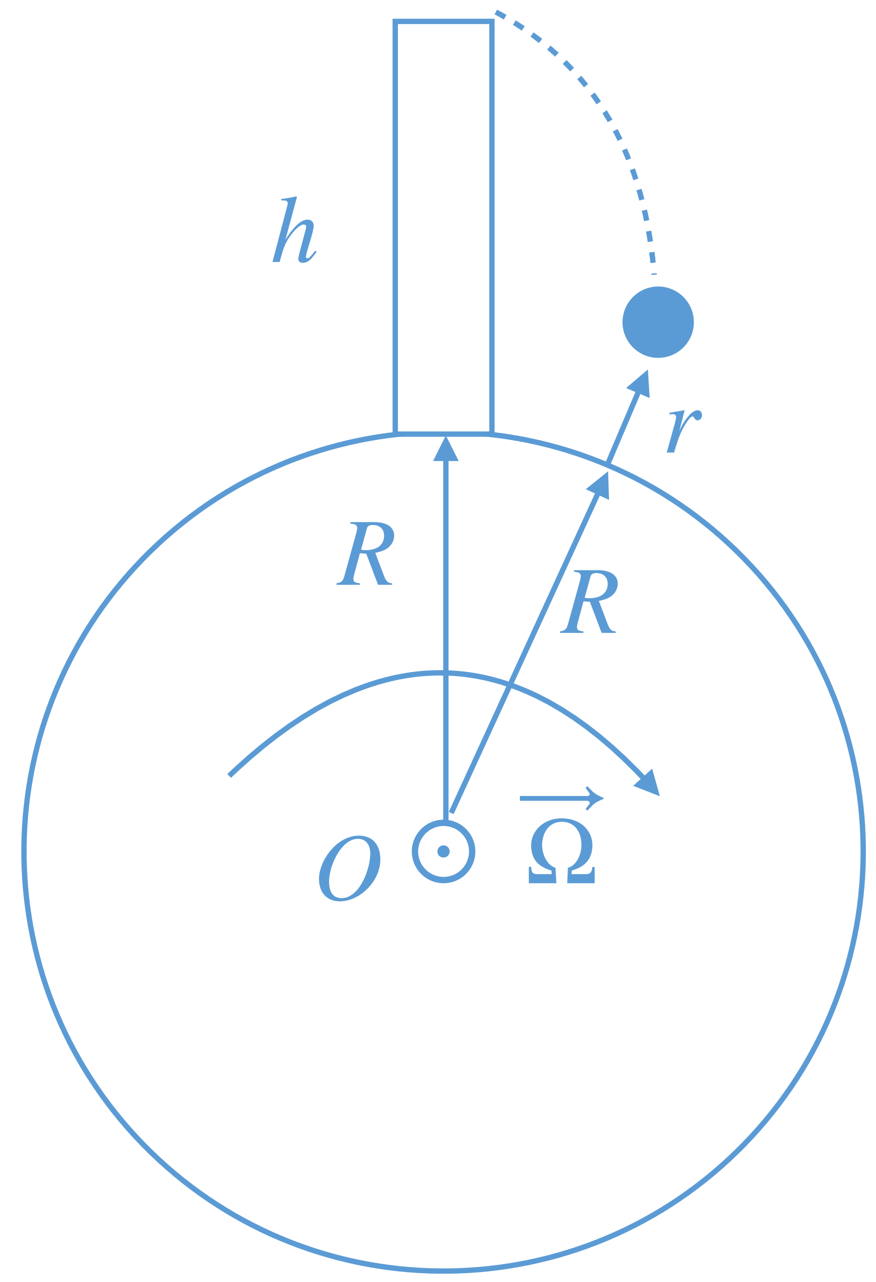

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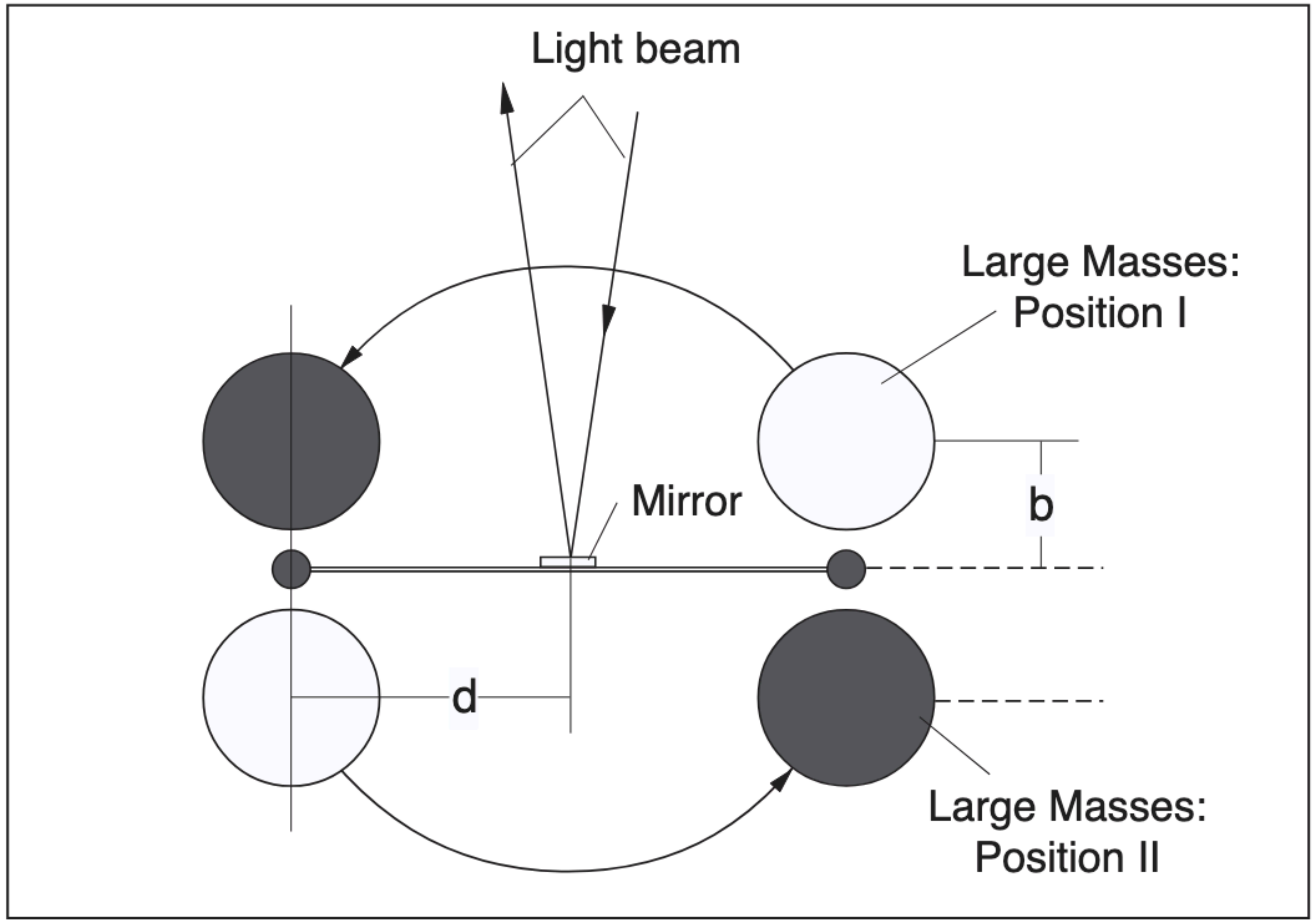
Bonus exercise: calculate the displacement of the stone from the base of the tower, using conservation of angular momentum

Hint: from ballistics $t_f = \sqrt{\frac{2h}{g}}$ $r(t) = h - \frac{1}{2}gt^2$

GRAVITATIONAL FORCE

$$\vec{F}_G = -\frac{GMm}{r^2}\vec{e}_r$$

CAVENDISH BALANCE



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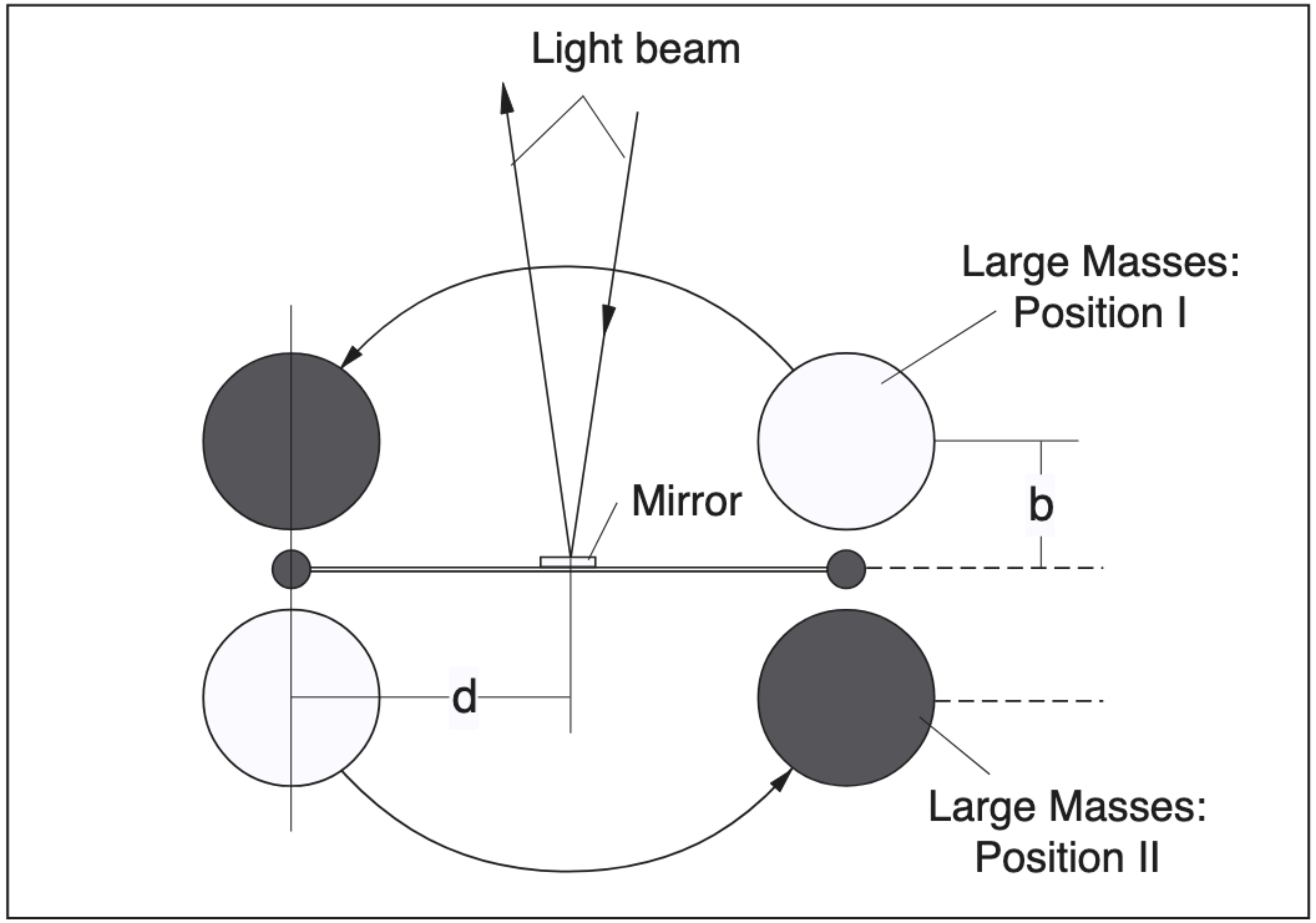

