

Consider a mass m suspended from a spring of rest length l_0 and spring constant k , attached vertically. The origin of the coordinate system O is placed at the position of the end of the spring when the spring is at rest. When the mass is attached to the spring, the equilibrium position is denoted d_0 . The mass is pulled downwards by a distance x_0 , in addition to d_0 , and released from rest.

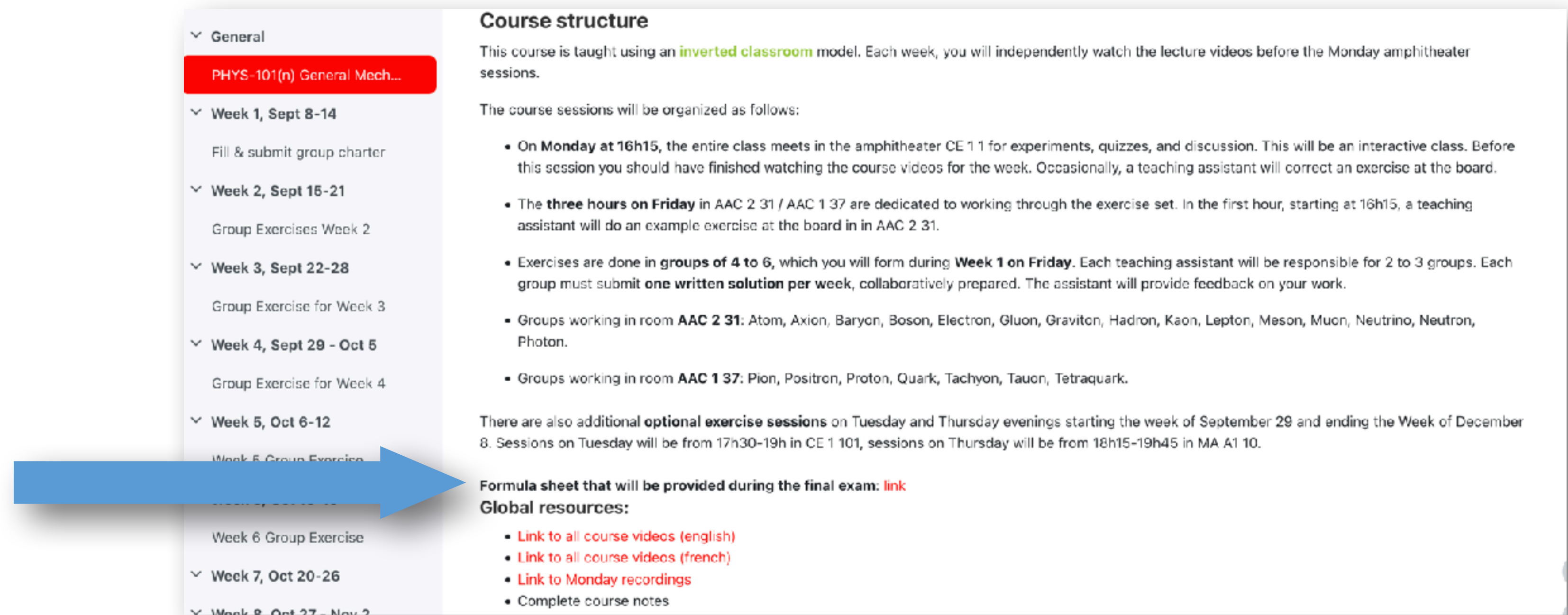
Calculate the position of the mass as a function of time

PHYS-101 WEEK 10

EXAMEN BLANC / PRACTICE EXAM

Available next week! No room booked, but we will put a practice final exam on the course website

You can use the formula sheet linked on the Moodle, and one 1-sided hand-written sheet. No calculators, computers, phones, etc



The screenshot shows a Moodle course page for 'PHYS-101(n) General Mech...'. The left sidebar contains a navigation menu with sections for 'General', 'Week 1, Sept 8-14', 'Week 2, Sept 15-21', 'Week 3, Sept 22-28', 'Week 4, Sept 29 - Oct 5', 'Week 5, Oct 6-12', 'Week 6 Group Exercise', 'Week 7, Oct 20-26', and 'Week 8, Oct 27 - Nov 2'. The main content area is titled 'Course structure' and contains the following text:

Course structure

This course is taught using an **inverted classroom** model. Each week, you will independently watch the lecture videos before the Monday amphitheater sessions.

