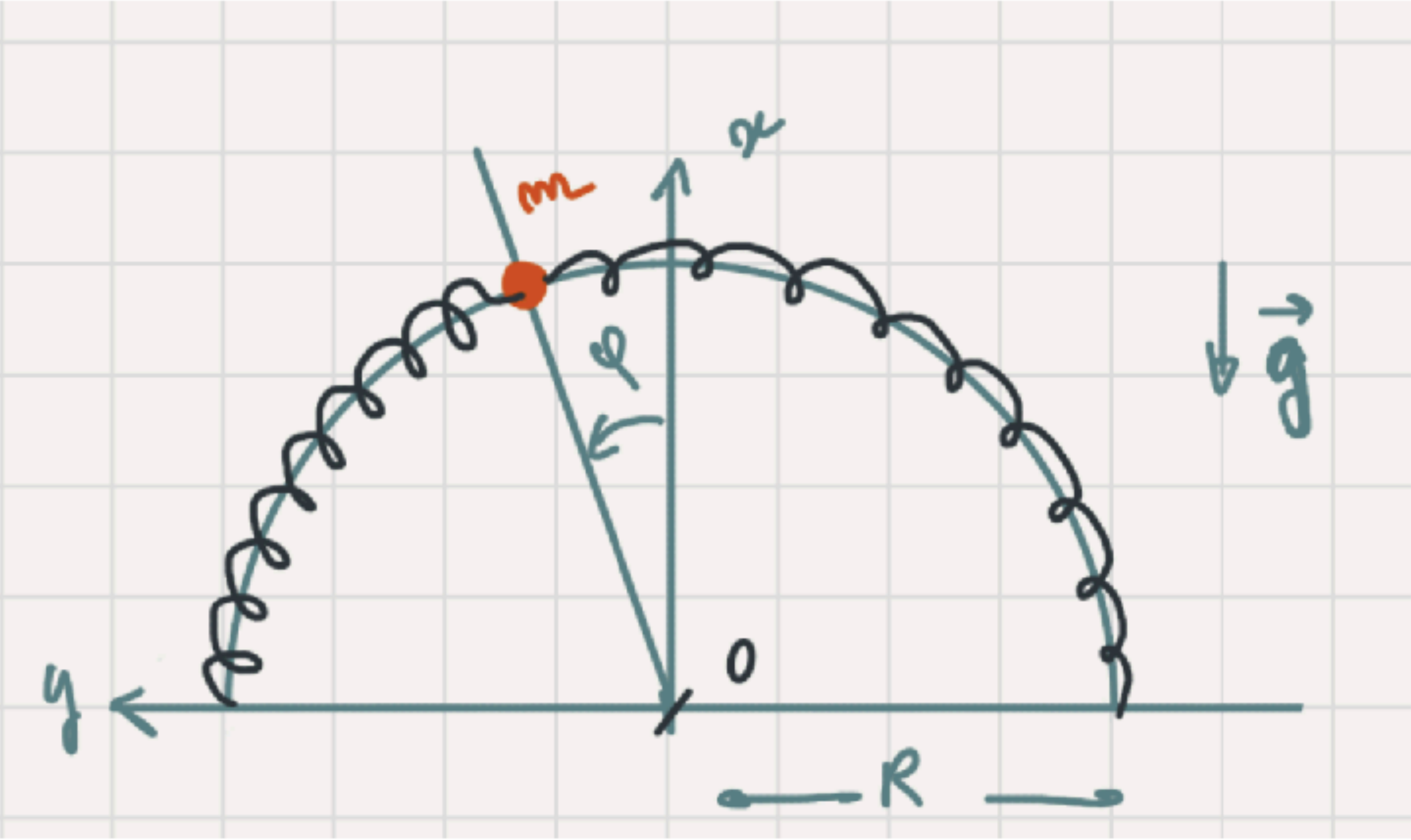
EXERCISE: METRONOME

(Question from an old exam)

We want to study a device intended to serve as a metronome (an instrument used in music to keep time). It consists of a rigid semicircular rail of radius R, fixed vertically, a mass m that slides on the rail, and two springs on the rail. All friction is neglected. The springs have a spring constant k and a rest length $l_0 = R\pi/2$



a) List the forces acting on the system and represent them on the diagram.

b) Express the acceleration of mass m in the polar coordinates of the drawing, taking into account the characteristics of the device.

c) Establish the equations of motion in polar coordinates.

d) Show that position $\phi = 0$ is an equilibrium position.

e) Establish the differential equation of motion for ϕ in the case where ϕ remains small.

f) Still in the case where ϕ remains small, what condition linking k, m, and R ensures that the system behaves as a harmonic oscillator?

g) Express the potential energy of the mass as it moves along the rod, as a function of the ϕ-coordinate, first in the general case, then in the case of small angles. Recall that for small angles, $\cos(\phi) \approx 1 - \phi^2/2$

h) In the approximation where ϕ remains small, (re)discover, using the potential energy, the condition on k for the system to behave as a harmonic oscillator.

i) Calculate the expression for the mechanical energy in the general case (no assumptions about ϕ), then use it to derive the differential equation of motion of the mass