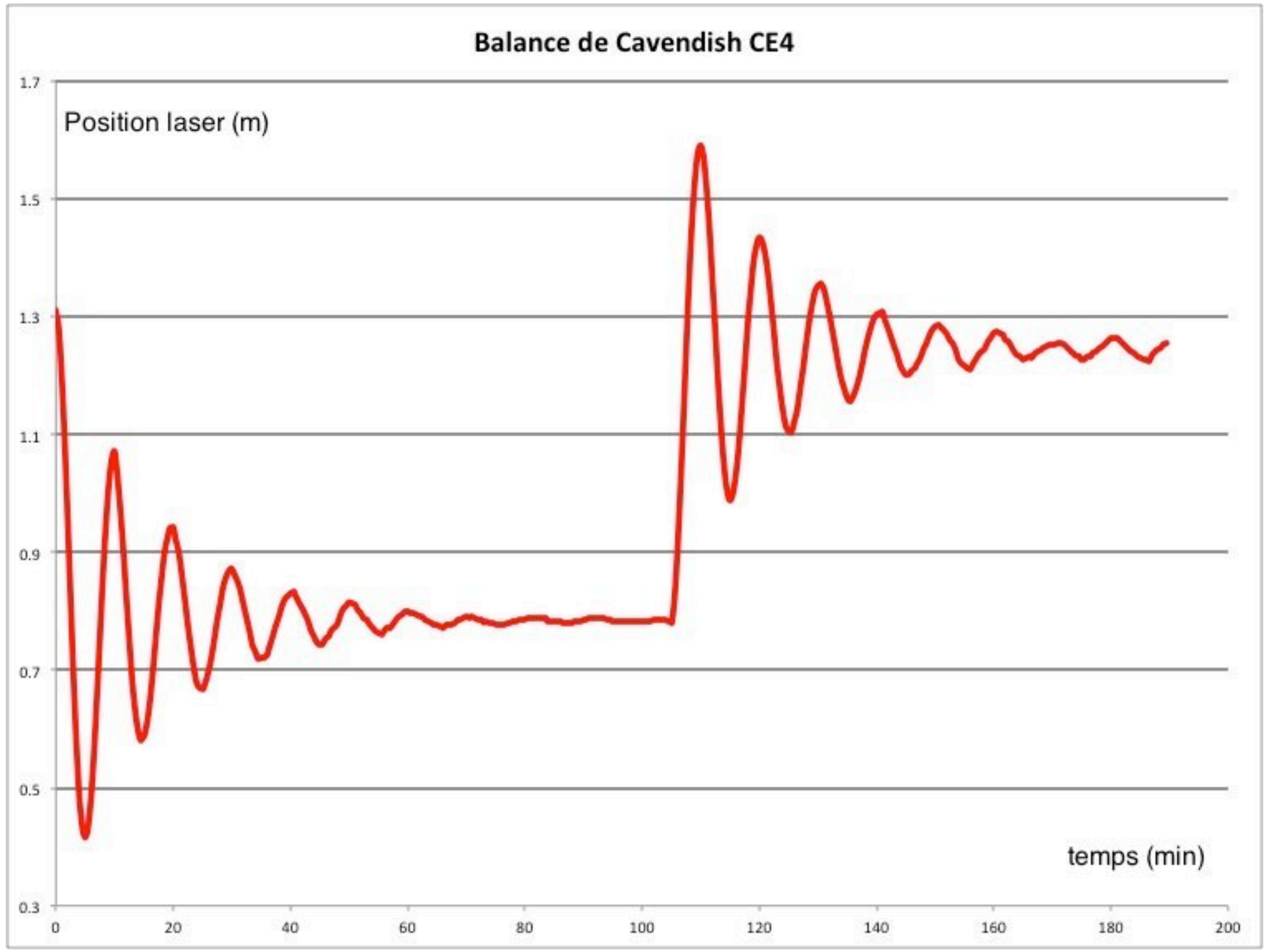

GRAVITATIONAL POTENTIAL ENERGY

$$E_P^G = -\frac{GMm}{r}$$

kinetic energy

gravitational potential energy

$$E_m = \frac{1}{2}m\dot{r}^2 + \frac{L_O^2}{2mr^2} - \frac{GMm}{r}$$

Only part of the kinetic energy

Effective gravitational potential energy

https://www.physicswithelliot.com/effective-potential-orbits


Lagoon Nebula
Galactic Center
Cat's paw Nebula
Antares


APEX+Planck/ATLASGAL/Submillimetre
NGC 6357
NGC 6334
Galactic Centre
M20
M8
Spitzer/GLIMPSE/Infrared
NGC 6357
NGC 6334
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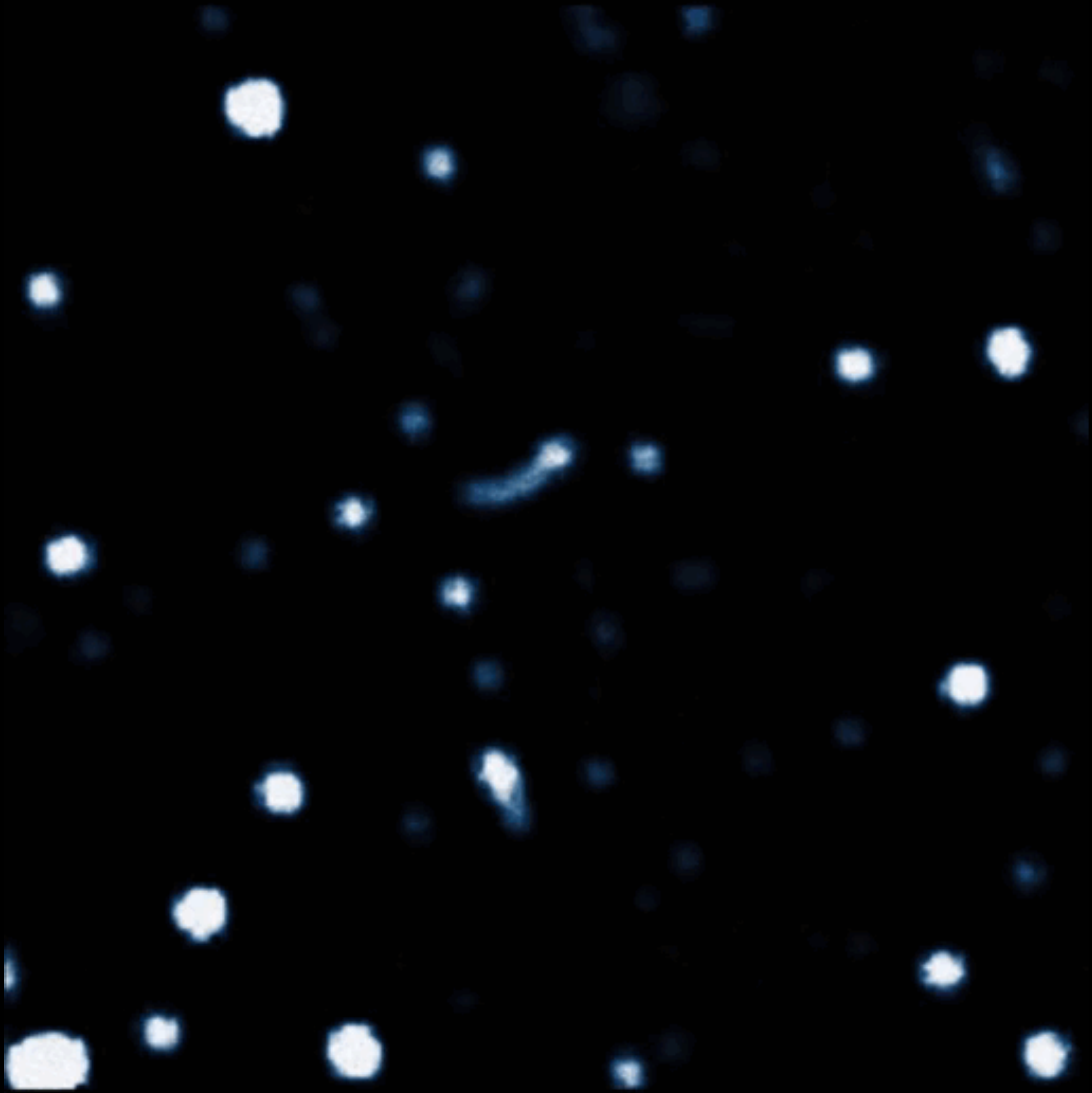

Model of stellar orbits:
https://www.youtube.com/watch?v=AZhUQl-wmq0

Want to find M in terms of T, v, and G

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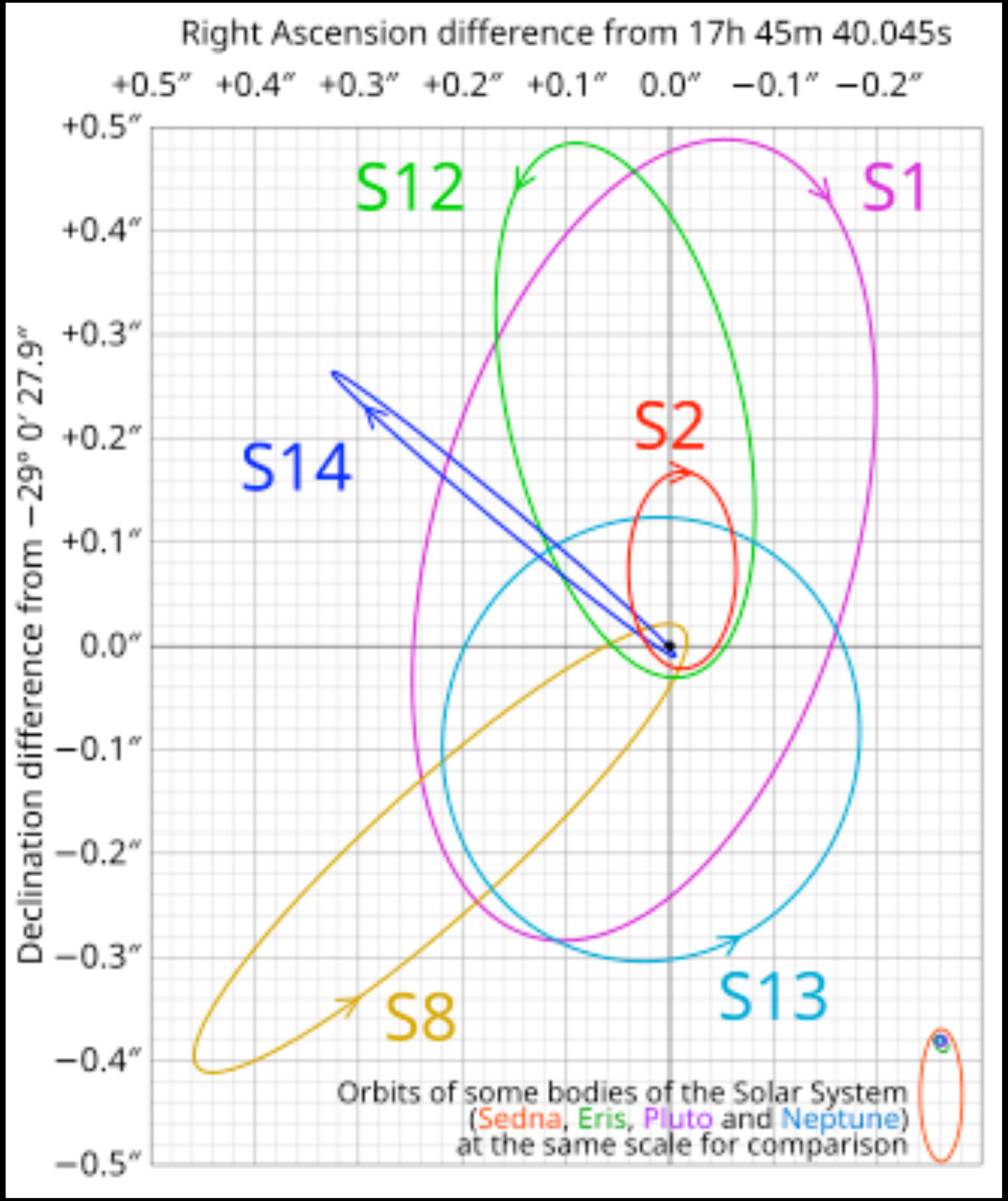