The course sessions will be organized as follows:

- On **Monday at 16h15**, the entire class meets in the amphitheater CE 1 1 for experiments, quizzes, and discussion. This will be an interactive class. Before this session you should have finished watching the course videos for the week. Occasionally, a teaching assistant will correct an exercise at the board.
- The **three hours on Friday** in AAC 2 31 / AAC 1 37 are dedicated to working through the exercise set. In the first hour, starting at 16h15, a teaching assistant will do an example exercise at the board in AAC 2 31.
- Exercises are done in **groups of 4 to 6**, which you will form during **Week 1 on Friday**. Each teaching assistant will be responsible for 2 to 3 groups. Each group must submit **one written solution per week**, collaboratively prepared. The assistant will provide feedback on your work.
- Groups working in room **AAC 2 31**: Atom, Axion, Baryon, Boson, Electron, Gluon, Graviton, Hadron, Kaon, Lepton, Meson, Muon, Neutrino, Neutron, Photon.
- Groups working in room **AAC 1 37**: Pion, Positron, Proton, Quark, Tachyon, Tauon, Tetraquark.

There are also additional **optional exercise sessions** on Tuesday and Thursday evenings starting the week of September 29 and ending the Week of December 8. Sessions on Tuesday will be from 17h30-19h in CE 1 101, sessions on Thursday will be from 18h15-19h45 in MA A1 10.

Formula sheet that will be provided during the final exam: [link](#)


Global resources:

- [Link to all course videos \(english\)](#)
- [Link to all course videos \(french\)](#)
- [Link to Monday recordings](#)
- Complete course notes

A blue arrow points from the 'Formula sheet' link in the screenshot to the text in the main slide.

