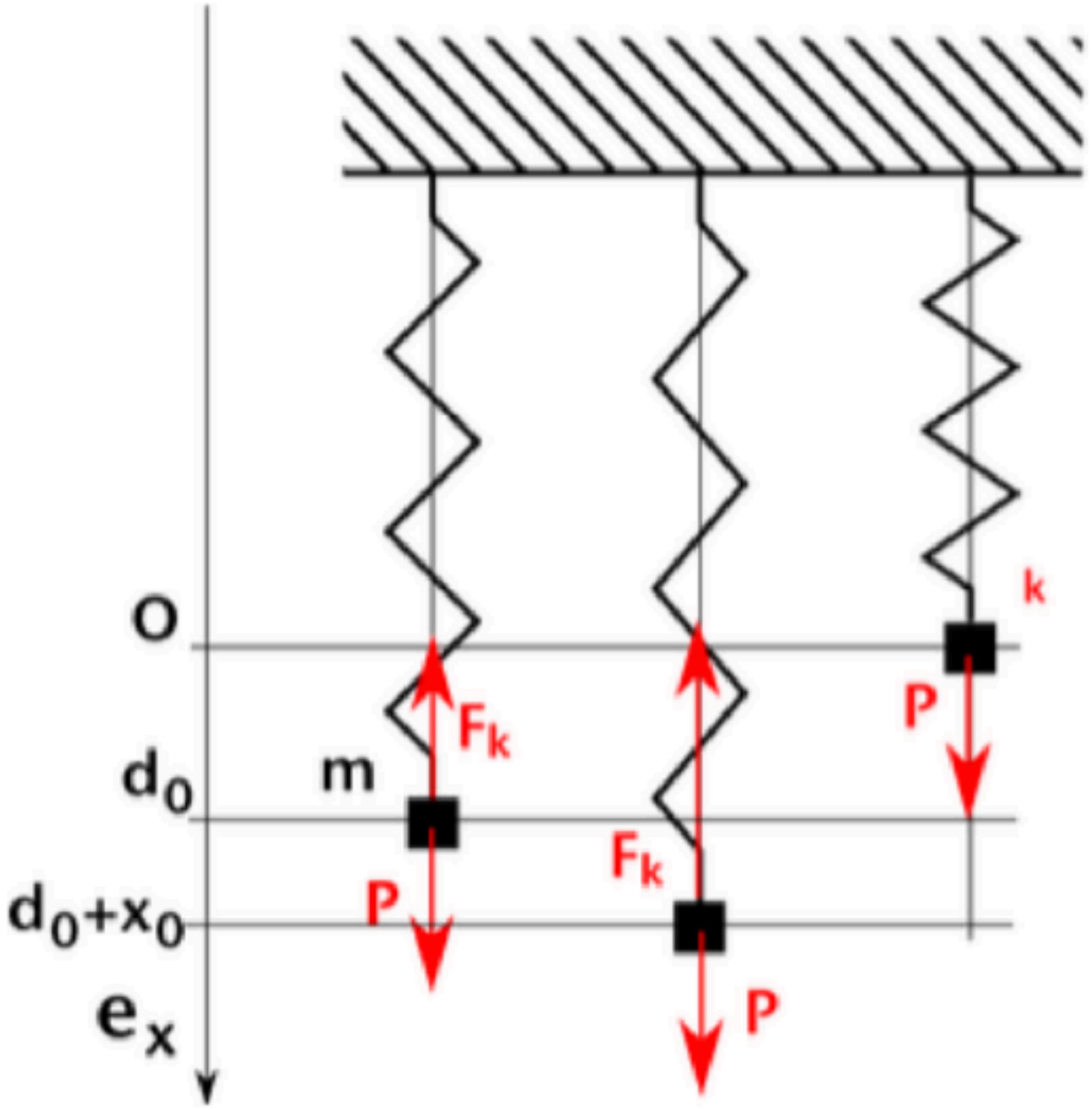
EXERCISE: VERTICAL SPRING



Consider a mass m suspended from a spring of rest length l_0 and spring constant k, attached vertically. The origin of the coordinate system O is placed at the position of the end of the spring when the spring at rest. When the mass is attached to the spring, the equilibrium position is denoted d_0. The mass is pulled downwards by a distance x_0, in addition to d_0, and released from rest.

1. Calculate d_0 as a function of the given data.
2. Find the the differential equation of motion using the in the coordinate system shown in the figure.
3. Make a change of variable on x to obtain the usual differential equation and solve it.
4. Calculate and draw the potential energy in the reference frame of the figure.
5. Use this potential energy to find the previous solution