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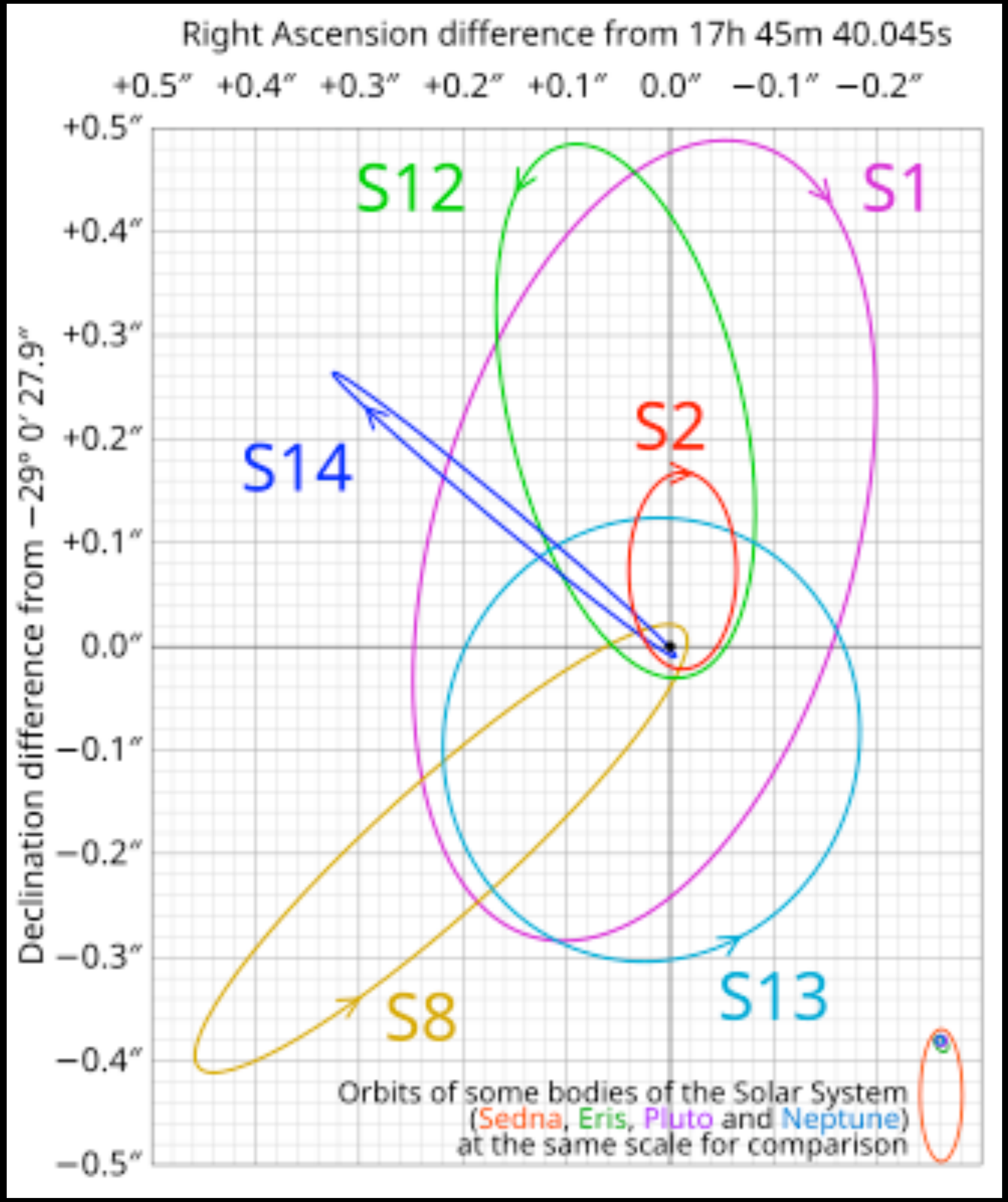

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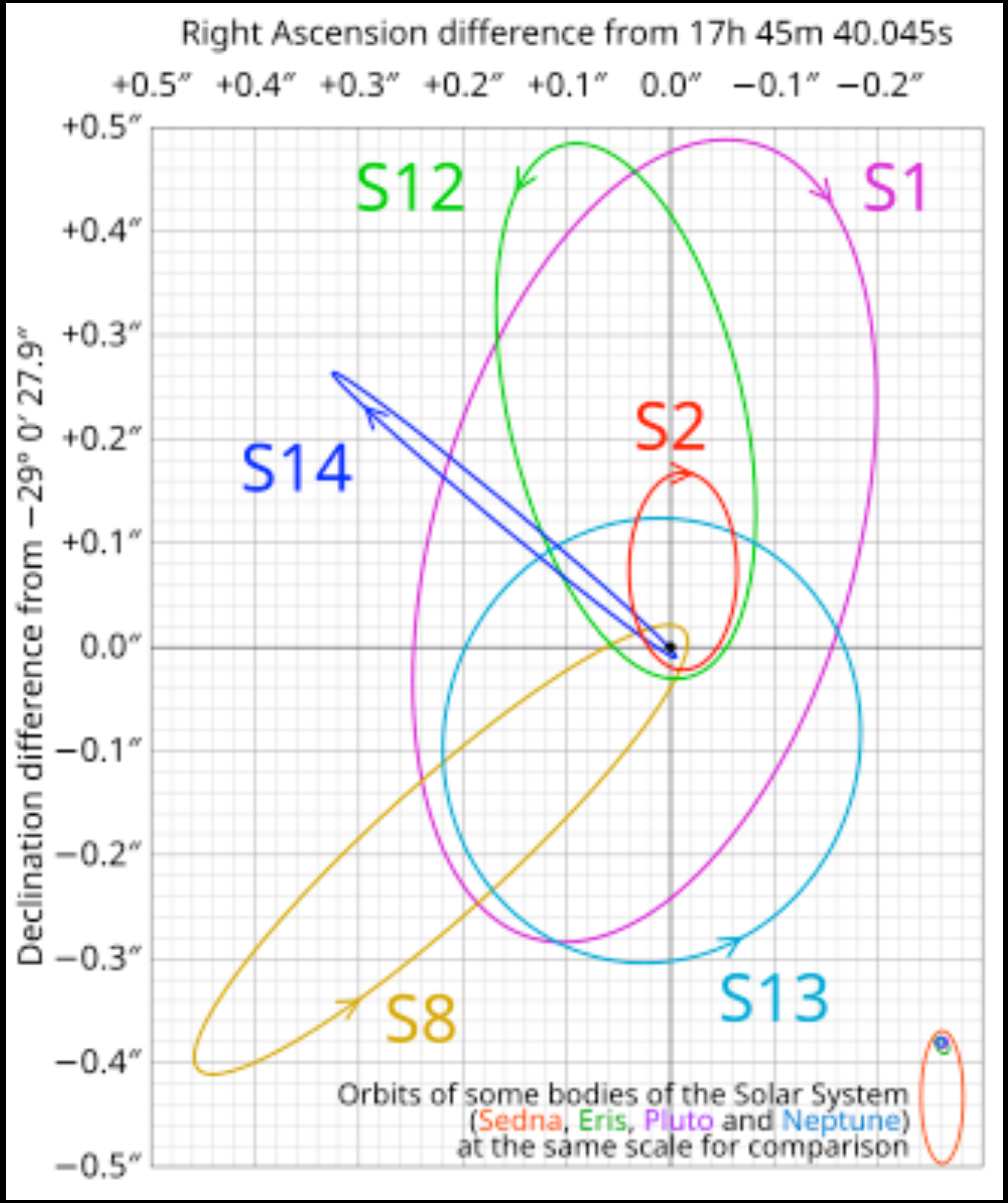

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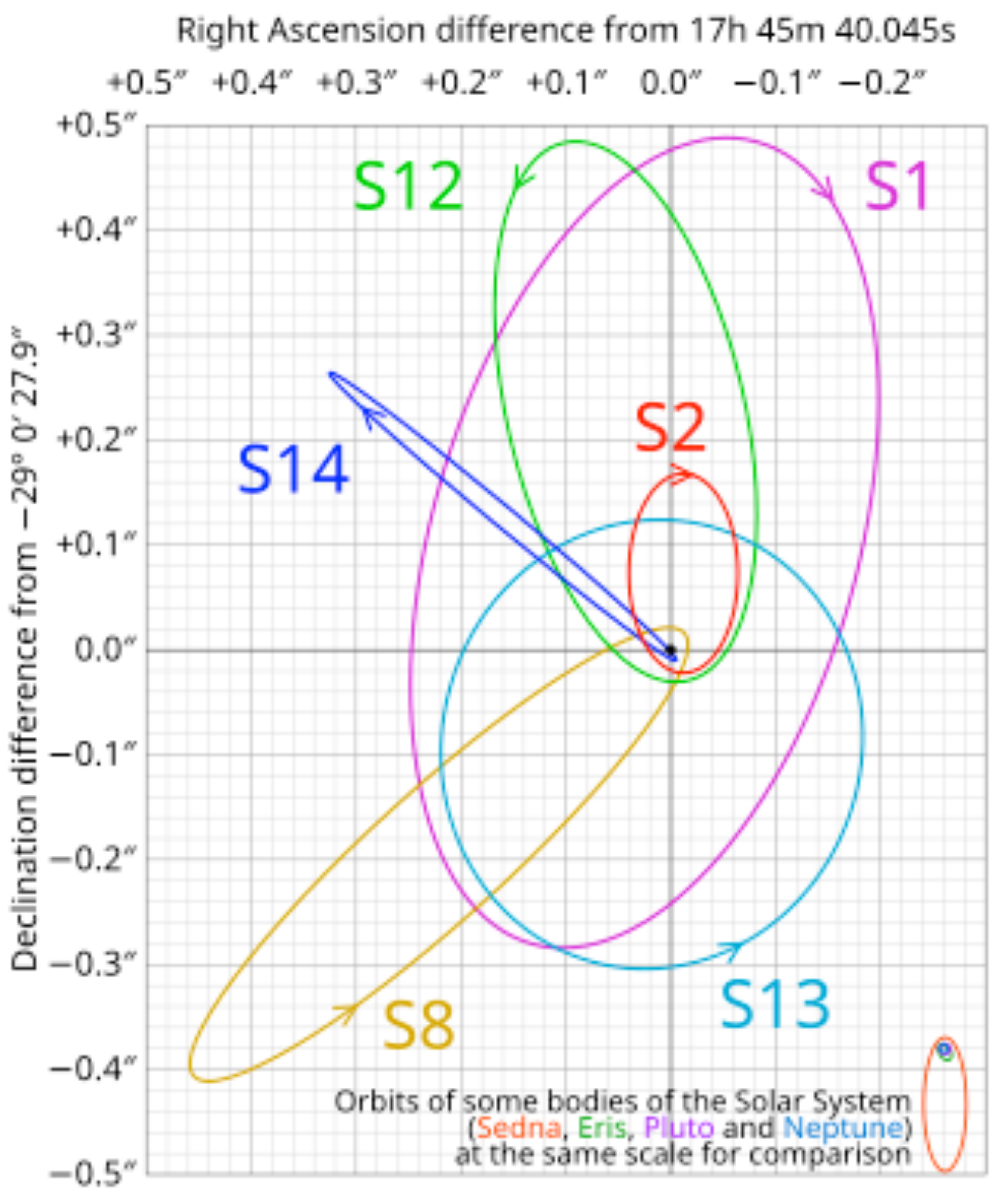
$$\implies M = \frac{v^2}{G}\frac{Tv}{2\pi} = \boxed{\frac{v^3T}{2\pi G}}$$



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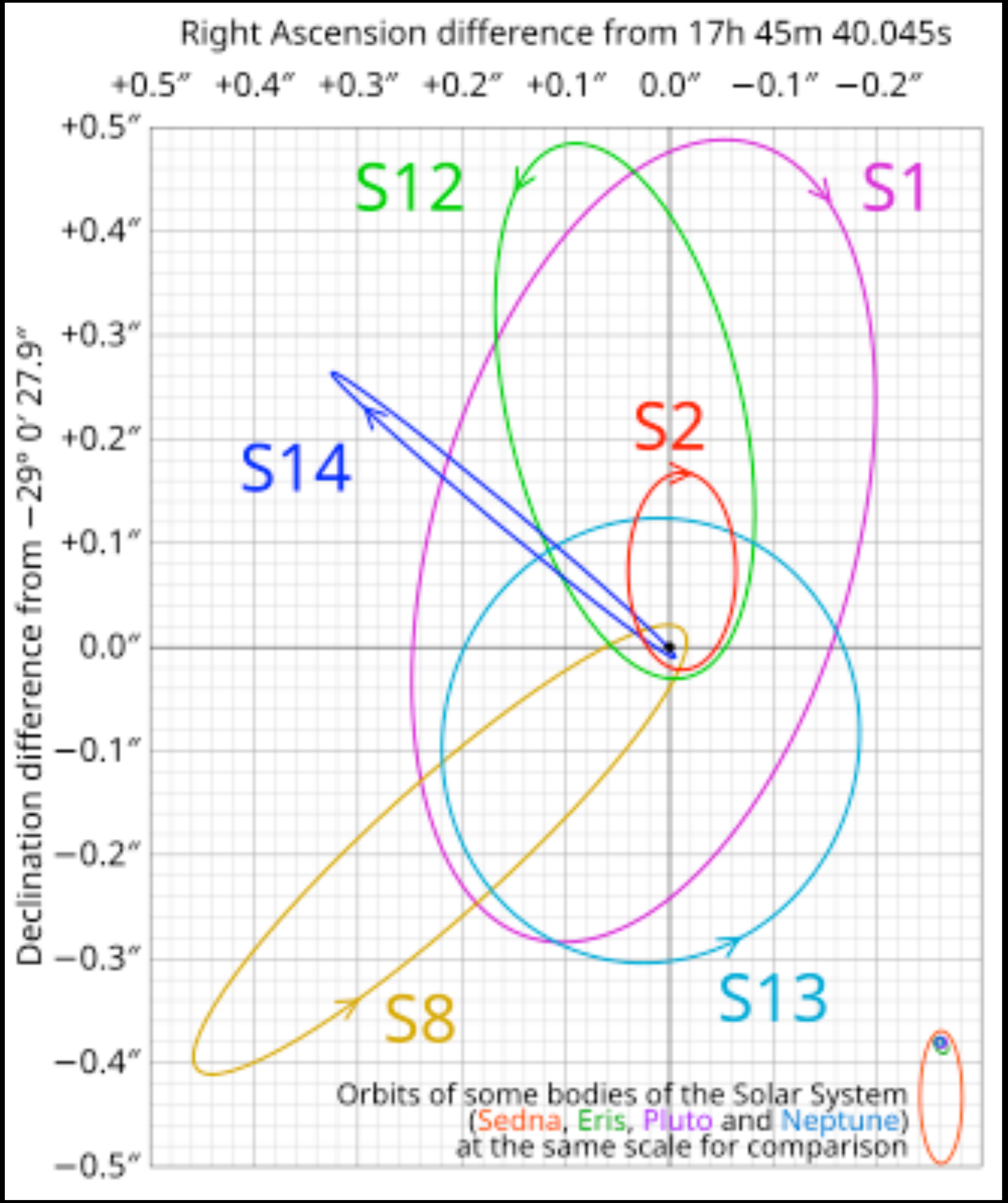