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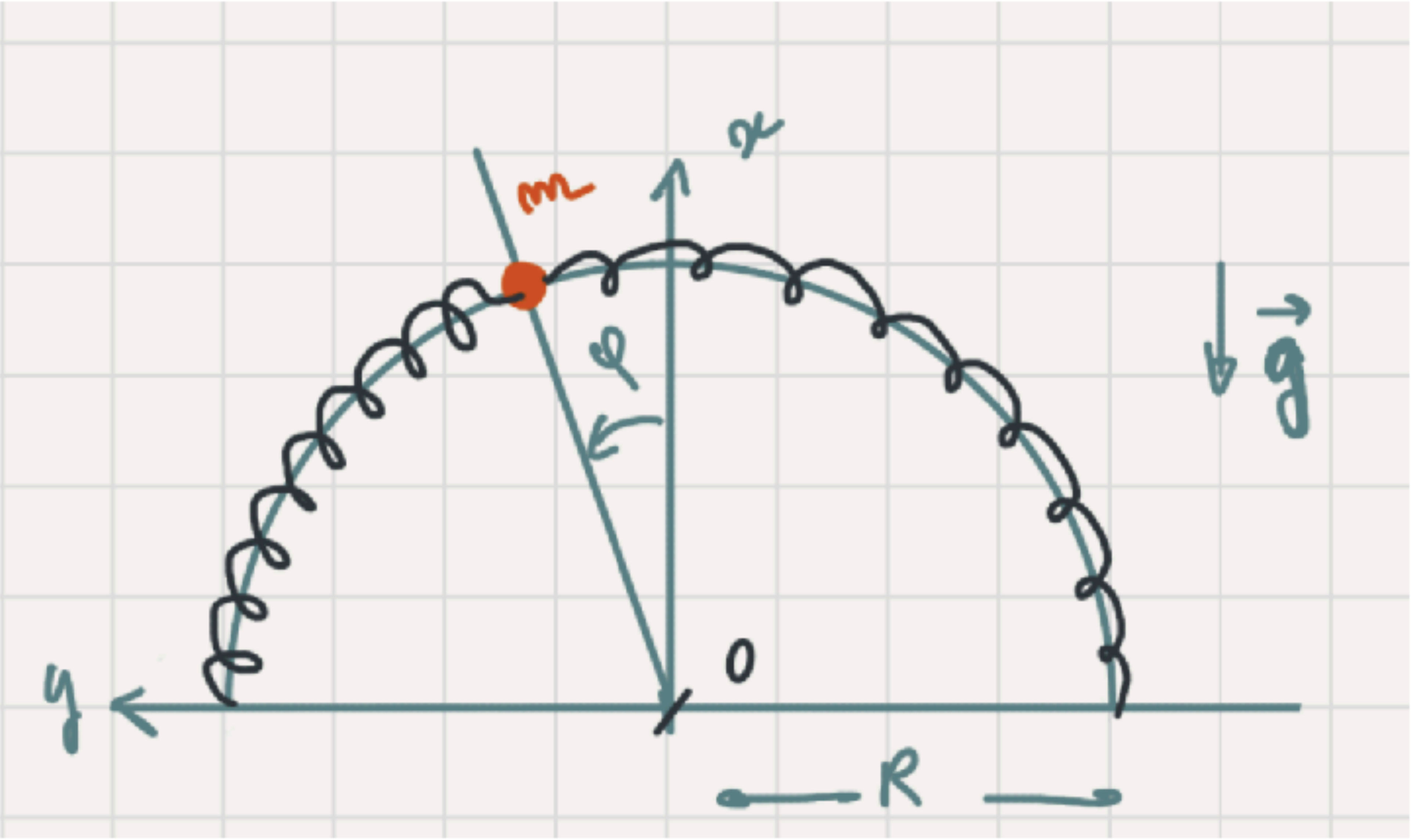
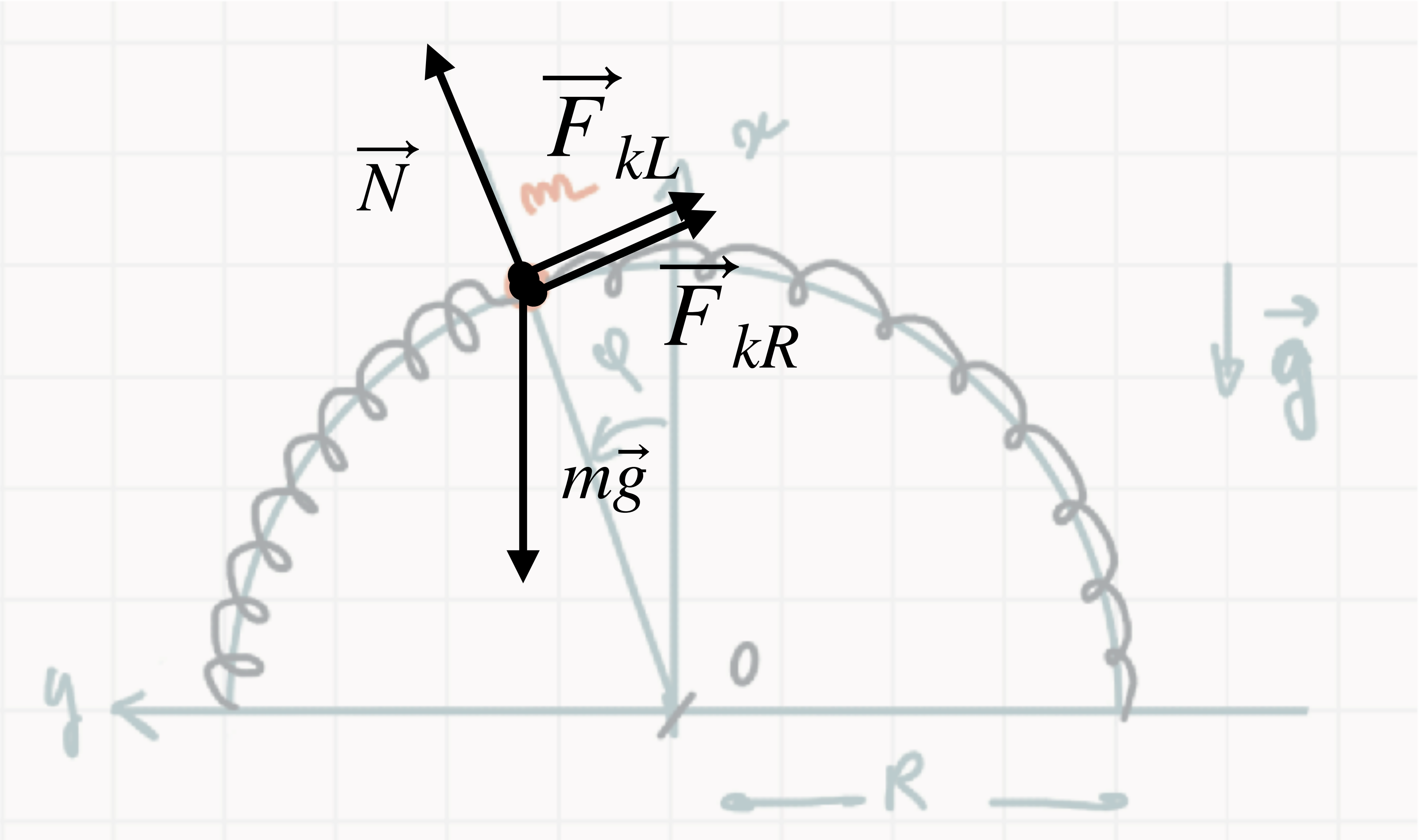


Forces are gravity, normal force of the rail, force from right spring and force from left spring



Both springs have the same spring constant k and same displacement $\Delta l = \phi R$, so the magnitudes of the two forces are the same. The springs are attached to the rail so the forces can only act in the direction along the rail (along $\vec{e}_\phi$). The rest position of the two springs is at the same position $\phi = 0$, so both forces act in the same direction.