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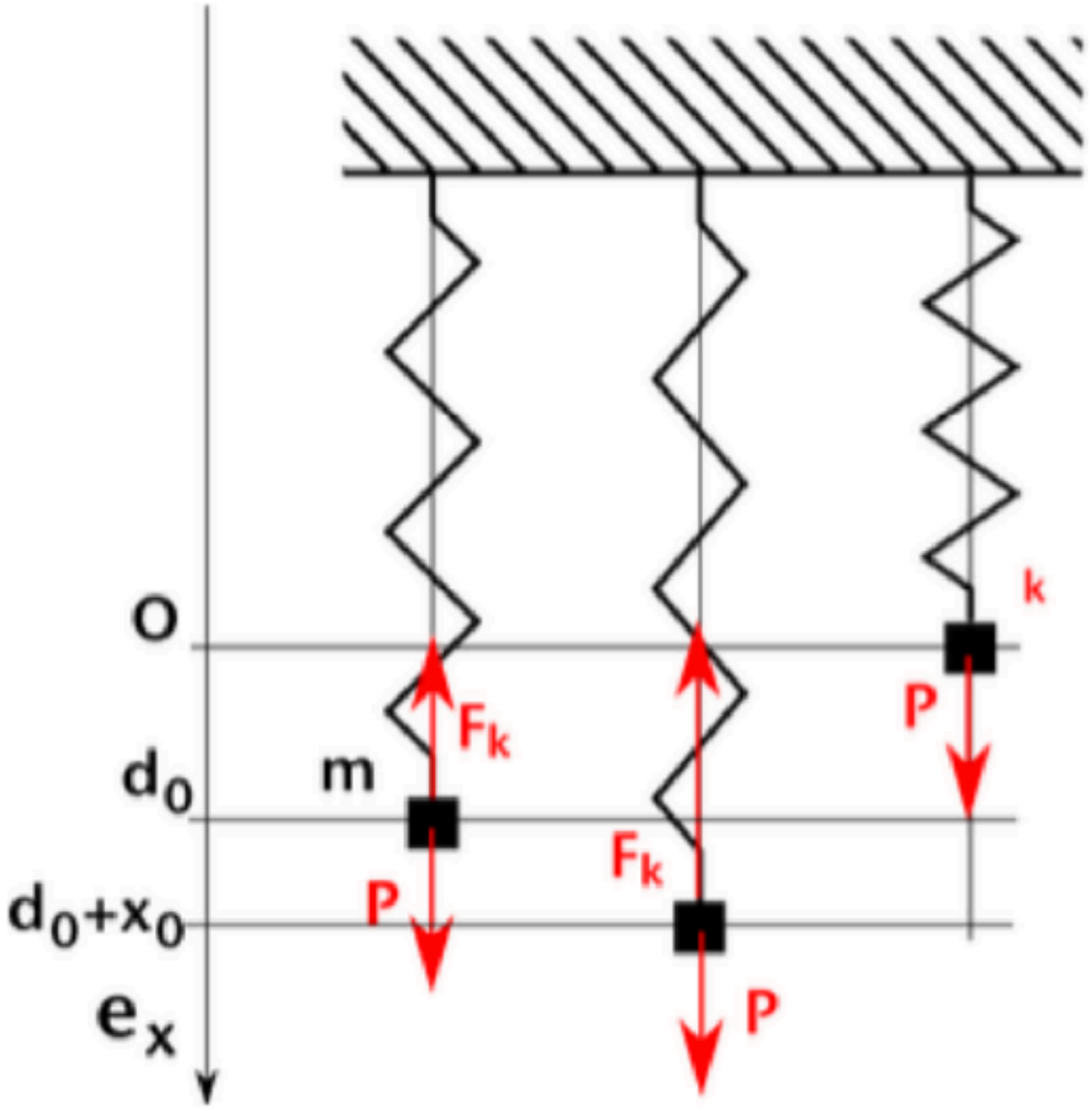

1. Calculate d_0 as a function of the given data.

First, determine all of the forces acing on the mass

$$\vec{F}_k = -k\overrightarrow{\Delta l} = -kx\vec{e}_x$$

$$\vec{F}_g = m\vec{g} = mg\vec{e}_x$$

Points up (negative x direction) when mass is pulled down (positive displacement x) and vice-versa