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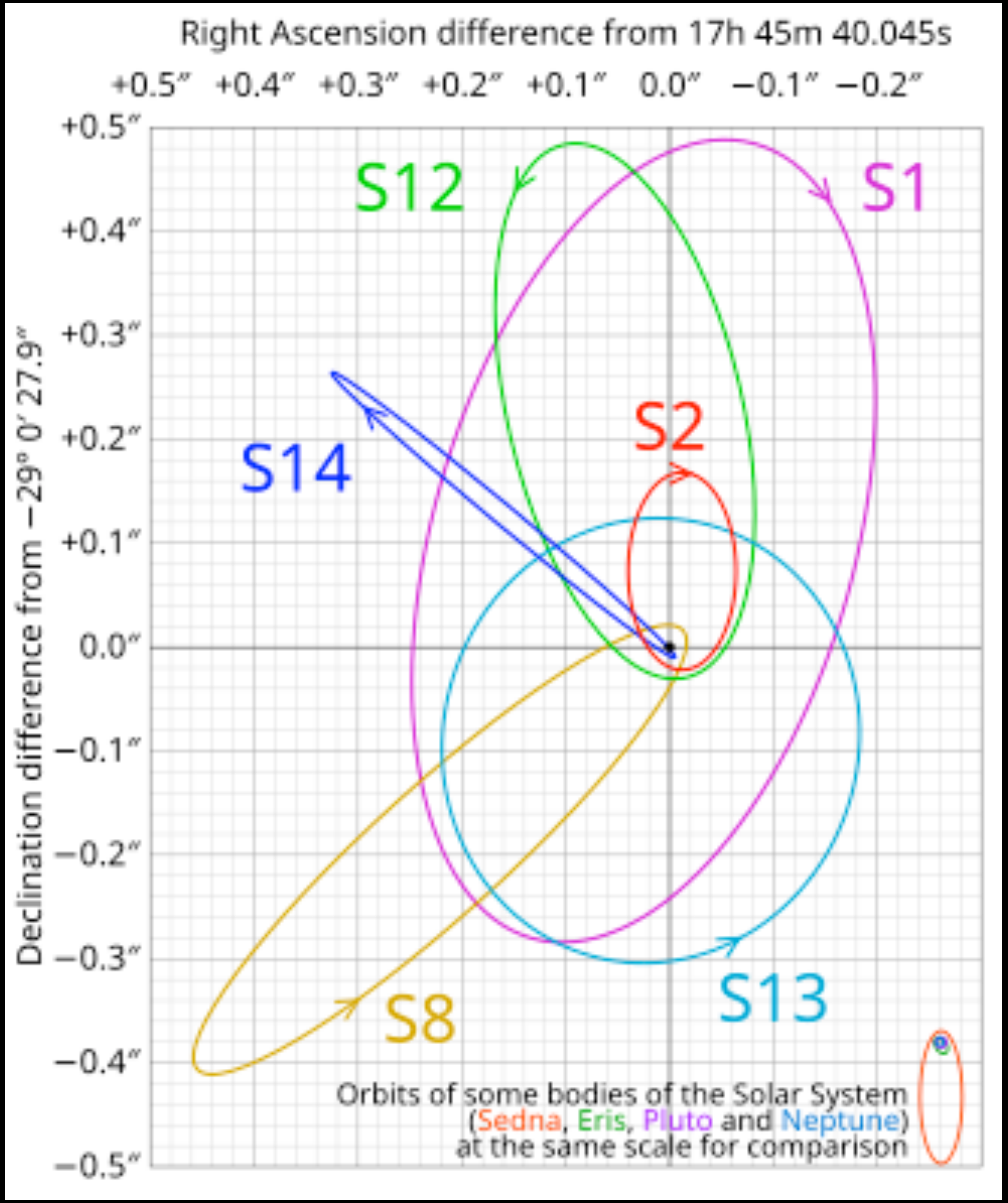
$$[M] = \mathrm{kg},\ [G] = \mathrm{m^3kg^{-1}s^{-2}},\ [v] = \mathrm{m\ s^{-1}},\ [T] = \mathrm{s}$$

$$\left[\frac{v^3T}{2\pi G}\right] = \frac{\mathrm{m^3s^{-3}s}}{\mathrm{m^3kg^{-1}s^{-2}}} = \mathrm{kg} \quad ✓$$



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4 million solar masses for Sagittarius A*

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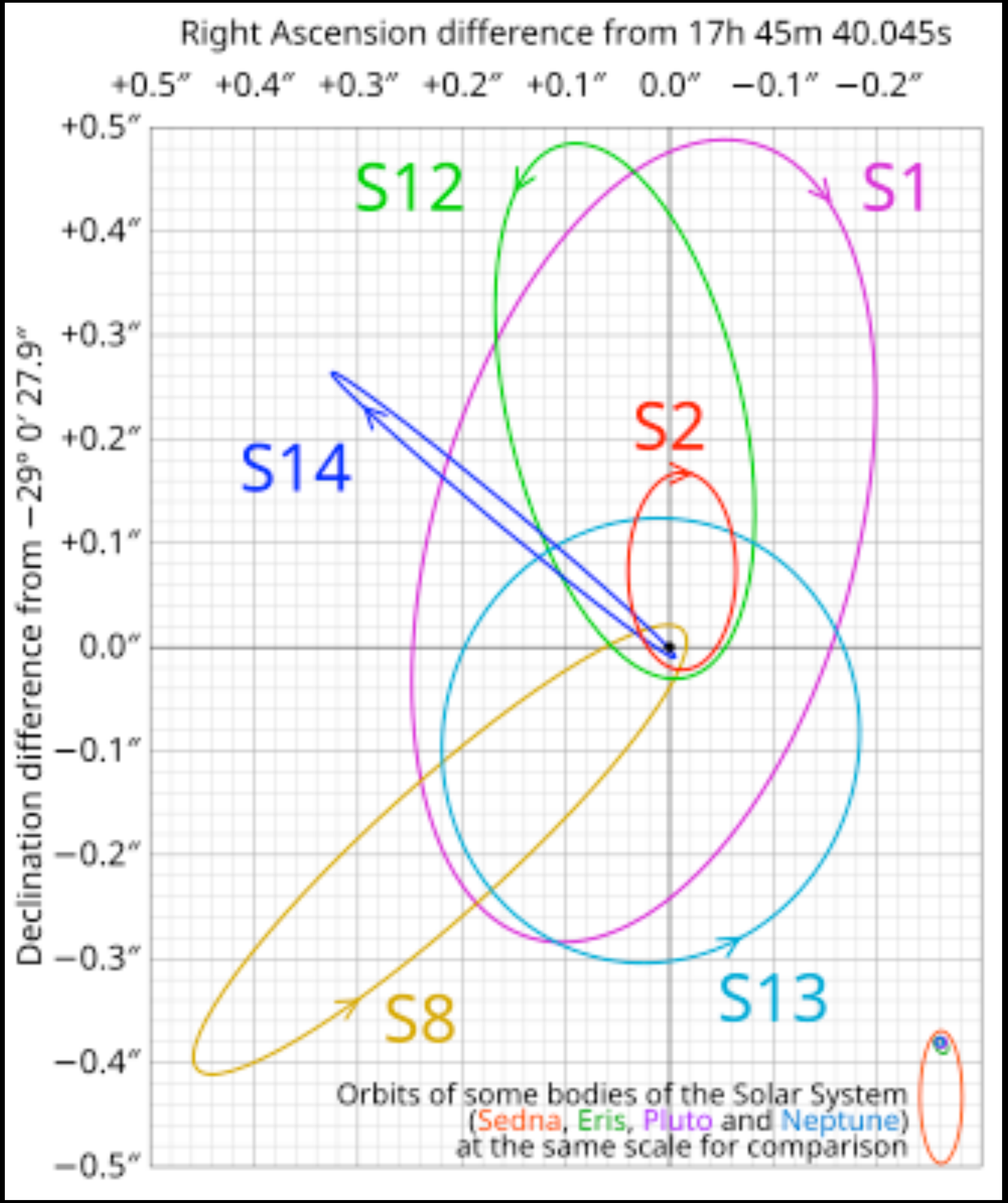
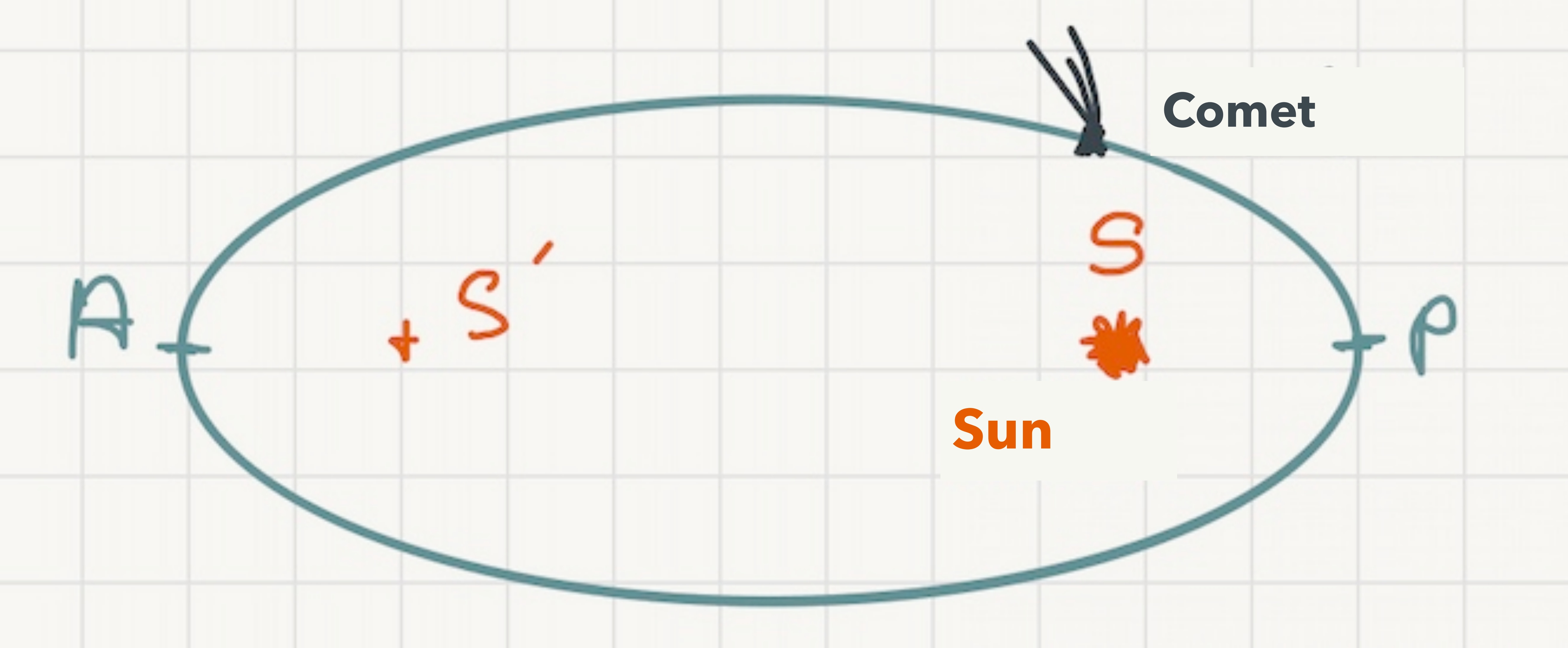
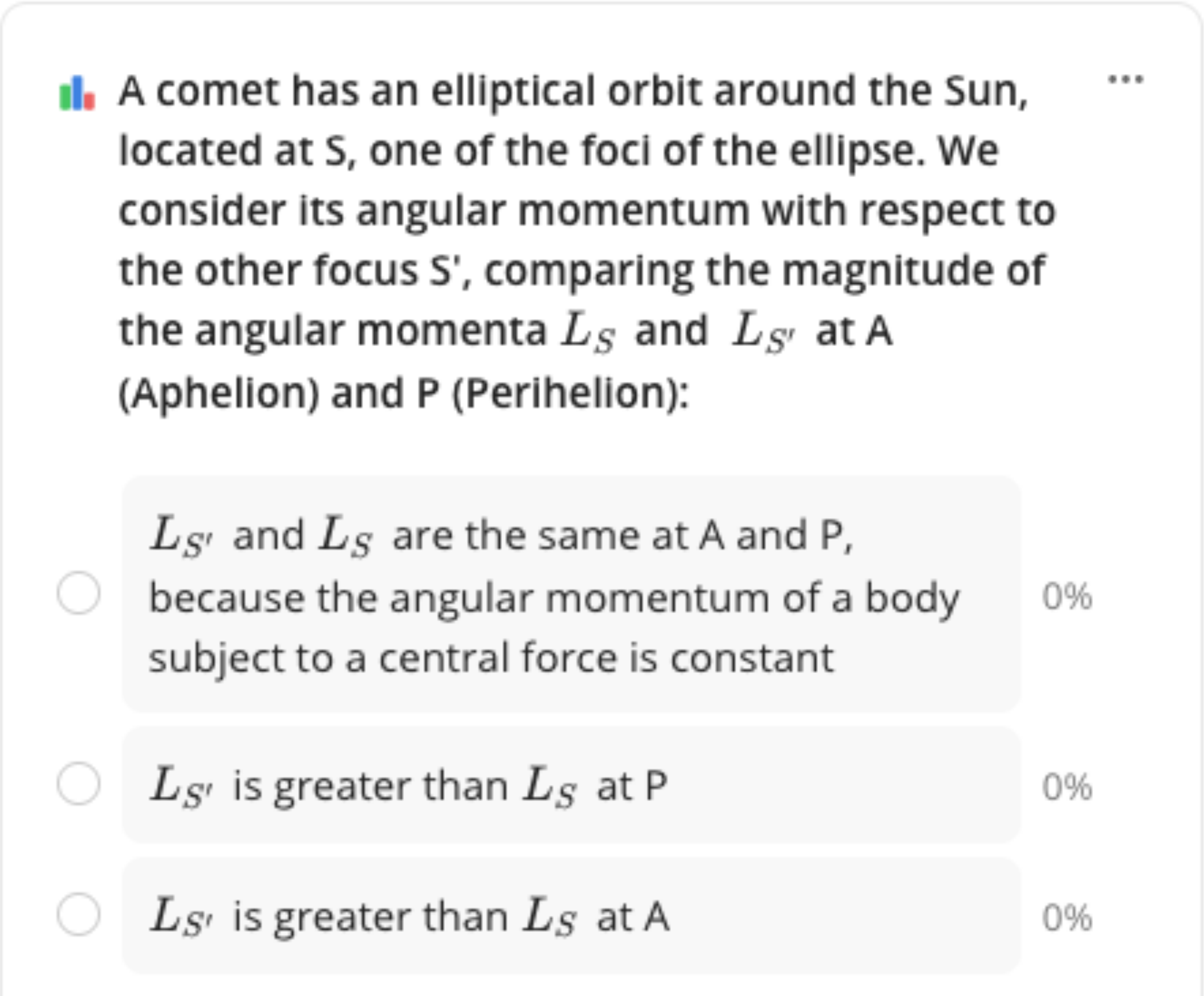