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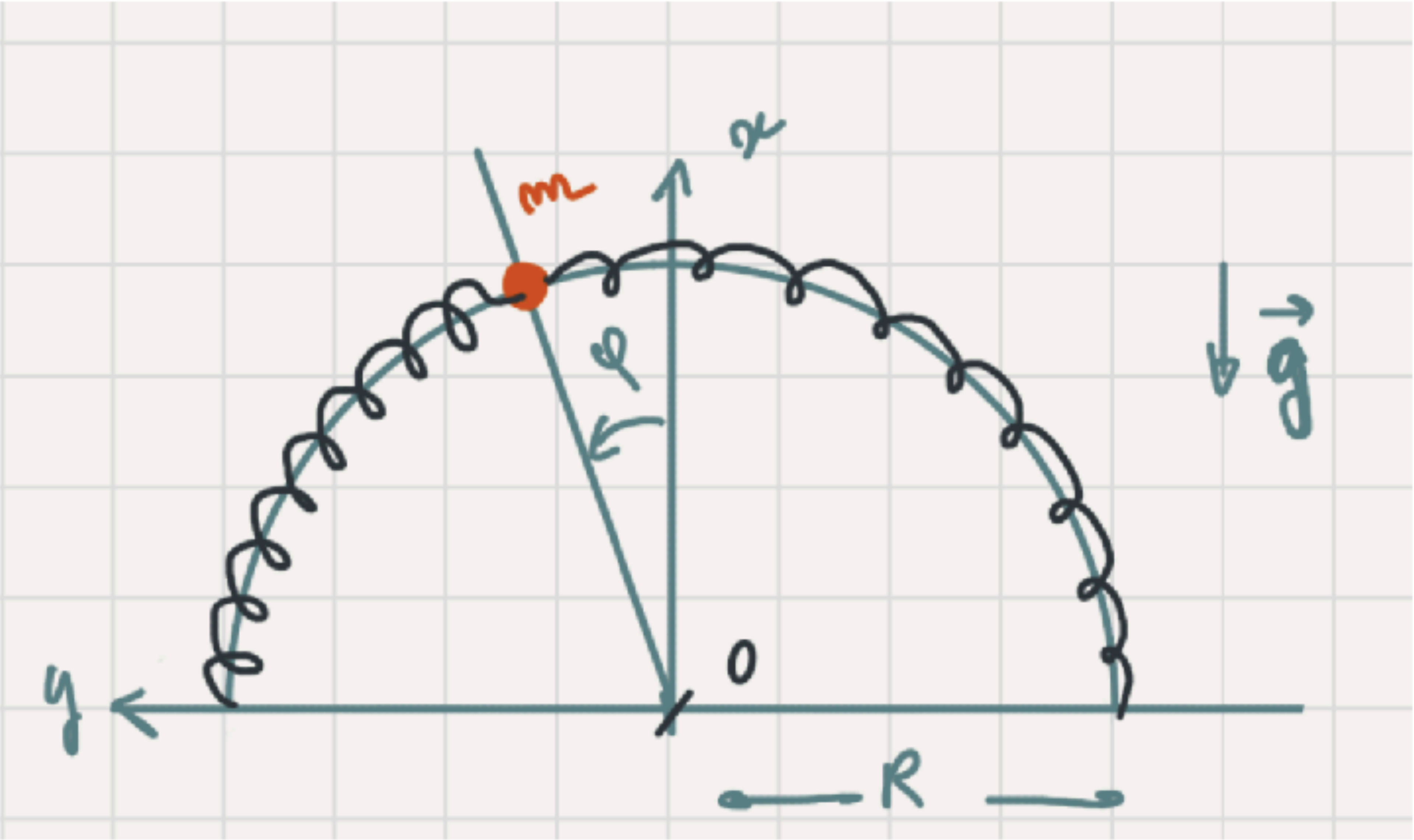

Acceleration in polar coordinates from formula sheet:

$$\vec{a}(t) = (\ddot{\rho} - \rho\dot{\varphi}^2)\,\vec{e}_\rho + (2\dot{\rho}\dot{\varphi} + \rho\ddot{\varphi})\,\vec{e}_\varphi$$

Mass can only move along curved rail so $\rho = R, \dot{\rho} = 0$

$$\vec{a} = -R\dot{\phi}^2\vec{e}_\rho + R\ddot{\phi}\vec{e}_\phi$$

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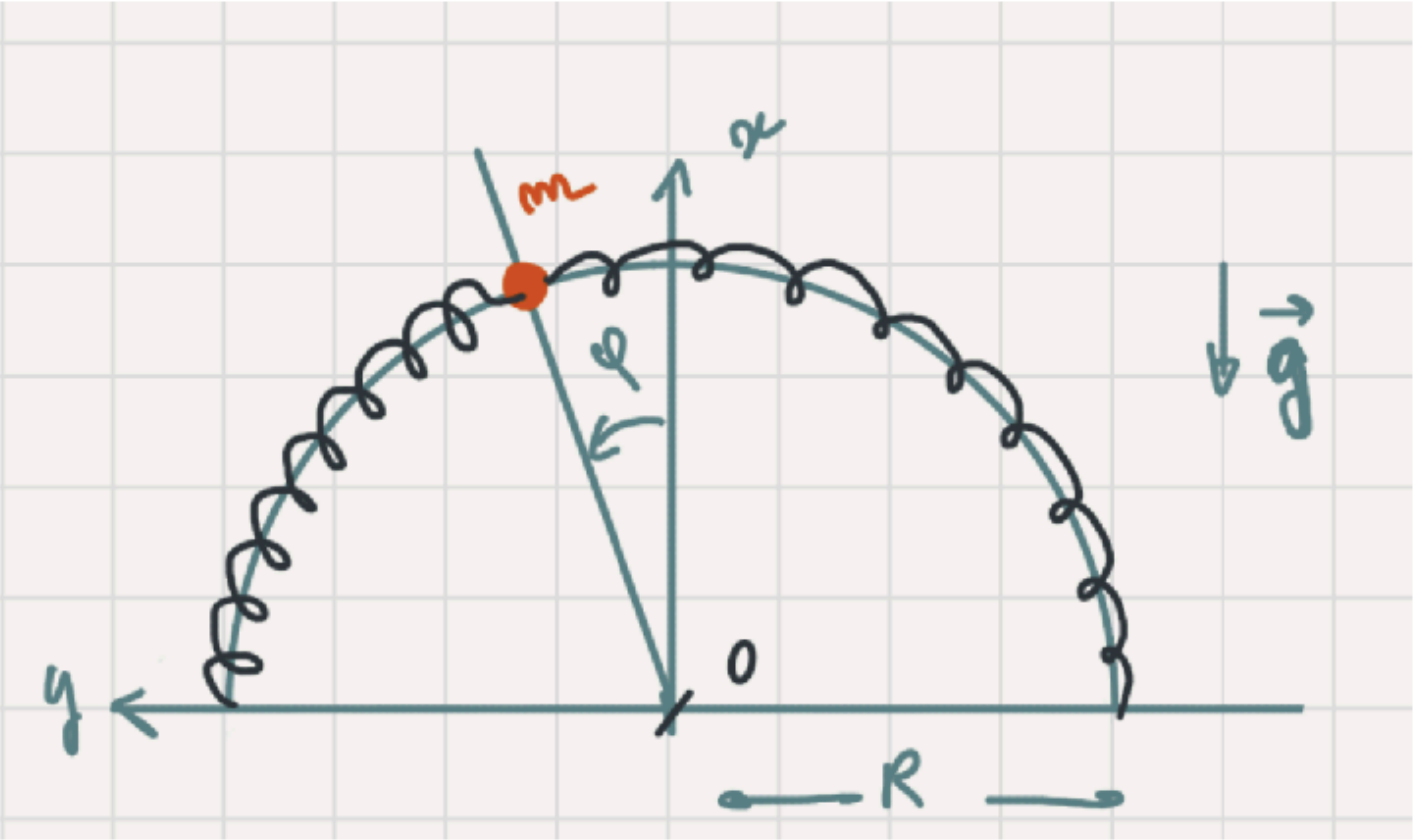
d) Show that position $\phi = 0$ is an equilibrium position.



How to show that something is an equilibrium position?