RECITATION CONTENT

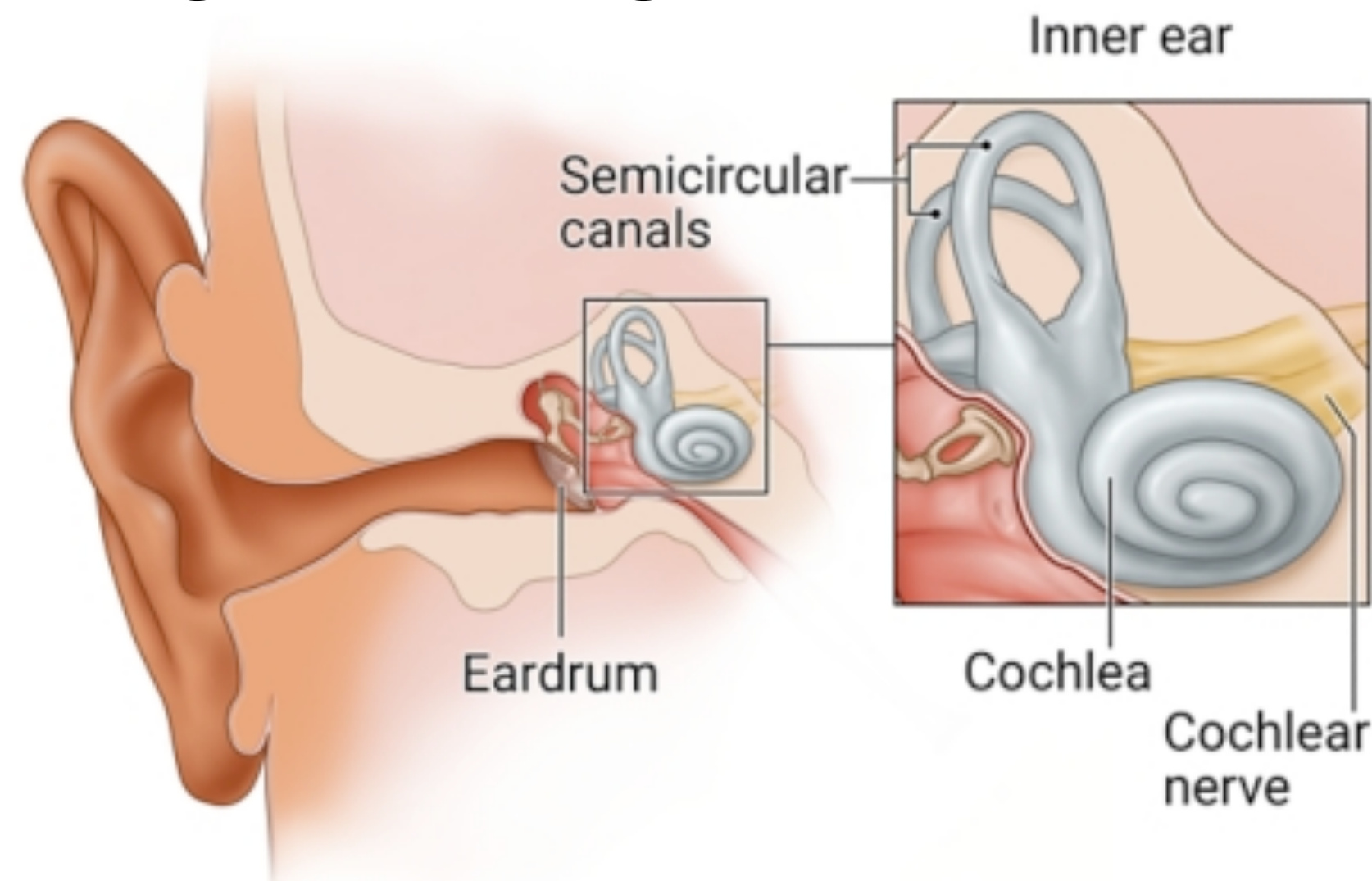