QUIZ: COMET





A comet has an elliptical orbit around the Sun, located at S, one of the foci of the ellipse. We consider its angular momentum with respect to the other focus S', comparing the magnitude of the angular momenta L_S and $L_{S'}$ at A (Aphelion) and P (Perihelion):

○ $L_{S'}$ and L_S are the same at A and P, because the angular momentum of a body subject to a central force is constant 0%

○ $L_{S'}$ is greater than L_S at P 0%

○ $L_{S'}$ is greater than L_S at A 0%

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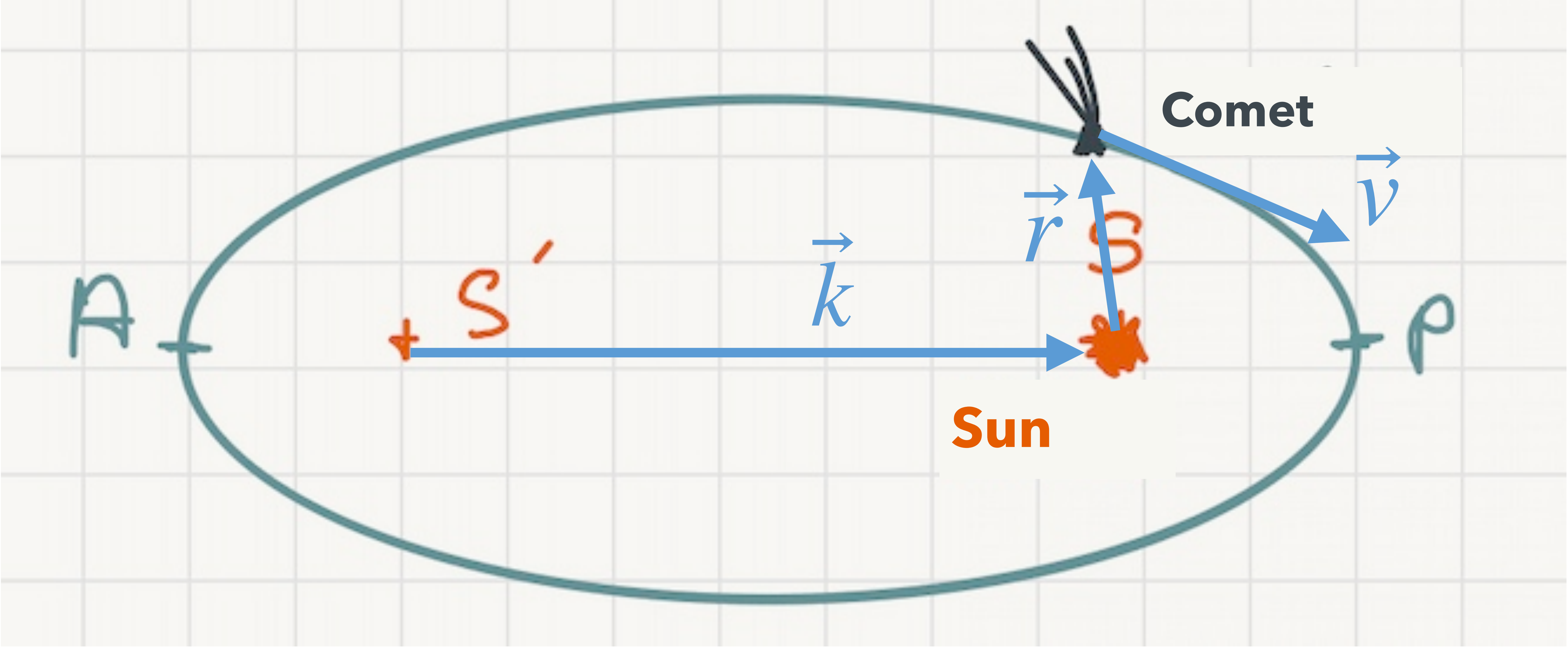
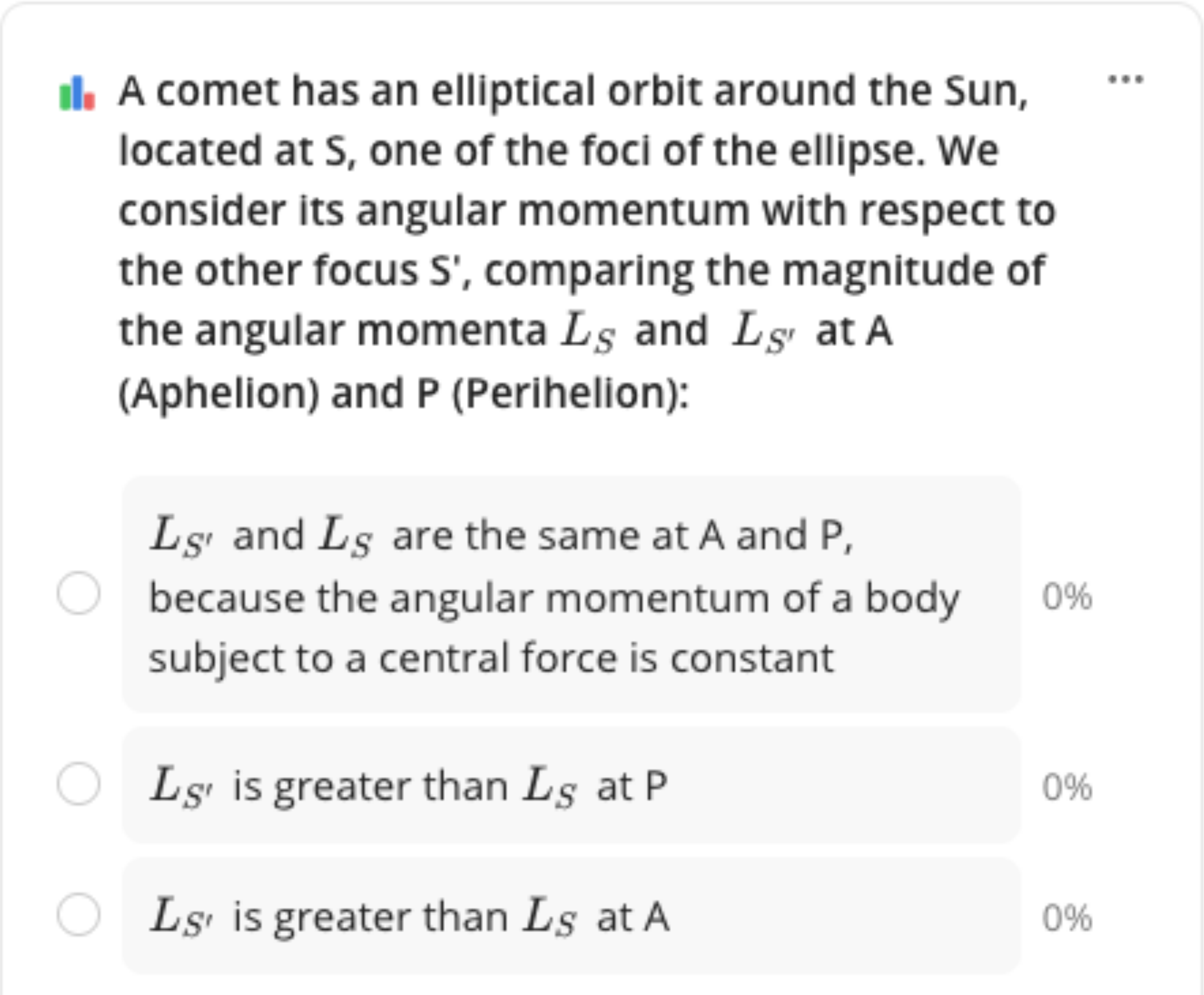




A comet has an elliptical orbit around the Sun, located at S, one of the foci of the ellipse. We consider its angular momentum with respect to the other focus S', comparing the magnitude of the angular momenta L_S and $L_{S'}$ at A (Aphelion) and P (Perihelion):