A) At the equilibrium position, show that if the velocity is zero, the acceleration will be zero

OR

B) Show that the gradient (spatial derivative) of the potential is zero at the equilibrium position

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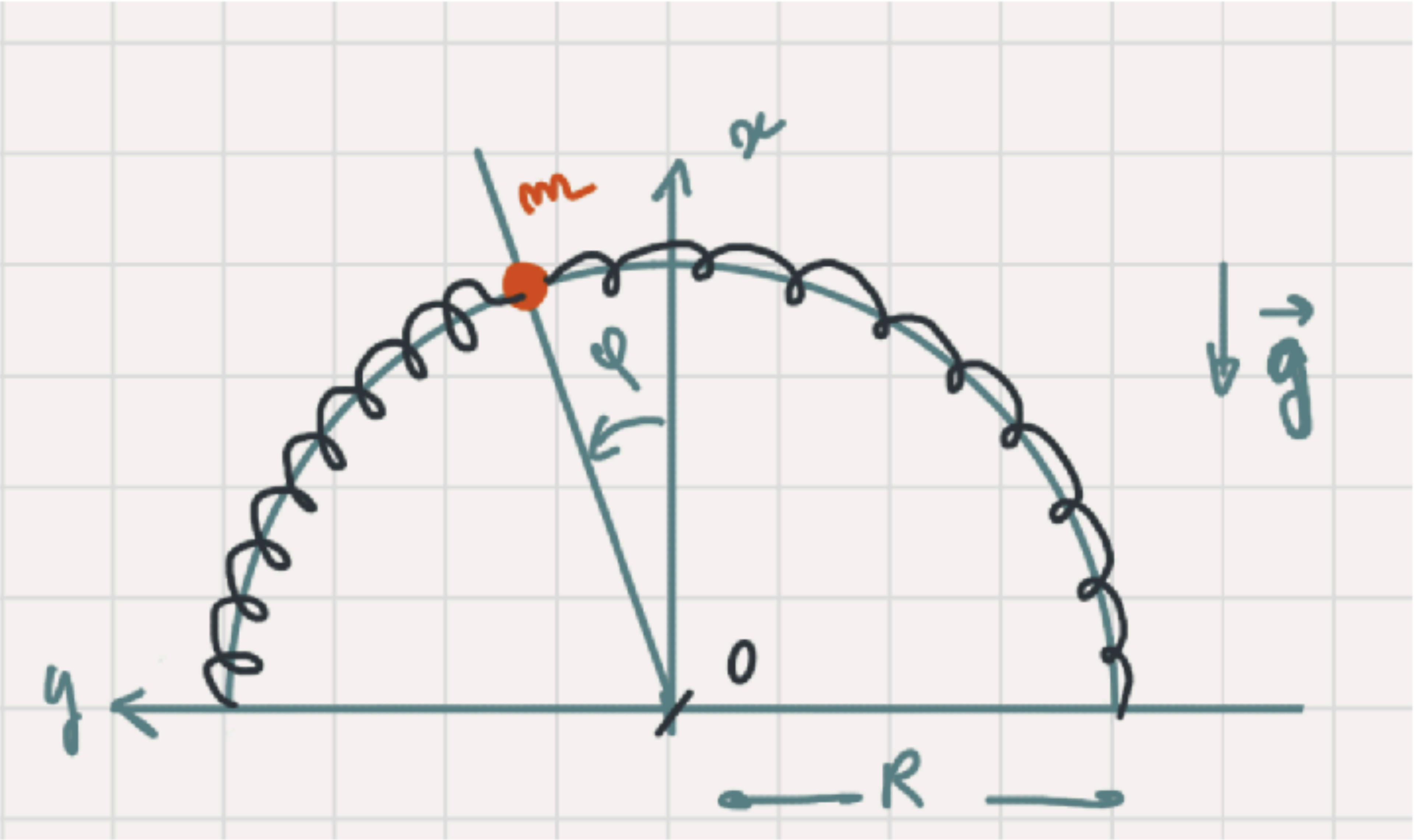

A) At the equilibrium position, show that if the velocity is zero, the acceleration will be zero

All forces at phi=0

$$m\vec{g} = -mg\vec{e}_\rho$$

$$\vec{N} = N\vec{e}_\rho$$

$$\vec{F}_{kR} = \vec{0}$$

$$\vec{F}_{kL} = \vec{0}$$

These are both zero at phi=0 because both springs are at their rest length

Acceleration equation from part C, if velocity is zero:

$$\vec{a} = R\ddot{\phi}\vec{e}_\phi$$

Newton's 2nd law on $\vec{e}_\rho$ $\quad 0 = N - mg$

Newton's 2nd law on $\vec{e}_\phi$

$$R\ddot{\phi} = 0 \implies \boxed{\ddot{\phi} = 0}$$

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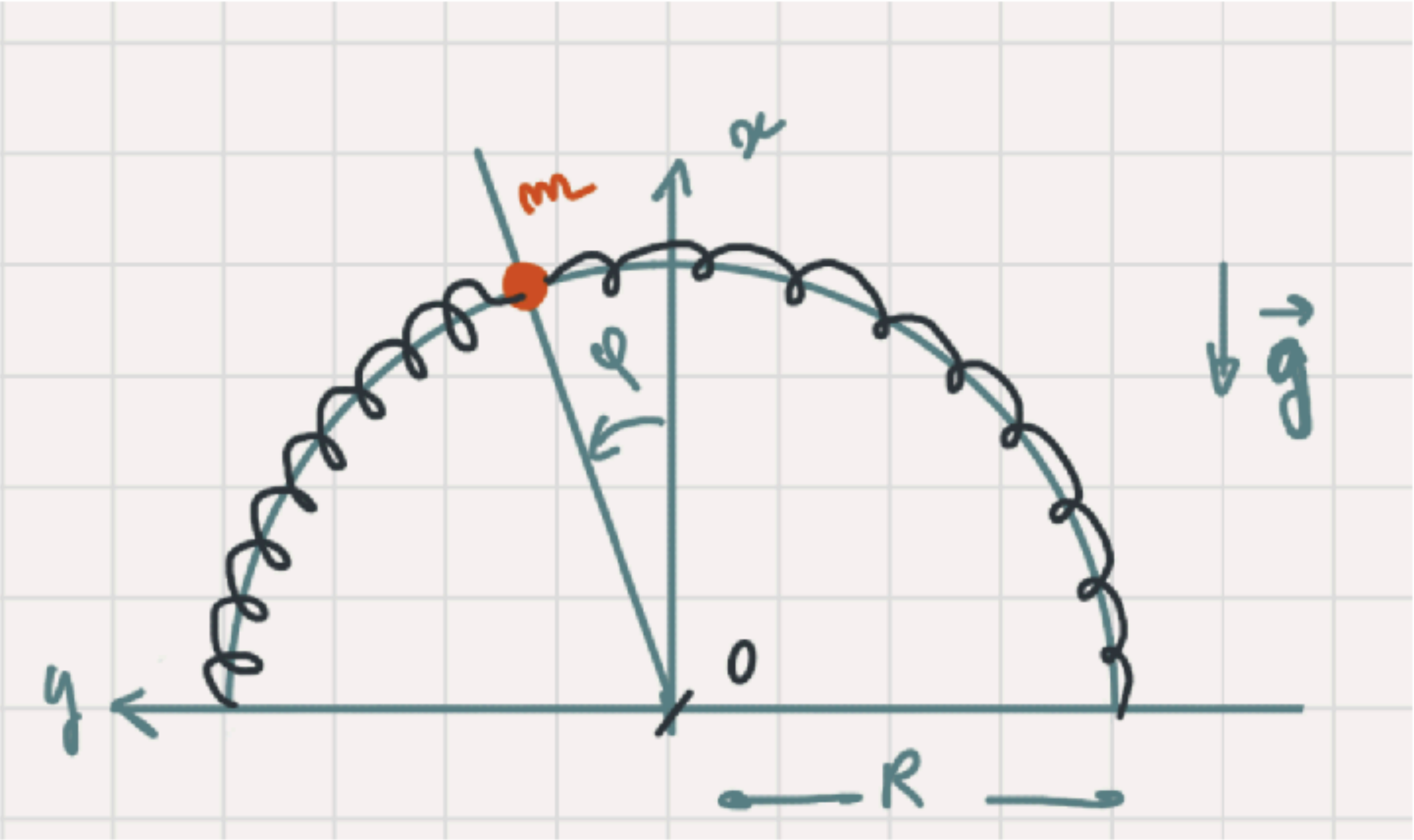