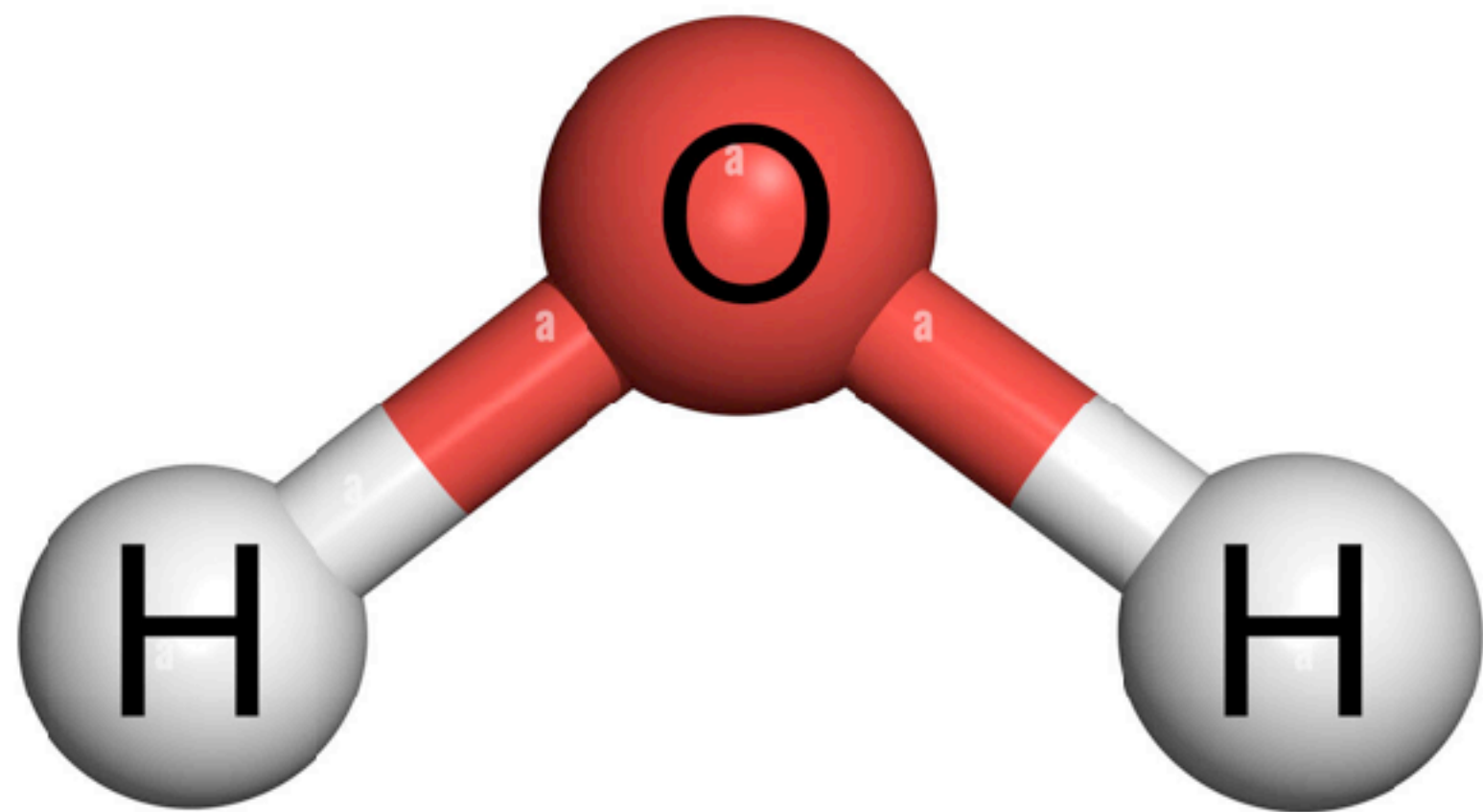