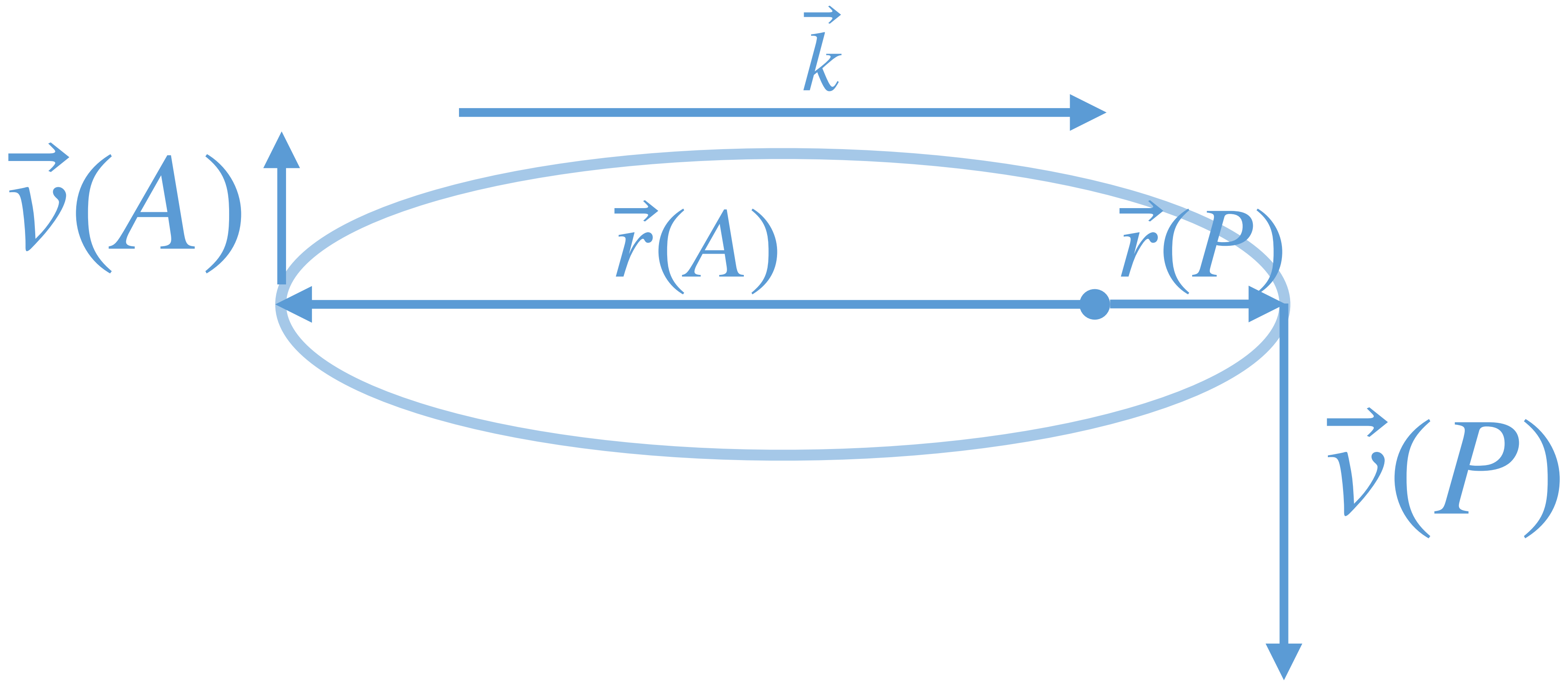
○ $L_{S'}$ and L_S are the same at A and P, because the angular momentum of a body subject to a central force is constant 0%

○ $L_{S'}$ is greater than L_S at P 0%

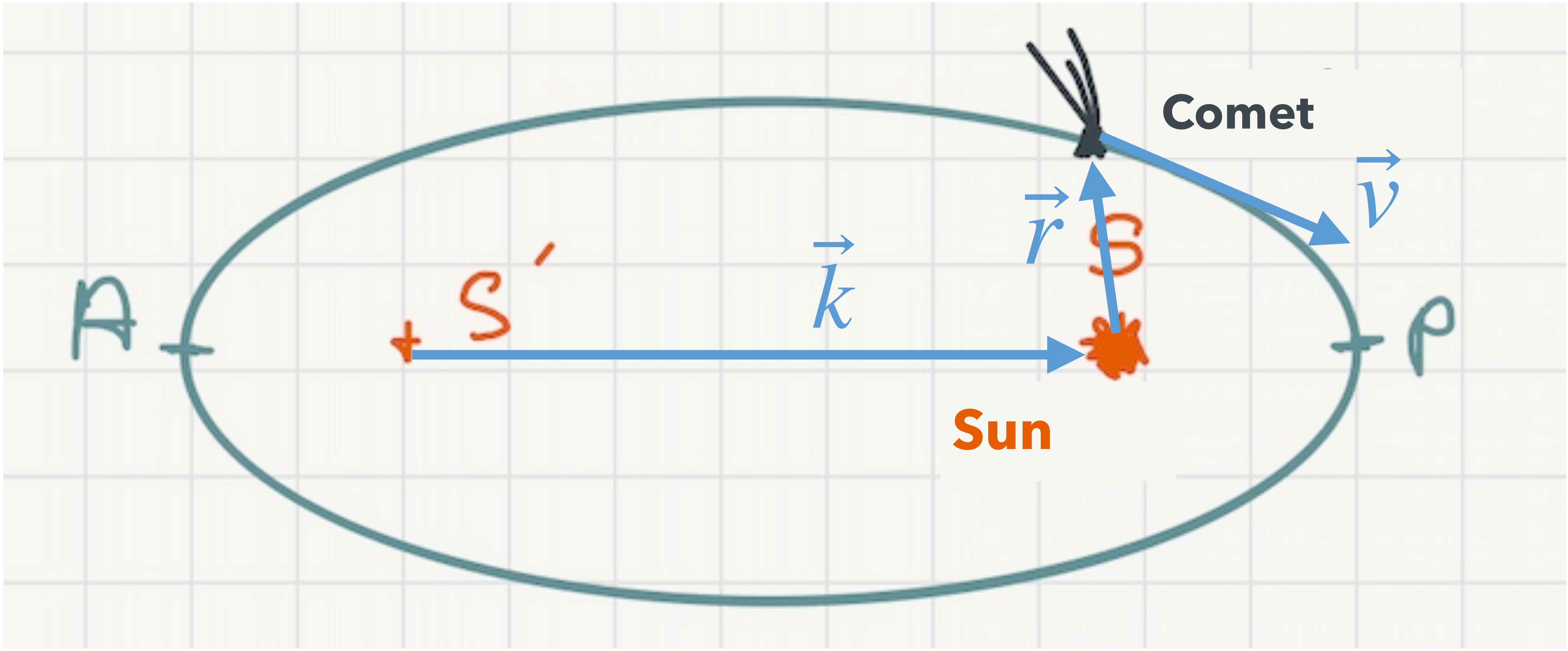
○ $L_{S'}$ is greater than L_S at A 0%

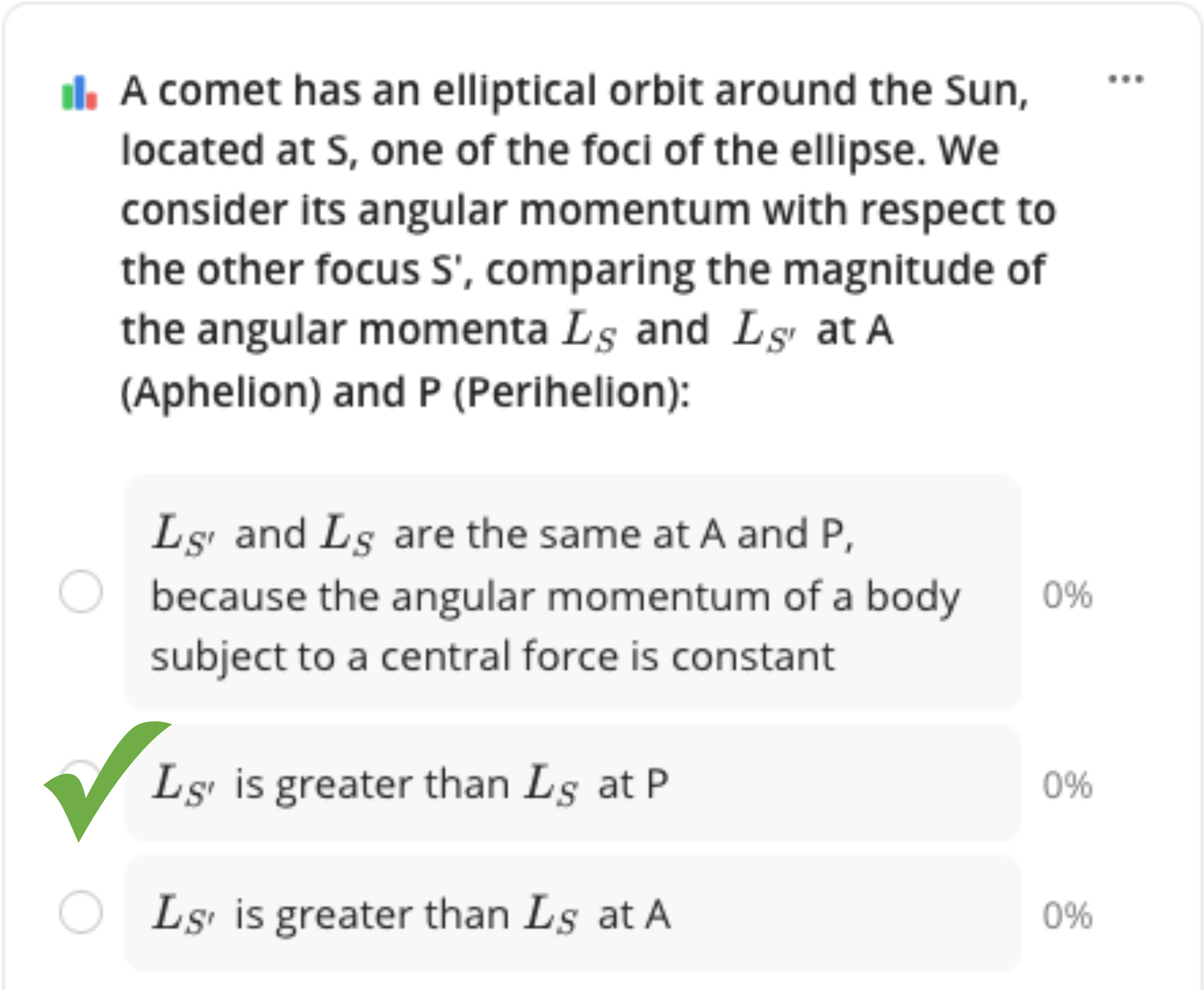
$$\overrightarrow{L}_S = \vec{r} \wedge m\vec{v} = cte$$

$$\overrightarrow{L}_{S'} = (\vec{k} + \vec{r}) \wedge m\vec{v} = \vec{k} \wedge m\vec{v} + \overrightarrow{L}_S$$