B) Show that the gradient (spatial derivative) of the potential is zero at the equilibrium position x_e

$$E_p^G = mgx = mgR\cos\phi$$

$$E_p^{kR} = \frac{1}{2}k(R\phi)^2$$

$$E_p^{kL} = \frac{1}{2}k(R\phi)^2$$

$$E_p^{total} = k(R\phi)^2 + mgR\cos\phi$$

Need to calculate the gradient before we plug in phi=0

$$\frac{d}{d\phi}E_p^{total} = 2kR^2\phi - mgR\sin\phi$$

Now plug in phi=0

$$\frac{dE_p^{total}}{d\phi}(\phi = 0) = 0$$

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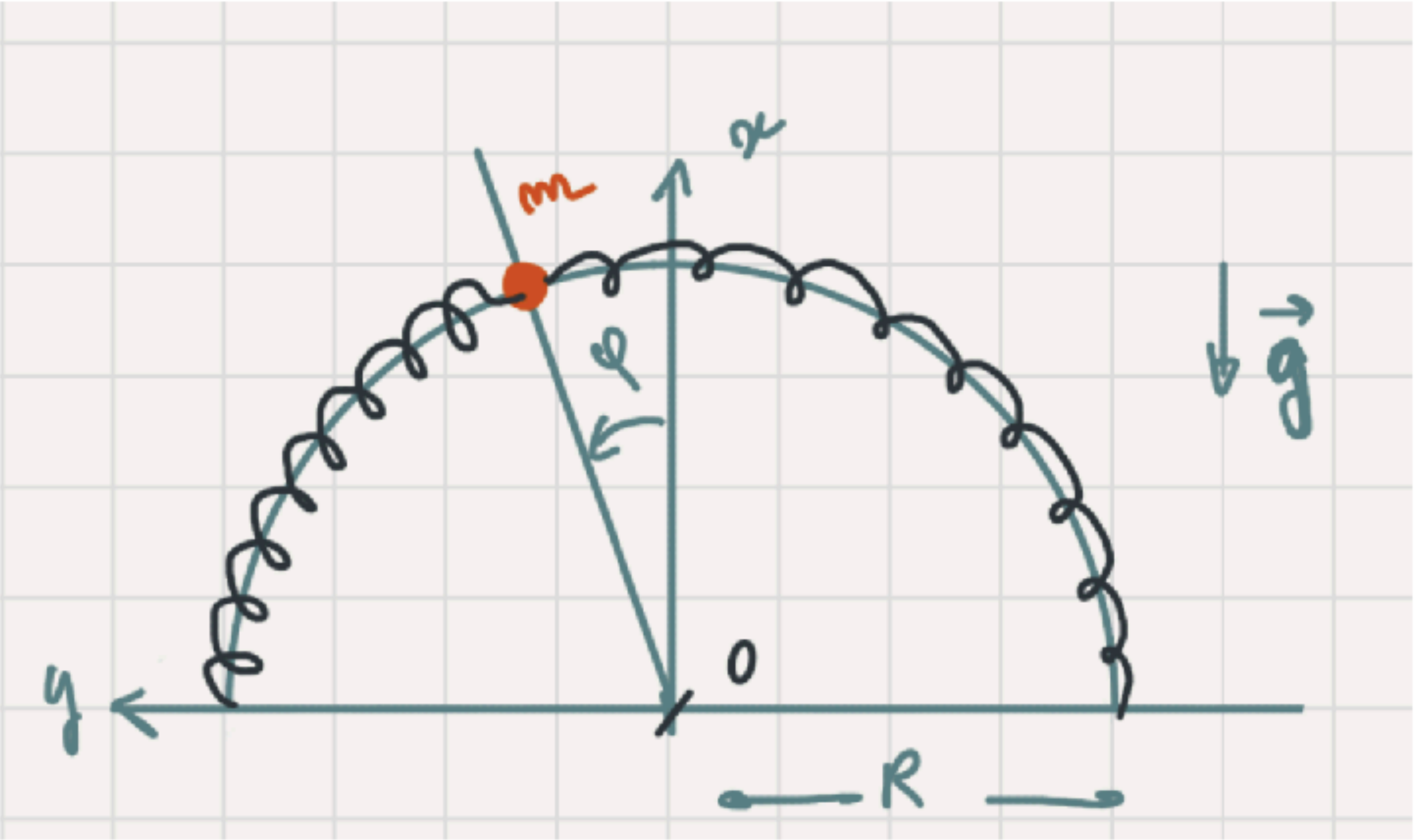
e) Establish the differential equation of motion for ϕ in the case where ϕ remains small.



All forces:

$$m\vec{g} = mg(-\cos\phi\vec{e}_\rho + \sin\phi\vec{e}_\phi)$$

$$\vec{N} = N\vec{e}_\rho$$

$$\vec{F}_{kR} = -kR\phi\vec{e}_\phi$$

$$\vec{F}_{kL} = -kR\phi\vec{e}_\phi$$

Acceleration from part B

$$\vec{a} = -R\dot{\phi}^2\vec{e}_\rho + R\ddot{\phi}\vec{e}_\phi$$

Newton's 2nd law

$$m\vec{a} = m\vec{g} + \vec{N} + \vec{F}_{kL} + \vec{F}_{kR}$$

$$-mR\dot{\phi}^2\vec{e}_\rho + mR\ddot{\phi}\vec{e}_\phi = mg(-\cos\phi\vec{e}_\rho + \sin\phi\vec{e}_\phi) + N\vec{e}_\rho - 2kR\phi\vec{e}_\phi$$