 This week in recitation, I would like the TA to review... ⋮

- Week 9 problem set exercise 1 0%
- Week 9 problem set exercise 2 0%
- Week 9 problem set exercise 3 0%
- Week 9 problem set exercise 4 0%
- A new example using material from week 10 (harmonic oscillator) 0%
- An example using non-inertial reference frames 0%
- Other (please comment below) 0%

HARMONIC OSCILLATOR

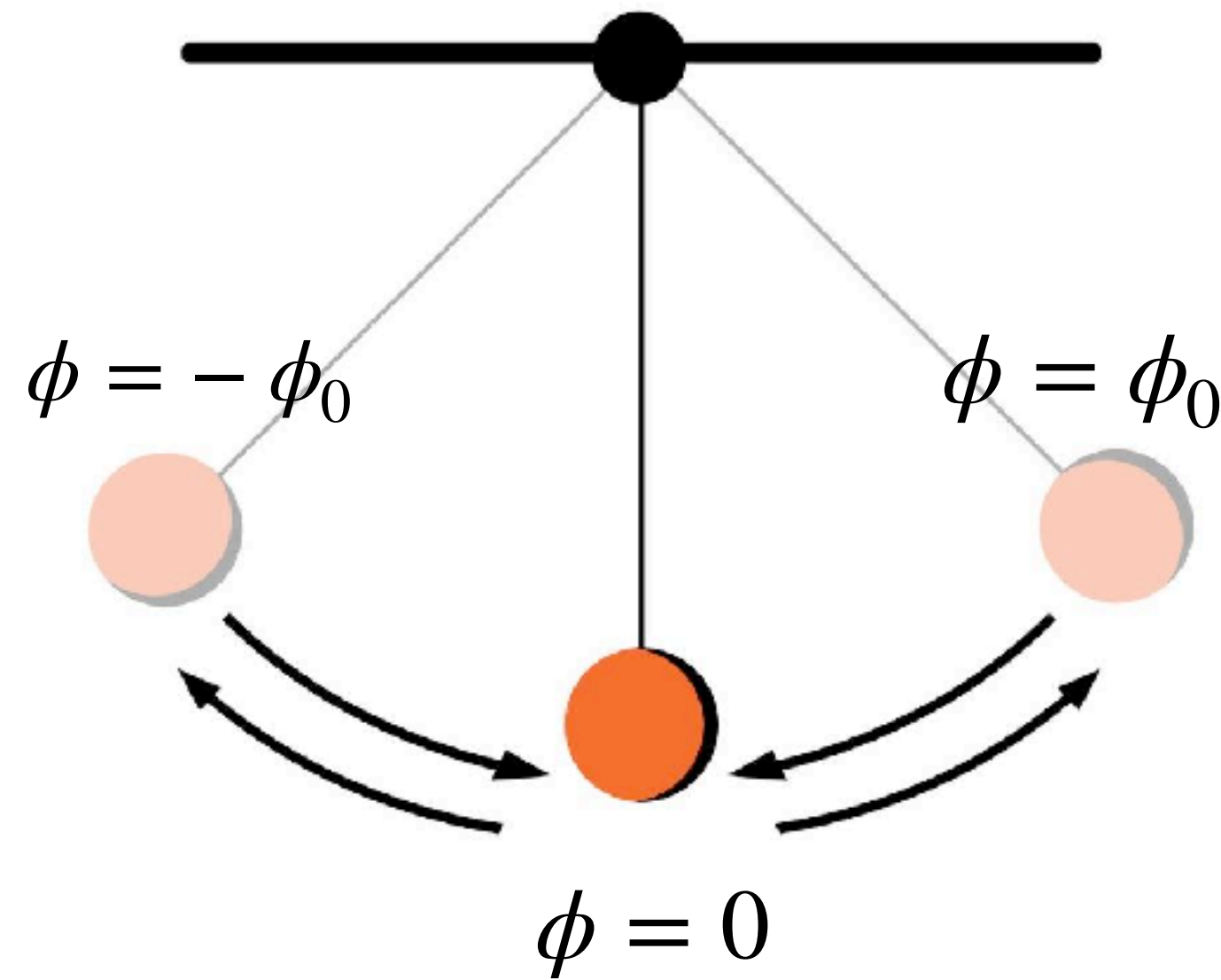
- Requirements for a simple harmonic oscillator:
 - Restoring force proportional to displacement $F_k = -kx$
 - Quadratic potential $E_p = A(x - x_0)^2 + E_{p,0}$
- Damped oscillator adds a force proportional to negative velocity $F_f = -\beta\dot{x}$
- In this course: spring & pendulum, but many other examples of oscillators in nature & engineering



QUIZ: PENDULUM



Consider a simple pendulum with mass m attached to a string. The mass oscillates between $+\phi_0$ and $-\phi_0$. Which statements are correct?