QUIZ: COMET

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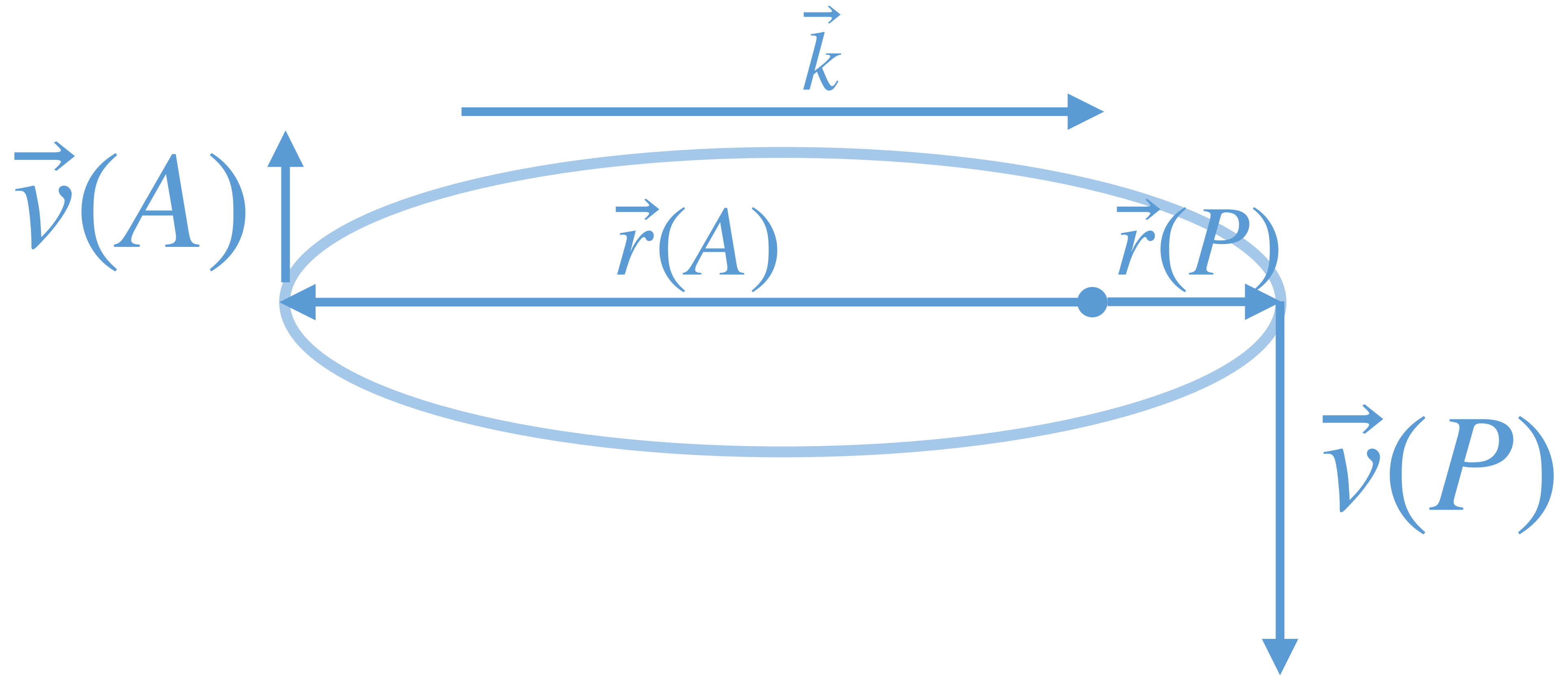
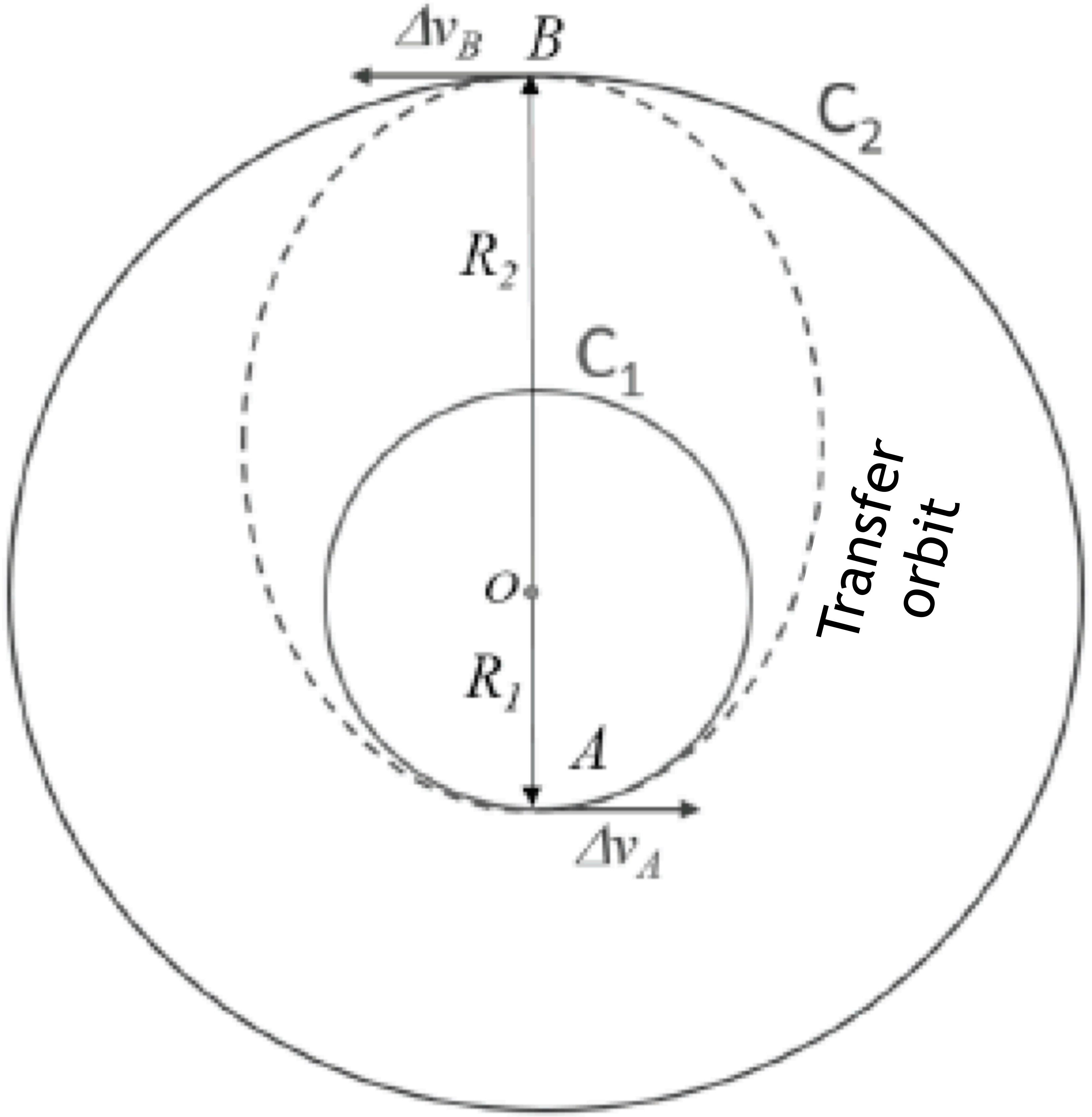


EXERCISE: TRANSFER ORBIT

(Question from 2018 exam)

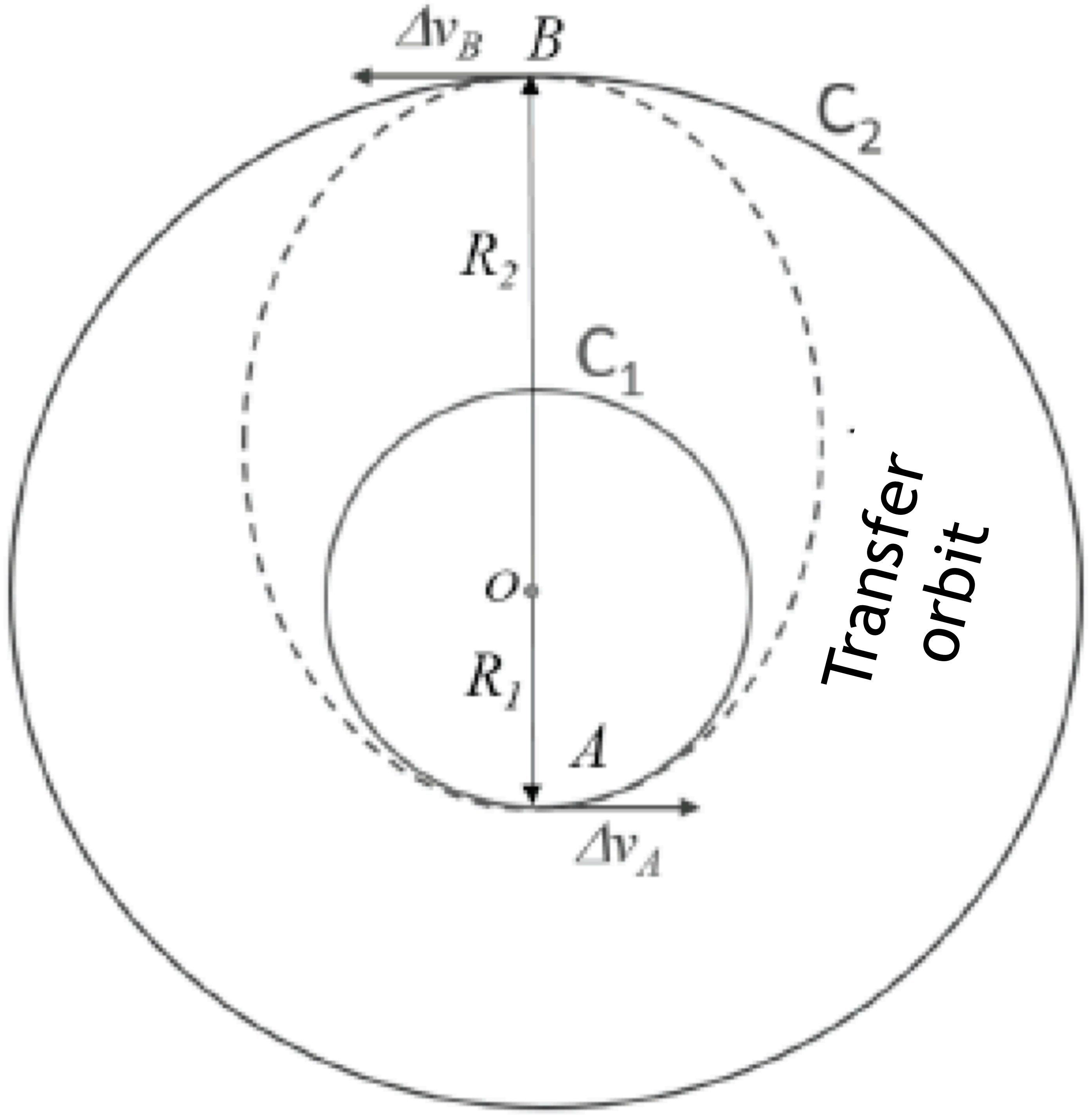


The International Space Station is a satellite orbiting the Earth. Astronauts are periodically resupplied by a shuttle launched by a rocket. Let G be the universal gravitational constant and M the mass of the Earth. After being released by the rocket, the shuttle of mass m is placed in a circular orbit C_1 of radius R_1, which is smaller than the circular orbit C_2 of radius R_2 of the space station.

https://phyanim.sciences.univ-nantes.fr/Meca/Planetes/transfert.php

EXERCISE: TRANSFER ORBIT

(Question from 2018 exam)



1. Prove that the speed of a satellite in a circular orbit is constant.

2. Express the speed v_1 of the shuttle in the circular orbit C_1 as a function of the given information.

3. Give the expression for the mechanical energy E_1 in orbit C_1 as a function of G, m, M, and R_1.

4. Calculate the work W_{12} done by the gravitational force F acting on the shuttle when it moves from orbit C_1 to orbit C_2. In practice, to reach the circular orbit C_2, it is first necessary to pass through a transfer orbit which is elliptical, as shown by the dotted line in the diagram

5. The shuttle is in the transfer orbit. Express the shuttle's velocity v_B at point B as a function of its velocity v_A at point A.