Newton's 2nd Law

$$m\vec{a} = mg\vec{e}_x - kx\vec{e}_x$$

On x-axis

$$m\ddot{x} = mg - kx$$

$$0 = mg - kd_0 \implies d_0 = \frac{mg}{k}$$

d_0 is the equilibrium position, so at $x = d_0, \ddot{x} = 0$

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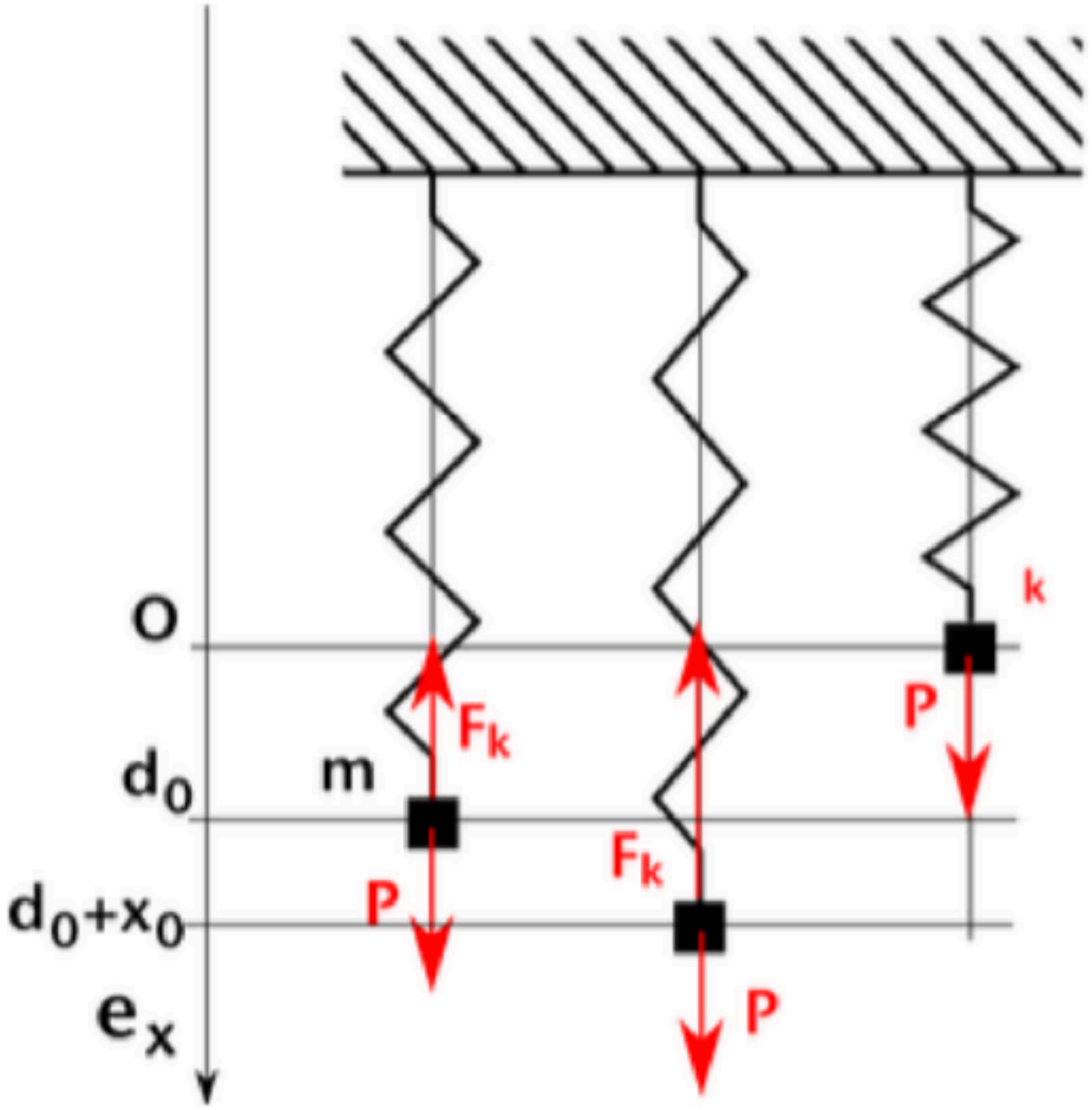

2. Find the the differential equation of motion using the in the coordinate system shown in the figure.

Newton's 2nd Law on the x-axis

$$m\ddot{x} = mg - kx$$

$$\ddot{x} + \frac{k}{m}x = g$$

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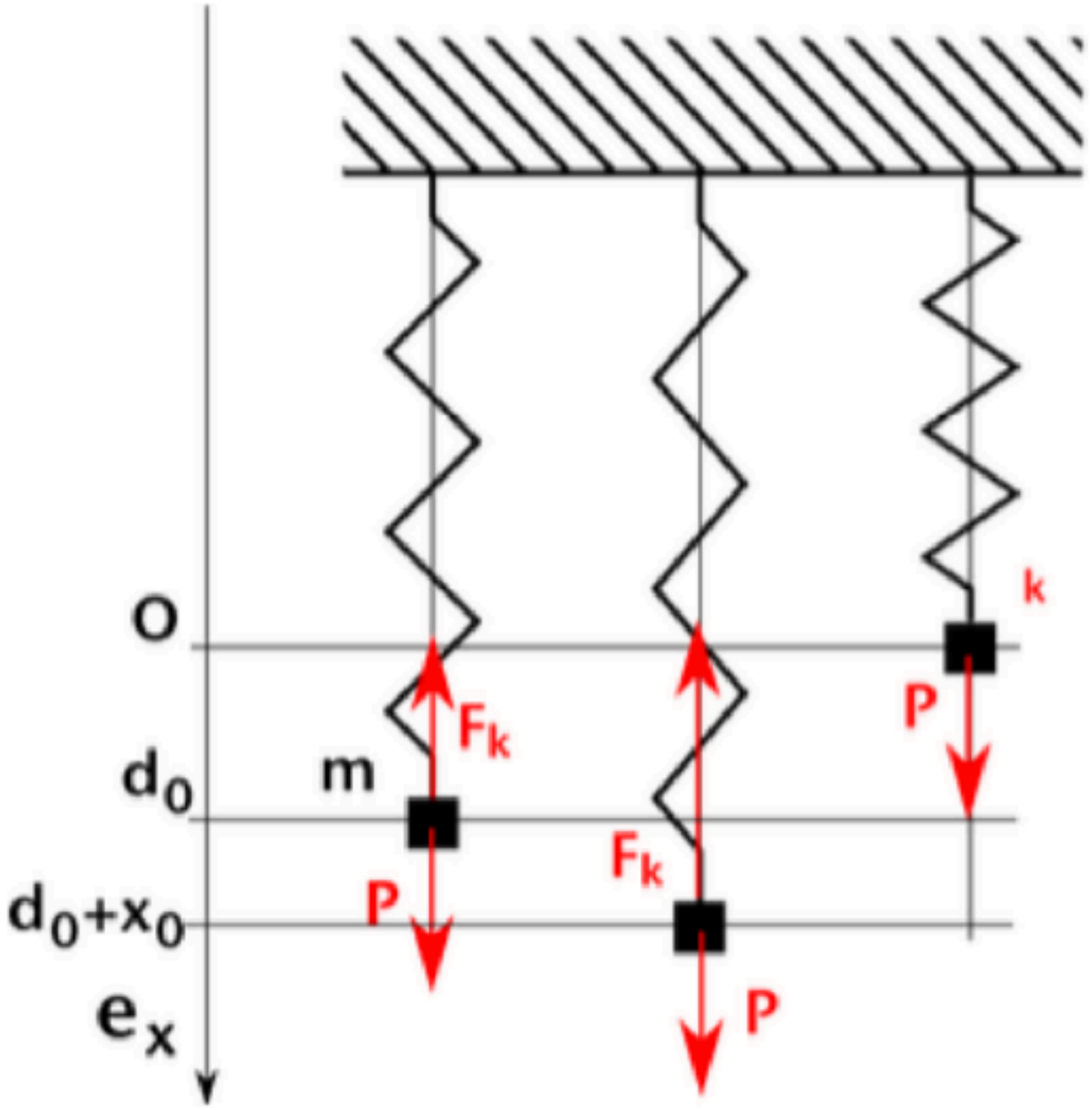