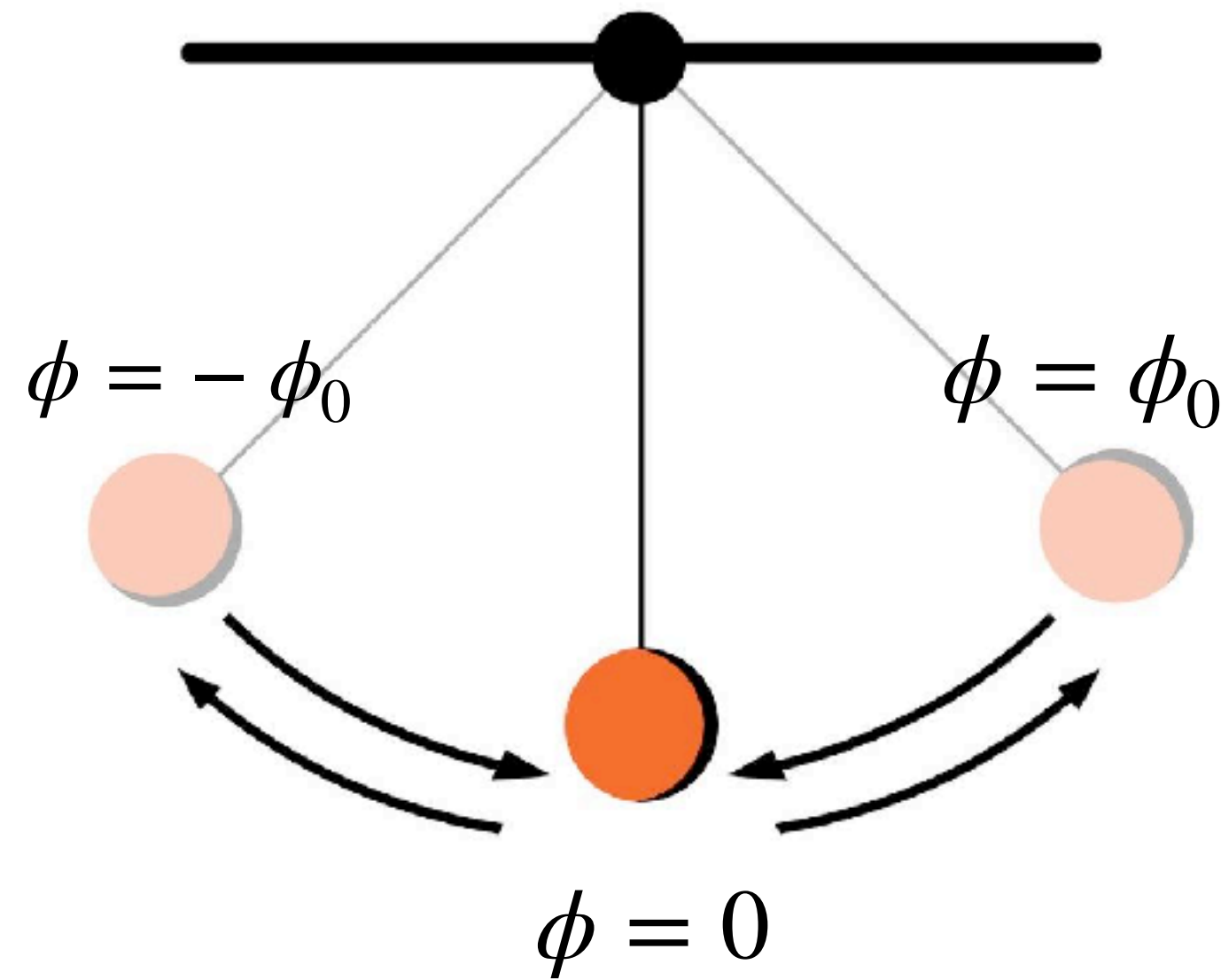


- The tension in the string is constant 0%
- The tension in the string is maximum at $+$ or $-\phi_0$ 0%
- The tension in the string is maximum at $\phi = 0$ (vertical) 0%
- The tension in the string at $\phi = \phi_0$ is equal to mg 0%
- The tension in the string depends on m , ϕ_0 , and ϕ 0%
- The maximum tension depends on ϕ_0 0%

QUIZ: PENDULUM



Consider a simple pendulum with mass m attached to a string. The mass oscillates between $+\phi_0$ and $-\phi_0$. Which statements are correct?



- The tension in the string is constant 0%
- The tension in the string is maximum at $+$ or $-\phi_0$ 0%
- The tension in the string is maximum at $\phi = 0$ (vertical) 0%
- The tension in the string at $\phi = \phi_0$ is equal to mg 0%
- The tension in the string depends on m , ϕ_0 , and ϕ 0%
- The maximum tension depends on ϕ_0 0%