Newton's 2nd law on $\vec{e}_\rho$ $\quad -mR\dot{\phi}^2 = -mg\cos\phi + N$

Newton's 2nd law on $\vec{e}_\phi$ $\quad mR\ddot{\phi} = mg\sin\phi - 2kR\phi$

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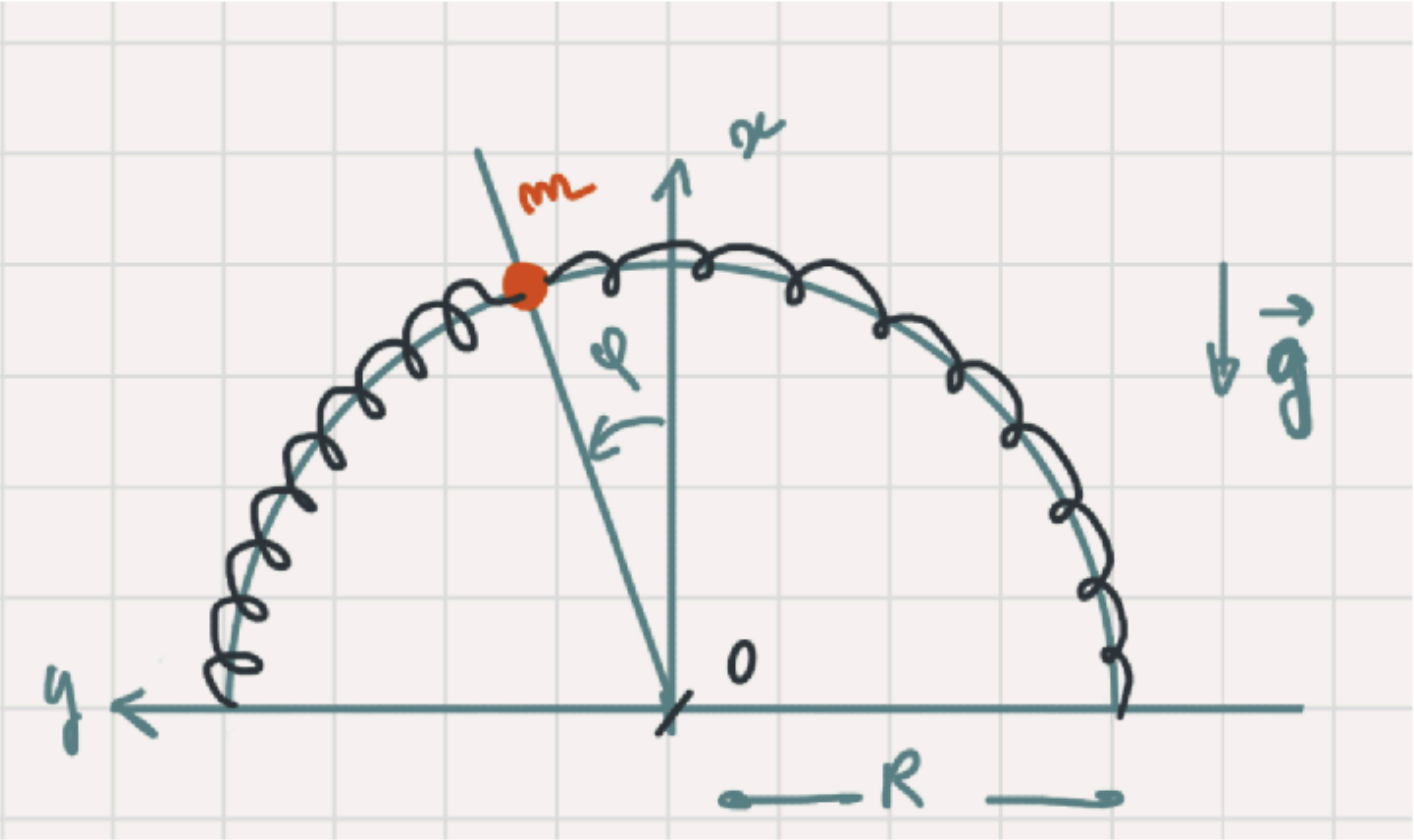

Newton's 2nd law on $\vec{e}_\rho$ $\quad -mR\dot{\phi}^2 = -mg\cos\phi + N$

Not that interesting here because N is a free parameter which could be positive or negative (motion along a rail)

Newton's 2nd law on $\vec{e}_\phi$ $\quad mR\ddot{\phi} = mg\sin\phi - 2kR\phi$

$$\phi \ll 1 \implies \sin\phi \approx \phi$$

$$mR\ddot{\phi} = mg\phi - 2kR\phi$$

$$\ddot{\phi} + \phi\frac{2kR - mg}{mR} = 0$$

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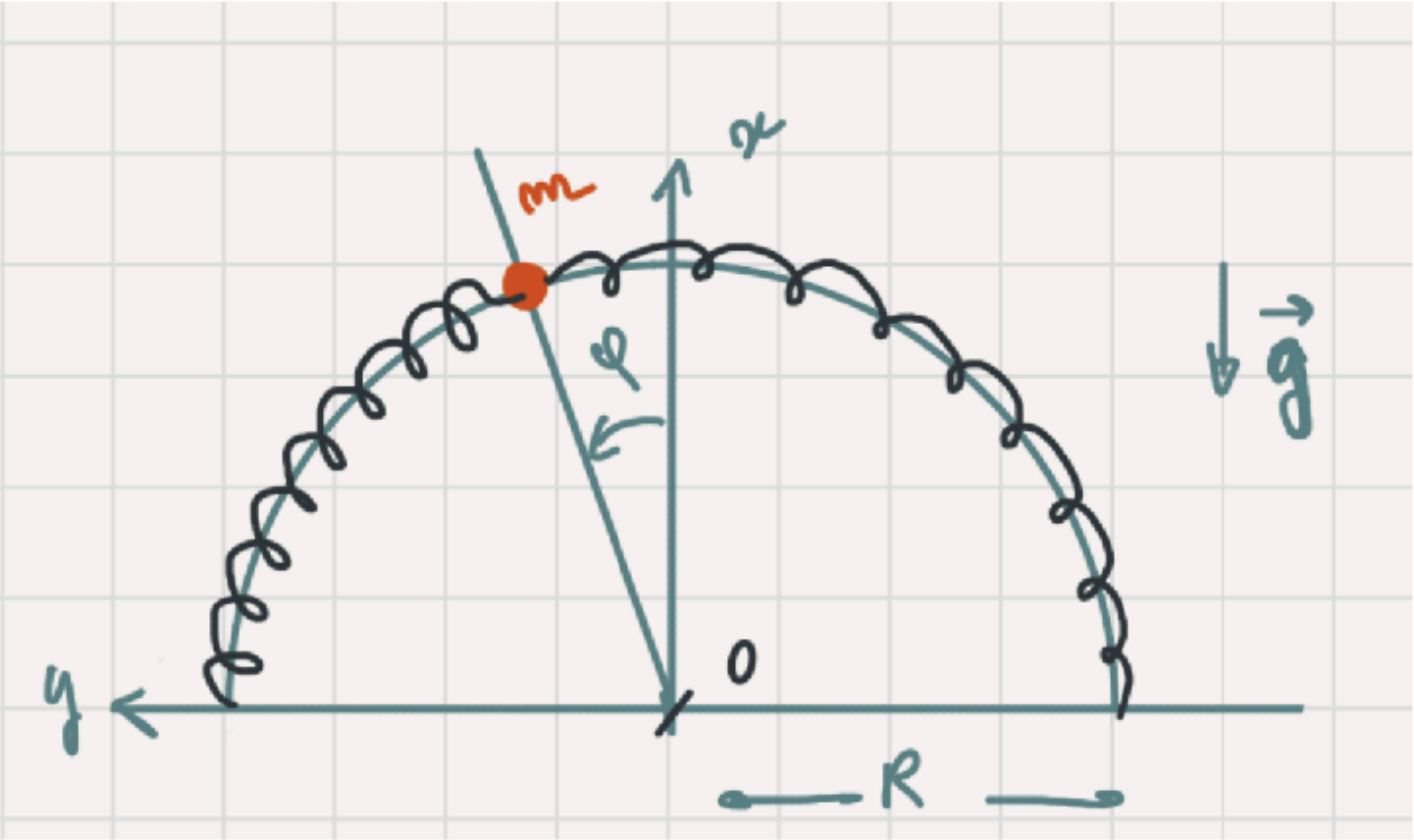
f) Still in the case where ϕ remains small, what condition linking k, m, and R ensures that the system behaves as a harmonic oscillator?