3. Make a change of variable on x to obtain the usual differential equation and solve it.

$$\ddot{x} + \frac{k}{m}x = g$$

Define $x = y + \Delta, \quad y = x - \Delta$ for some unknown constant Δ

Find the derivatives $\dot{x} = \dot{y}, \quad \ddot{x} = \ddot{y}$

Plug y into the differential equation

$$\ddot{y} + \frac{k}{m}(y + \Delta) = g$$

$$\ddot{y} + \frac{k}{m}y = g - \Delta\frac{k}{m}$$

$$\ddot{y} + \frac{k}{m}y = 0$$

Choose $\Delta = \dfrac{mg}{k} = d_0$

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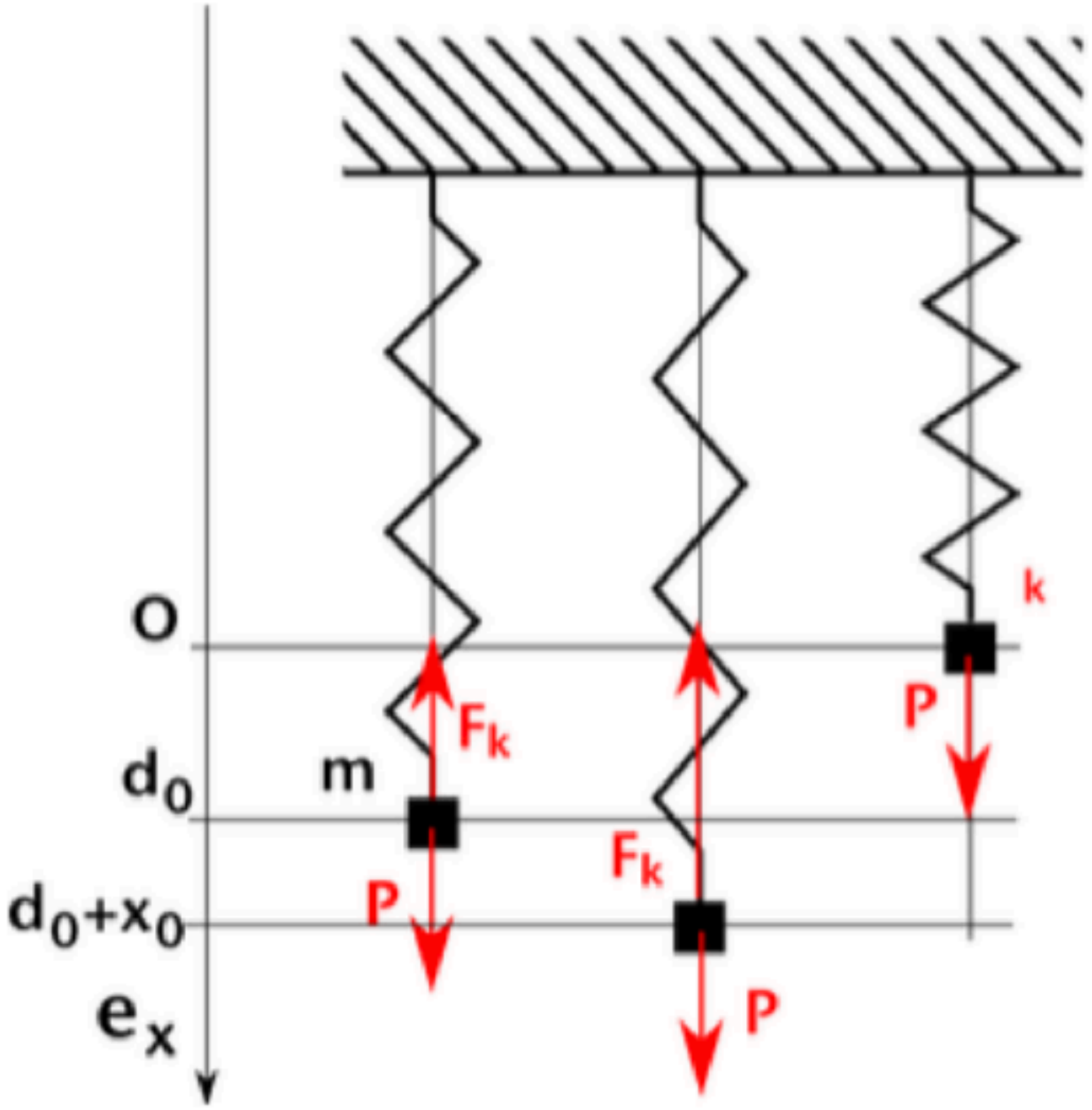