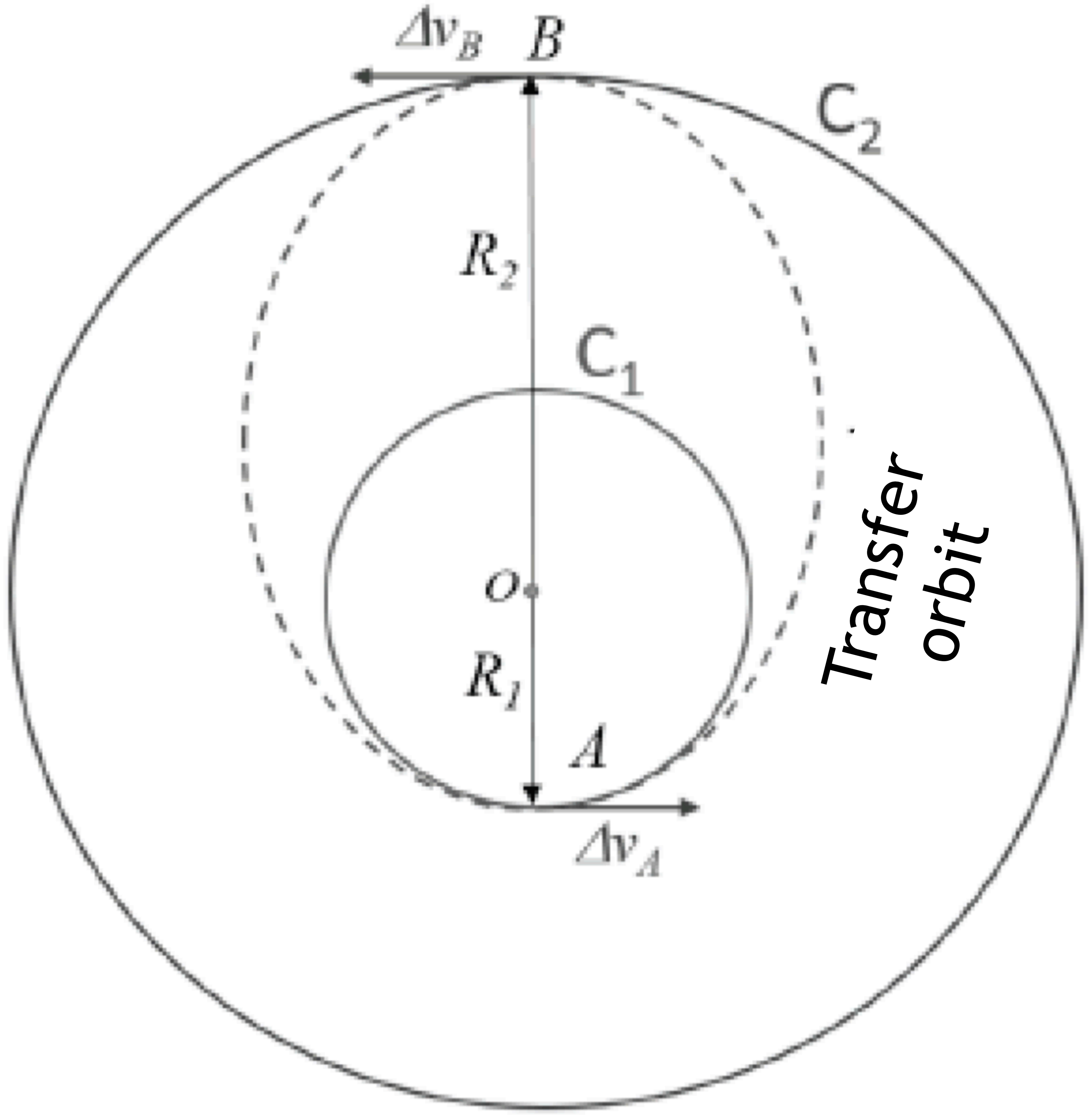
6. Determine the expression for the mechanical energy E_T in the transfer orbit as a function of G, m, M, R_1, and R_2.

7. Express the velocity $v_A = v_1 + \Delta v_1$ that must be imparted to the shuttle to move from the circular orbit C_1 to the transfer orbit. The result should be expressed as a function of E_T, E_1, and m.

8. Is the change in velocity Δv_2 of the shuttle at B, to go from the transfer orbit to C_2, positive or negative? Justify your answer without calculation

EXERCISE: TRANSFER ORBIT

(Question from 2018 exam)



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EXERCISE: TRANSFER ORBIT

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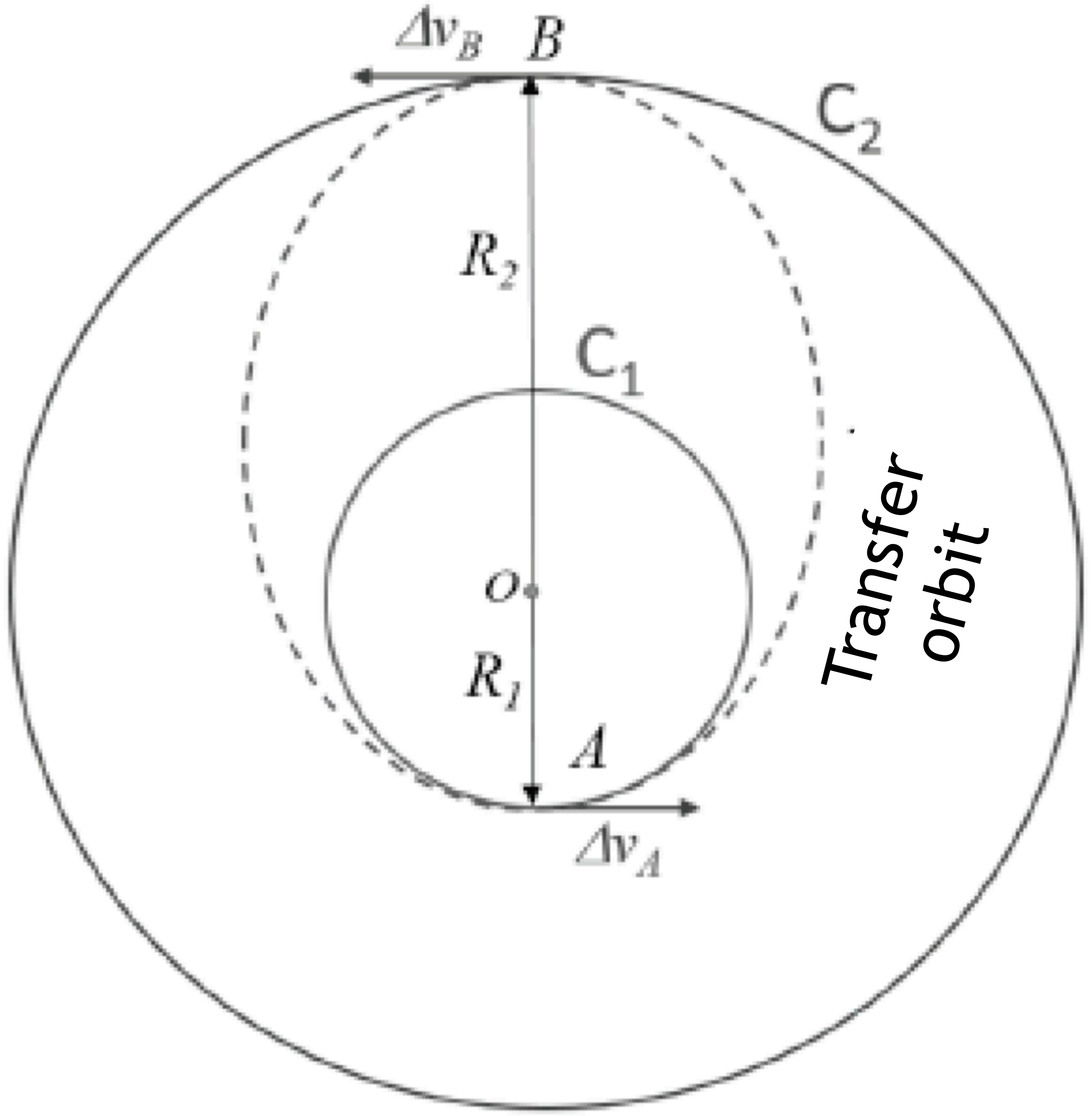

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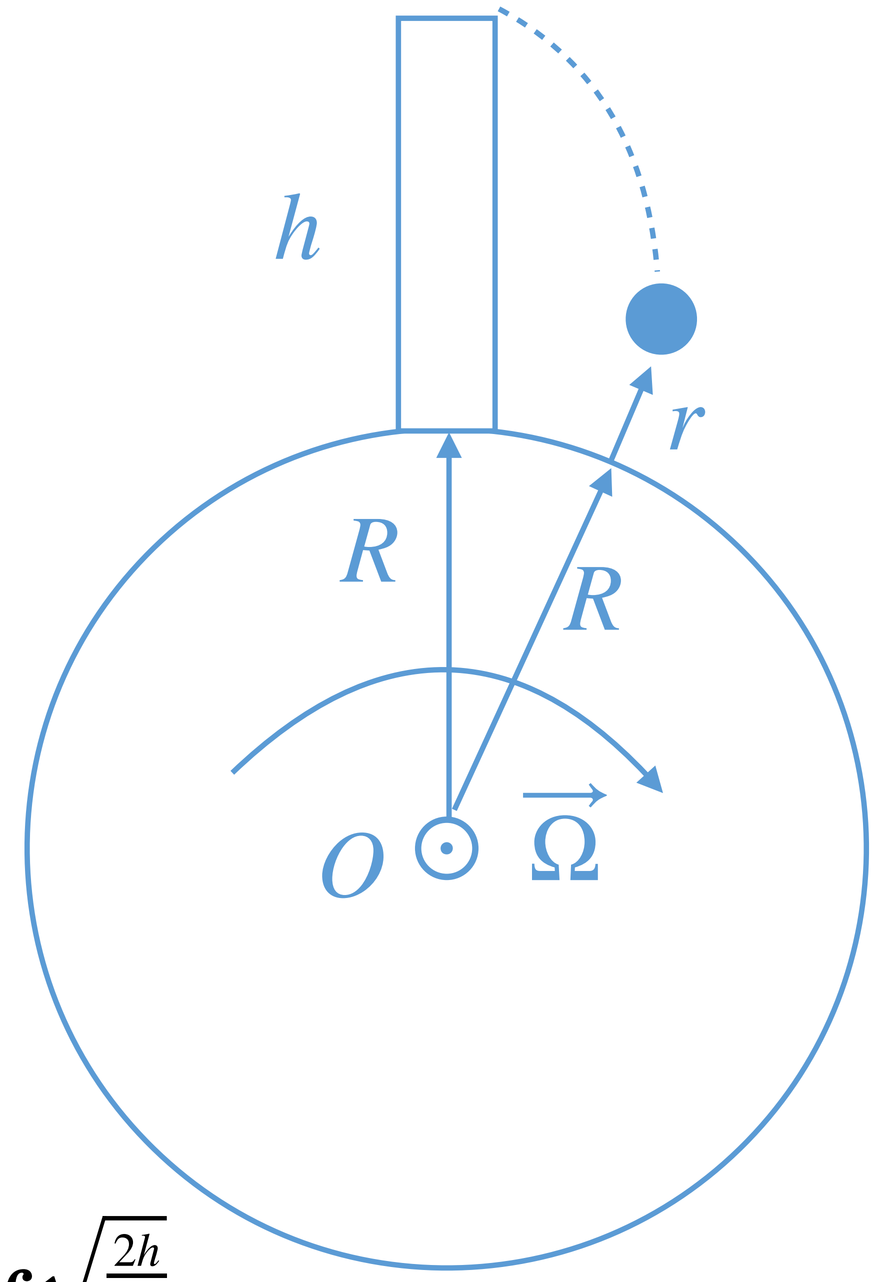
REVISITING THE TOWER: ANGULAR MOMENTUM



Initial angular momentum $\vec{L}_o^i = m(R+h)^2\vec{\Omega} = m(R+h)^2\Omega E\vec{e}_z$

Angular momentum as a function of height r $\vec{L}_o = m(R+r)^2\omega(r)\vec{e}_z$

When the stone is dropped, it will only move under gravity which is a central force. Therefore angular **momentum is conserved**

$$m(R+r)^2\omega(r)\vec{e}_z = m(R+h)^2\Omega\vec{e}_z$$

$$\omega(r) = \frac{(R+h)^2}{(R+r)^2}\Omega = \frac{(1+h/R)^2}{(1+r/R)^2}\Omega$$

Function	Truncated Taylor series
$(1+x)^n$	$1+nx$
$1/(1+x)^n$	$1-nx$

$$\omega(r) \approx (1+2h/R)(1-2r/R)\Omega \approx \Omega + \frac{2\Omega}{R}(h-r)$$

From ballistics $t_f = \sqrt{\frac{2h}{g}}$ $r(t) = h - \frac{1}{2}gt^2$

$$\omega(t) = \Omega + \frac{2\Omega}{R}(h-h+gt^2/2) = \Omega + \frac{\Omega g t^2}{R}$$

$$\Delta\omega(t) = \omega(t) - \Omega = \frac{gt^2}{R}\Omega$$

$$\Delta\theta(t_f) = \int_0^{t_f}\Delta\omega(t)dt = \int_0^{\sqrt{\frac{2h}{g}}}\frac{gt^2}{R}\Omega dt = \frac{\Omega g}{R}\int_0^{\sqrt{\frac{2h}{g}}} t^2 dt$$

$$\Delta\theta(t_f) = \frac{\Omega g}{R}\frac{1}{3}t^3\Big|_0^{\sqrt{\frac{2h}{g}}} = \frac{\Omega g}{3R}\left(\sqrt{\frac{2h}{g}}\right)^3 = \frac{\Omega g}{3R}\frac{2h}{g}\sqrt{\frac{2h}{g}}$$

$$\Delta\theta(t_f) = \frac{2}{3}\frac{\Omega}{hR}\sqrt{\frac{2h}{g}}$$

$$\boxed{D = R\Delta\theta(t_f) = \frac{2}{3}\frac{\Omega}{h}\sqrt{\frac{2h}{g}}}$$