Differential equation for a harmonic oscillator $\ddot{\phi} + \Omega_0^2 \phi = 0, \ \Omega_0^2 > 0$

Our equation $\ddot{\phi} + \phi \dfrac{2kR - mg}{mR} = 0$

This will be a harmonic oscillator if we can show that the coefficient is greater than zero

$$\frac{2kR - mg}{mR} = \Omega_0^2 > 0$$

$$2kR - mg > 0$$

$$2kR > mg$$

$$k > \frac{mg}{2R}$$

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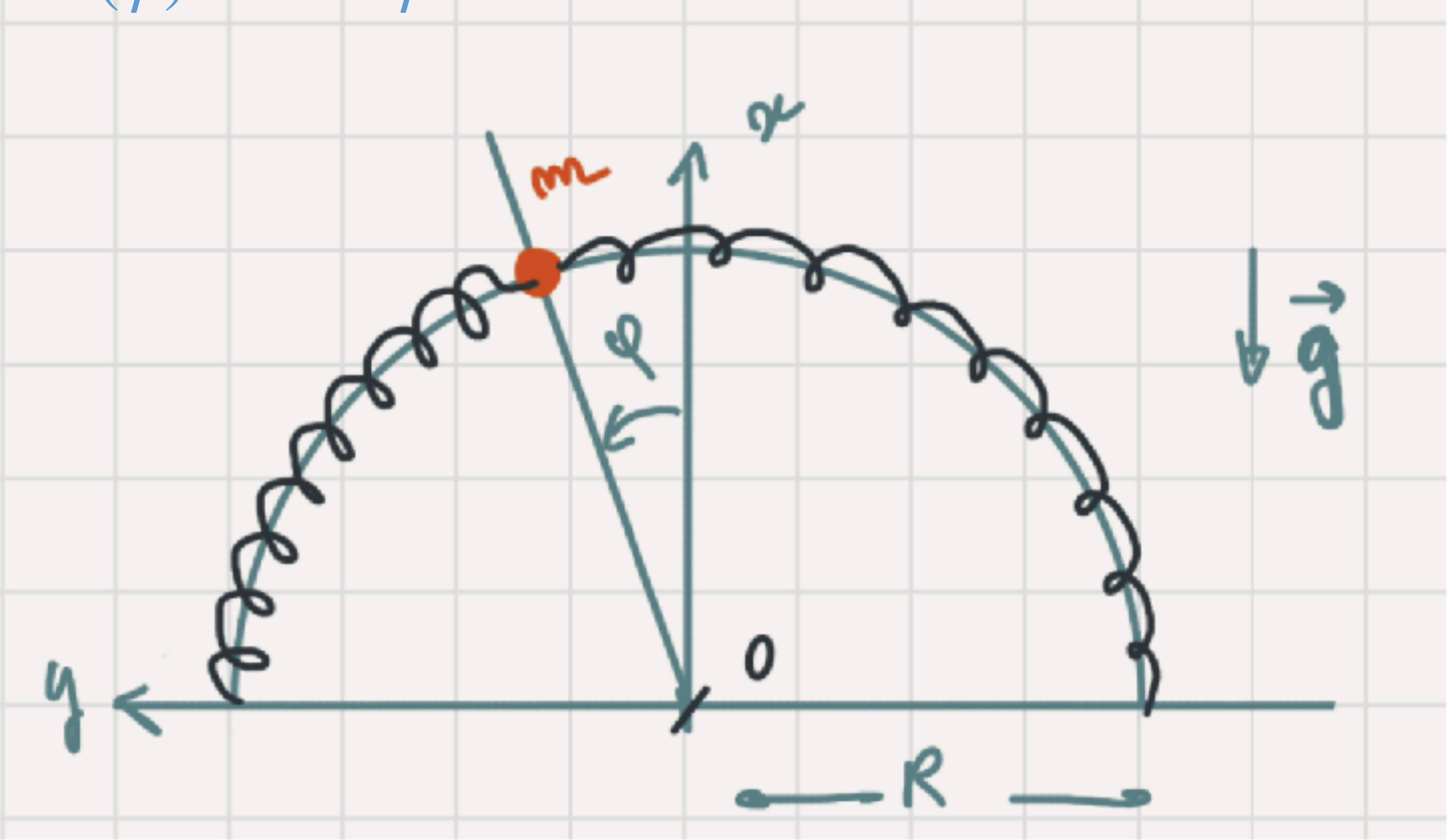

$$E_p^G = mgx = mgR\cos\phi$$

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$$E_p^{total} = kR^2\phi^2 + mgR\cos\phi$$

For small angles

$$E_p^{total} = kR^2\phi^2 + mgR(1 - \phi^2/2)$$

$$E_p^{total} = \phi^2(kR^2 - mgR/2) + mgR$$

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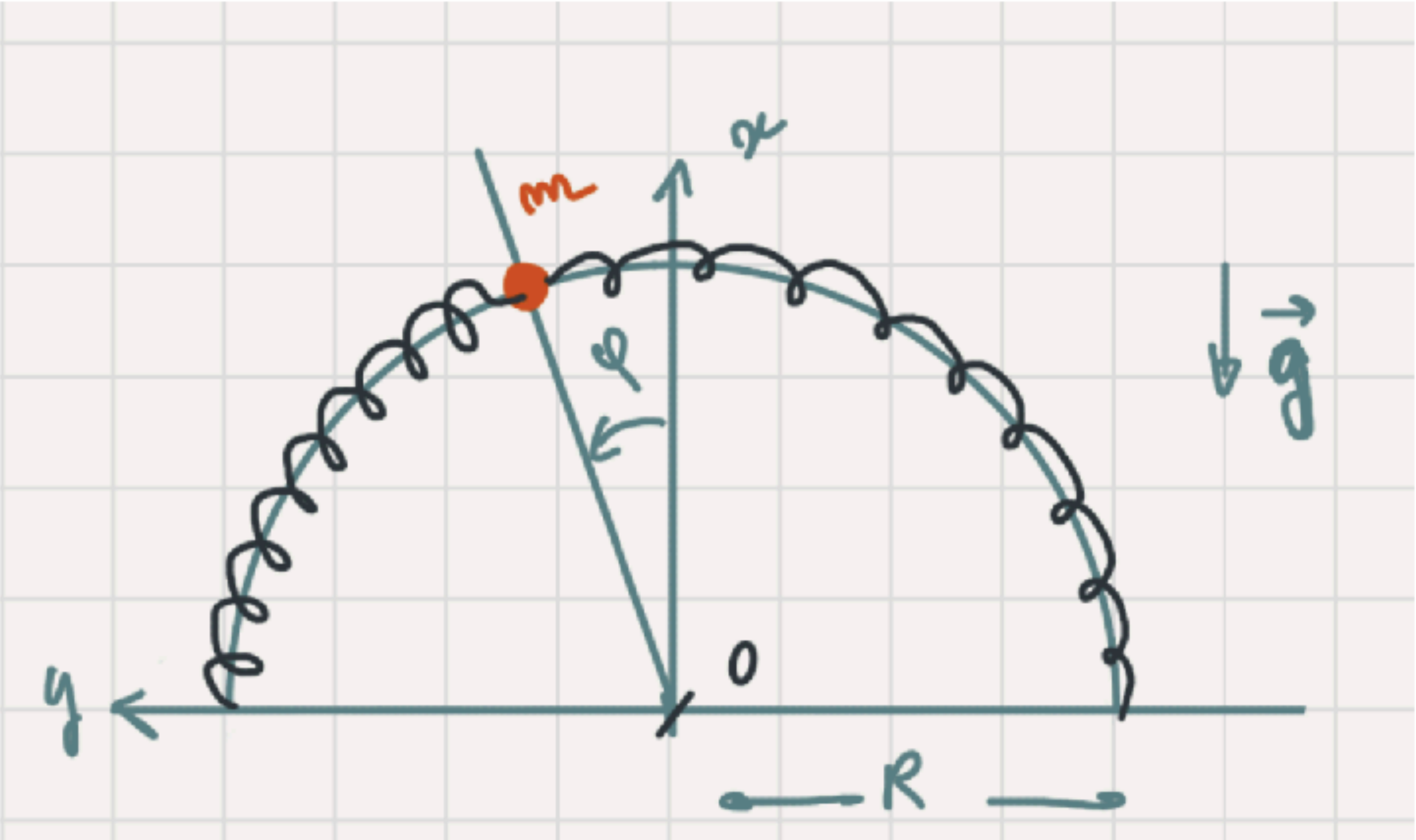