3. Make a change of variable on x to obtain the usual differential equation and solve it.

$$\ddot{x} + \frac{k}{m}x = g \qquad x = y + \Delta, \quad \Delta = \frac{mg}{k} = d_0$$

$$\ddot{y} + \frac{k}{m}y = 0$$

Differential equation for a harmonic oscillator

$$y(t) = A\sin(\Omega_0 t) + B\cos(\Omega_0 t)$$

General solution

$$\Omega_0 = \sqrt{\frac{k}{m}}$$

For this problem

Initial conditions:

$$x(t=0) = d_0 + x_0 \qquad y(t=0) = x(t=0) - d_0 = x_0 \qquad y(t=0) = A*0 + B*1 = x_0$$

$$\dot{x}(t=0) = 0 \qquad \dot{y}(t=0) = 0 \qquad \dot{y}(t=0) = A\Omega_0 * 1 - B\Omega_0 * 0 = 0$$

$$B = x_0, \quad A = 0$$

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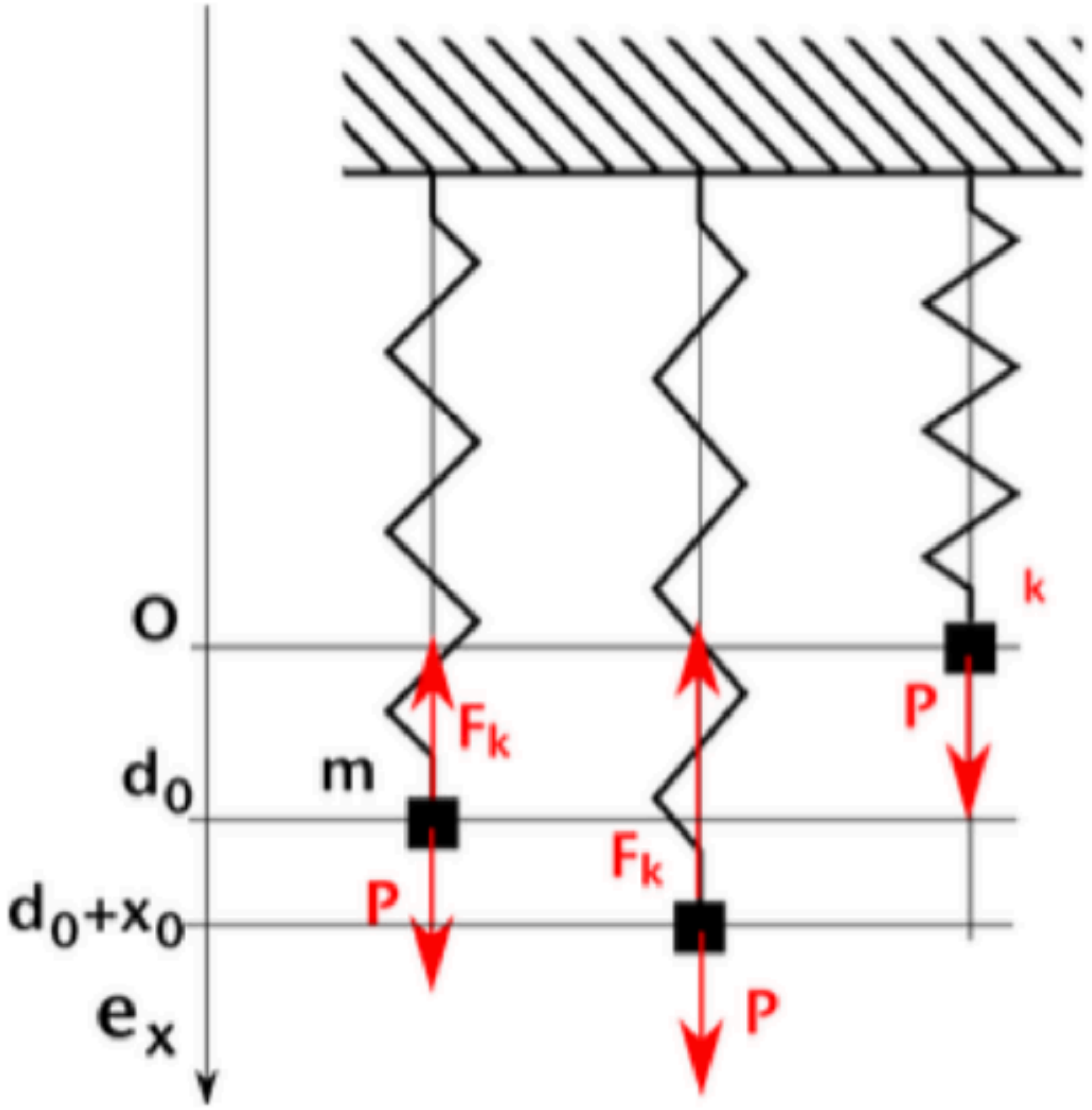

3. Make a change of variable on x to obtain the usual differential equation and solve it.

$$\ddot{x} + \frac{k}{m}x = g \qquad x = y + \Delta, \quad \Delta = \frac{mg}{k} = d_0$$

$$\ddot{y} + \frac{k}{m}y = 0$$

Differential equation for a harmonic oscillator

$$y(t) = x_0 \cos(\Omega_0 t)$$

Solution

$$\Omega_0 = \sqrt{\frac{k}{m}}$$

For this problem

$$x(t) = y(t) + \Delta$$

$$x(t) = x_0 \cos\left(\sqrt{\frac{k}{m}}t\right) + d_0$$

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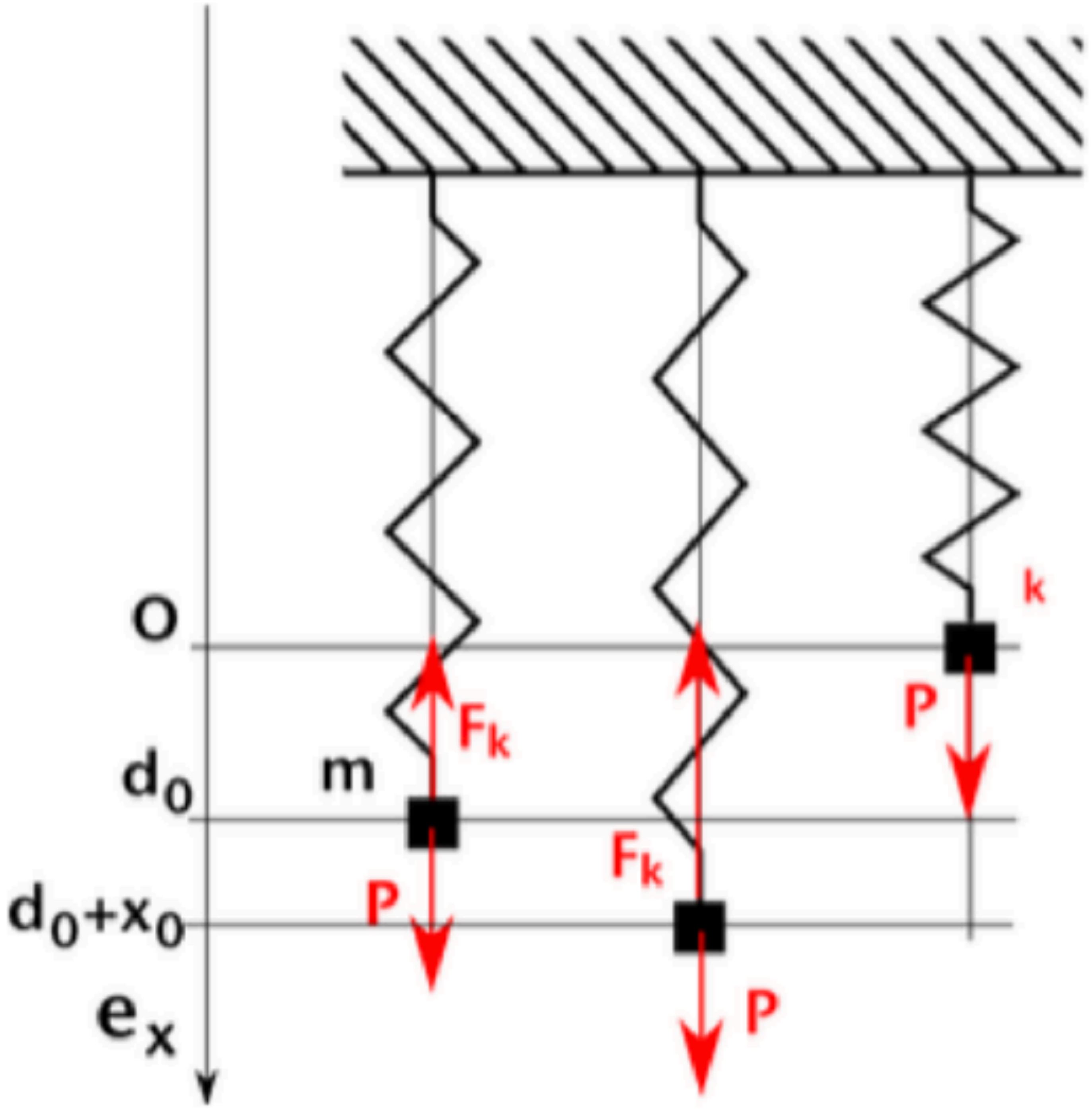


4.Calculate and draw the potential energy in the reference frame of the figure.