For small angles

$$E_p^{total} = \phi^2(kR^2 - mgR/2) + mgR$$

Potential energy of a simple harmonic oscillator will have the form

$$E_p = A(x - x_0)^2 + E_{p,0}$$

With $A > 0$

Matching terms with our equation, we find

$$E_{p,0} = mgR$$

$$x_0 = 0$$

$$A = kR^2 - mgR/2 \overset{?}{>} 0$$

$$kR - mg/2 > 0$$

$$k > \frac{mg}{2R}$$

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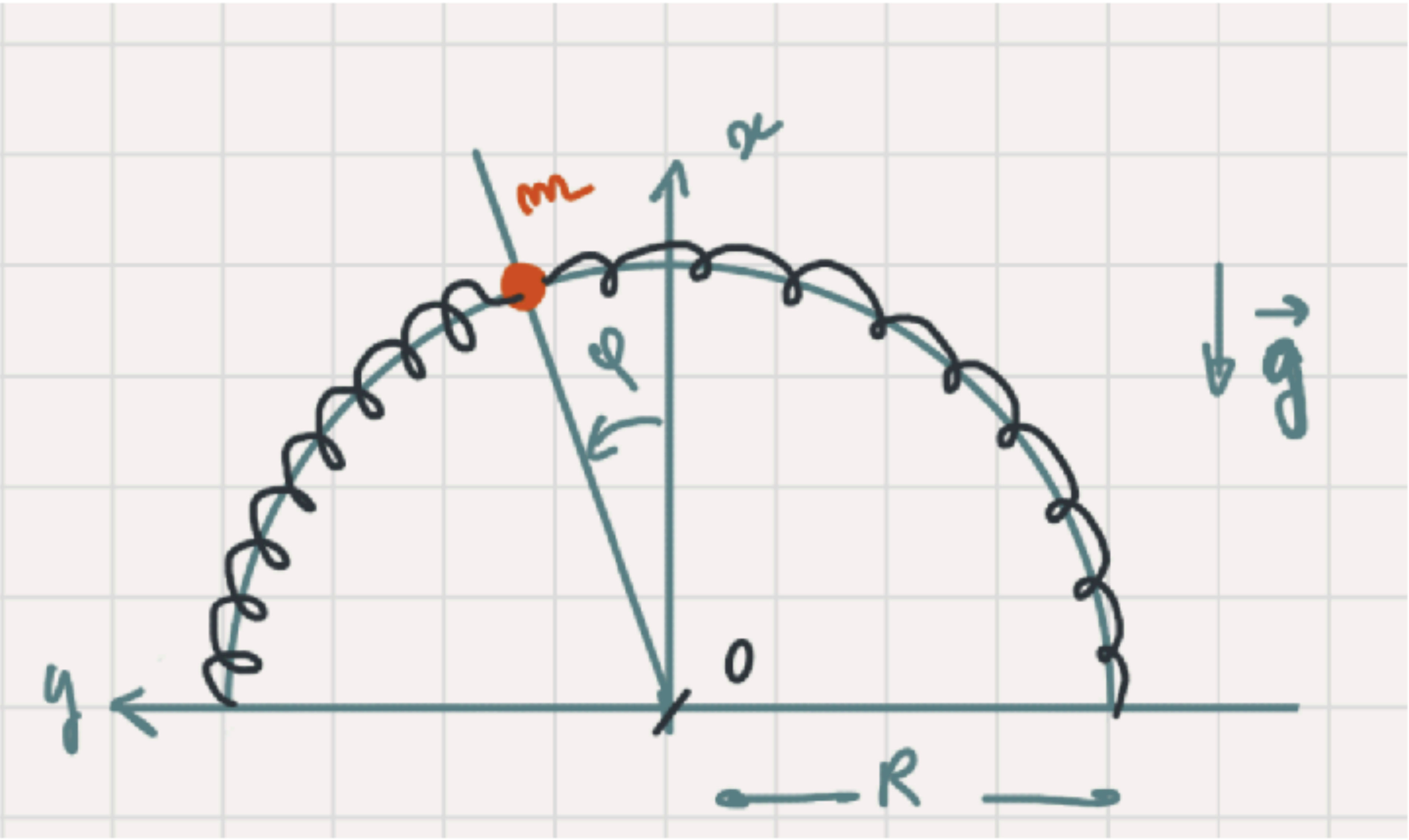

General expression for potential energy

$$E_p^{total} = k(R\phi)^2 + mgR\cos\phi$$

Kinetic energy

$$E_c = \frac{1}{2}mv^2 = \frac{1}{2}m(R\dot{\phi})^2$$

Mechanical energy

$$E_m = E_c + E_p^{total} = \frac{1}{2}m(R\dot{\phi})^2 + k(R\phi)^2 + mgR\cos\phi$$

Conservation of energy => mechanical energy does not change over time

$$\frac{d}{dt}E_m = 0 = \frac{d}{dt}\left(\frac{1}{2}m(R\dot{\phi})^2 + k(R\phi)^2 + mgR\cos\phi\right)$$

$$0 = \frac{1}{2}mR^2(2\ddot{\phi}\dot{\phi}) + kR^2(2\dot{\phi}\phi) - mgR\dot{\phi}\sin\phi$$

$$0 = mR\ddot{\phi} + 2kR\phi - mg\sin\phi$$

$$\ddot{\phi} + \frac{2k}{m}\phi - \frac{g}{R}\sin\phi = 0$$