$$E_p^G = -mgx$$

As x increases, gravitational potential energy decreases

$$E_p^G = \frac{1}{2}kx^2$$

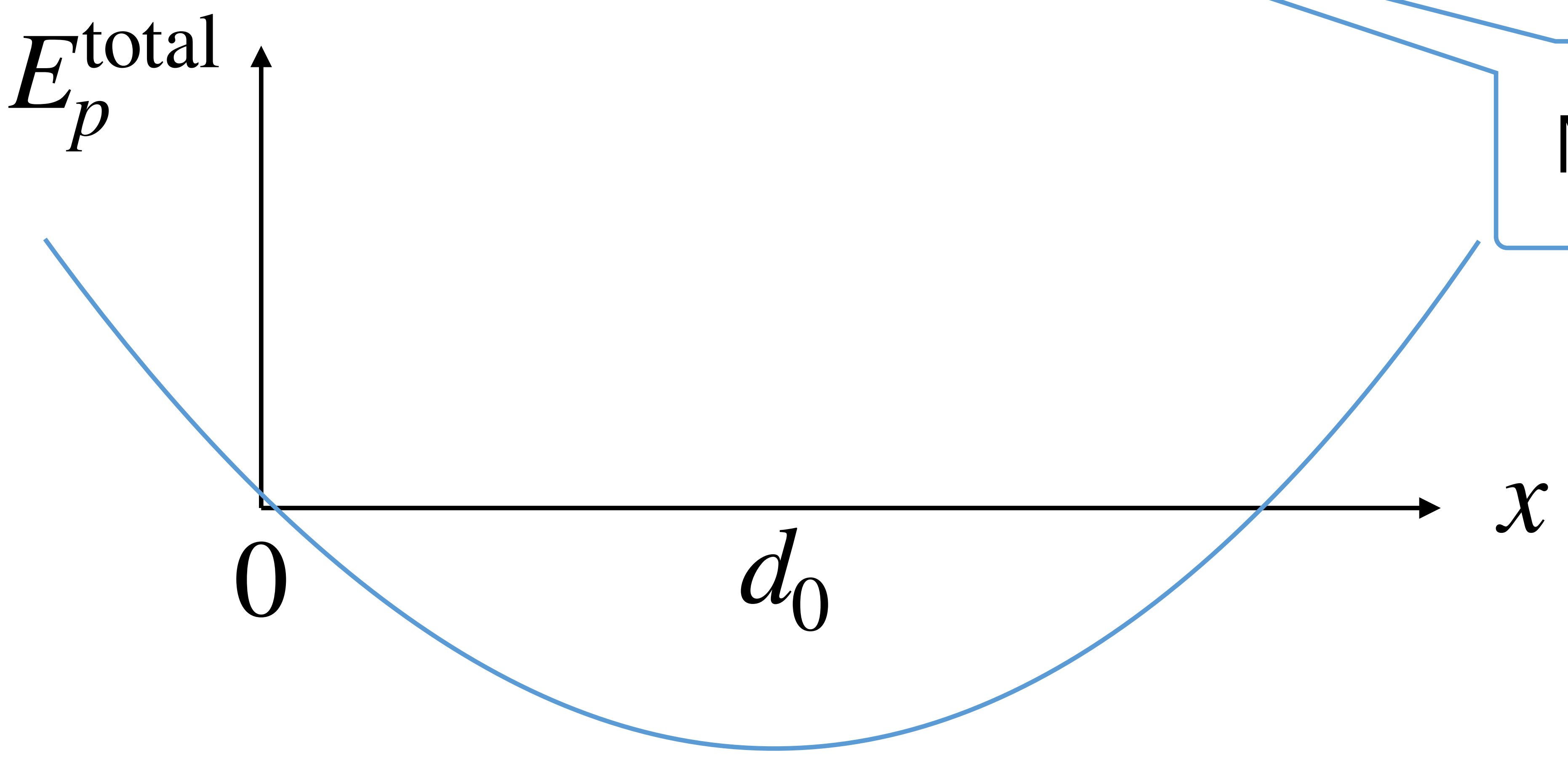
As x increases, spring potential energy increases

$$E_p^{total} = \frac{1}{2}kx^2 - mgx$$

Parabola

Zero at $x = 0$ or $x = 2mg/k$

Minimum at $x = mg/k = d_0$



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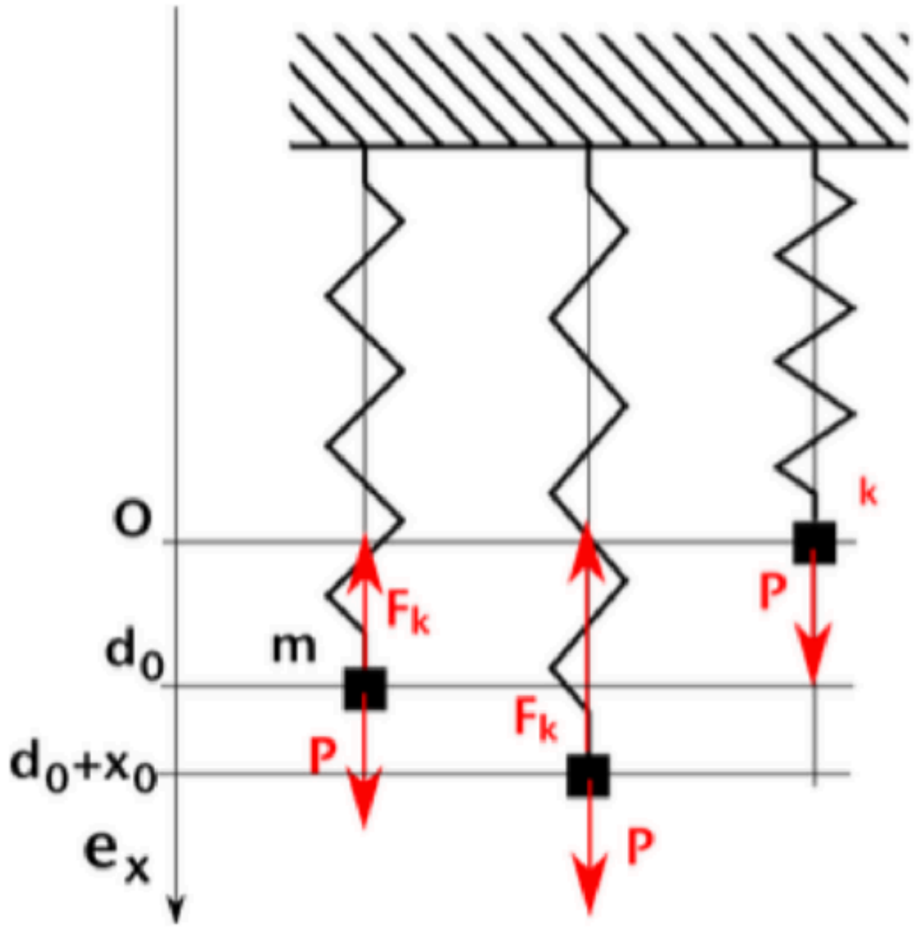

5. Use this potential energy to find the previous solution

$$E_p^{\text{total}} = \frac{1}{2}kx^2 - mgx$$

Method 1 using conservation of energy

Add kinetic and potential energy to get total mechanical energy

$$E_m^{\text{total}} = \frac{1}{2}kx^2 - mgx + \frac{1}{2}m\dot{x}^2$$

Time derivative

$$\frac{d}{dt}E_m^{\text{total}} = 0 = \frac{d}{dt}\left(\frac{1}{2}kx^2 - mgx + \frac{1}{2}m\dot{x}^2\right) = \frac{1}{2}k(2\dot{x}x) - mg\dot{x} + \frac{1}{2}m(2\ddot{x}\dot{x}) = k\dot{x}x - mg\dot{x} + m\ddot{x}\dot{x}$$

Divide by $\dot{x}$

$$kx - mg + m\ddot{x} = 0$$

$$\ddot{x} + \frac{k}{m}x = g$$

Recovered same differential equation and can solve it in the same way

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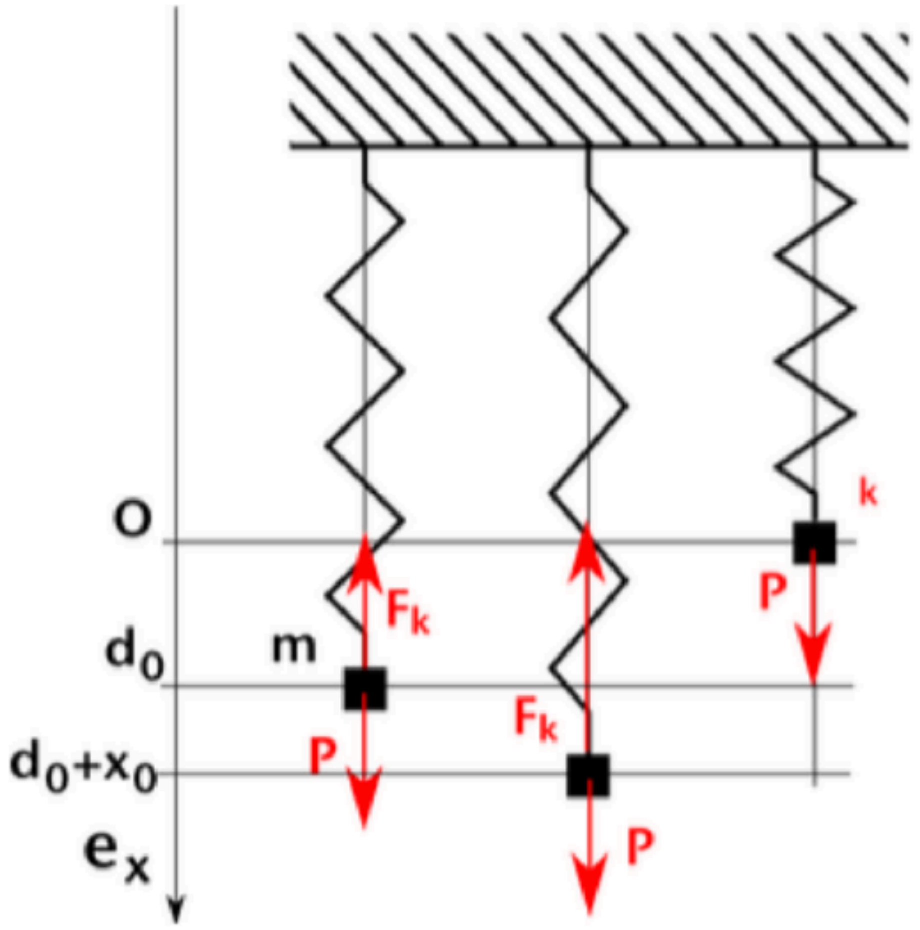

5. Use this potential energy to find the previous solution

$$E_p^{\text{total}} = \frac{1}{2}kx^2 - mgx$$

Method 1 using Force = -gradient of potential

$$m\vec{a} = \sum \vec{F}^{\text{conservative}} = -\frac{d}{dx}E_p^{\text{total}}$$

Derivative with respect to x

$$\frac{d}{dx}E_p^{\text{total}} = kx - mg$$

$$m\ddot{x} = -(kx - mg)$$

$$\ddot{x} + \frac{k}{m}x = g$$