DIFFERENTIAL EQUATIONS

Fluid friction alone

$$F_f = -\beta\dot{x}$$

Newton's 2nd Law

$$m\ddot{x} = -\beta\dot{x}$$

Differential equation

$$\frac{df}{dt} + \Lambda f = 0$$

$$f = \dot{x}, \Lambda = \beta/m$$

General solution

$$f(t) = Ae^{\lambda t}$$

DIFFERENTIAL EQUATIONS

Differential equation

General solution

$$\frac{df}{dt} + \Lambda f = 0 \quad \Lambda > 0$$

$$f(t) = Ae^{\lambda t}$$

$$\frac{df}{dt} = \lambda Ae^{\lambda t}$$

DIFFERENTIAL EQUATIONS

Differential equation

General solution

$$\frac{df}{dt} + \Lambda f = 0 \quad \Lambda > 0$$

$$f(t) = Ae^{\lambda t}$$

$$\frac{df}{dt} = \lambda Ae^{\lambda t}$$

$$(\lambda Ae^{\lambda t}) + \Lambda (Ae^{\lambda t}) = 0$$

$$\lambda + \Lambda = 0$$

DIFFERENTIAL EQUATIONS

Differential equation

General solution

$$\frac{df}{dt} + \Lambda f = 0 \quad \Lambda > 0$$

$$f(t) = Ae^{\lambda t}$$

$$\frac{df}{dt} = \lambda Ae^{\lambda t}$$

$$(\lambda Ae^{\lambda t}) + \Lambda (Ae^{\lambda t}) = 0$$

$$\lambda + \Lambda = 0$$

$$\lambda = -\Lambda$$

$$f(t) = Ae^{-\Lambda t}$$

Real number

DIFFERENTIAL EQUATIONS

Differential equation

Solution

$$\frac{d^2f}{dt^2} + \Omega_0^2 f = 0$$

$$f(t) = Ae^{\lambda t}?$$

$$\frac{df}{dt} = \lambda Ae^{\lambda t}$$

$$\frac{d^2f}{dt^2} = \lambda^2 Ae^{\lambda t}$$

$$(\lambda^2 Ae^{\lambda t}) + \Omega_0^2 (Ae^{\lambda t}) = 0$$

$$\lambda^2 + \Omega_0^2 = 0 \quad \Rightarrow \quad \lambda^2 = -\Omega_0^2 \quad \Rightarrow \quad \lambda = \pm i\Omega_0$$

Two imaginary numbers

DIFFERENTIAL EQUATIONS

Differential equation

$$\frac{d^2f}{dt^2} + \Omega_0^2 f = 0$$

See slides 21/22 of the CH8 course video for this derivation

Solution

$$f(t) = Ae^{i\Omega_0 t} + Be^{-i\Omega_0 t}$$

If $f(t)$ is **real**, we take only the **real** part of this equation

$$\Re[f(t)] = A' \cos(\Omega_0 t) + B' \sin(\Omega_0 t) = C \cos(\Omega_0 t + \phi)$$

DIFFERENTIAL EQUATIONS

Differential equation

$$\frac{d^2f}{dt^2} + 2Y\frac{df}{dt} + \Omega_0^2 f = 0$$

$$\Omega_0^2 > 0, \quad Y > 0$$

General solution

$$f(t) = Ae^{\lambda t}?$$

$$\begin{aligned} (\lambda^2 Ae^{\lambda t}) + 2Y(\lambda Ae^{\lambda t}) + \Omega_0^2(Ae^{\lambda t}) &= 0 \\ \lambda^2 + 2Y\lambda + \Omega_0^2 &= 0 \end{aligned}$$

Characteristic equation

$$\lambda = -Y \pm \sqrt{Y^2 - \Omega_0^2}$$

Could be 2 real numbers, 2 complex numbers, or 1 real number

DIFFERENTIAL EQUATIONS

Differential equation

$$\frac{d^2f}{dt^2} + 2Y\frac{df}{dt} + \Omega_0^2 f = 0$$

$$\Omega_0^2 > 0, \quad Y > 0$$

General solution

$$\lambda_1 = -Y + \sqrt{Y^2 - \Omega_0^2} \quad \lambda_2 = -Y - \sqrt{Y^2 - \Omega_0^2}$$

2 roots

$$f(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

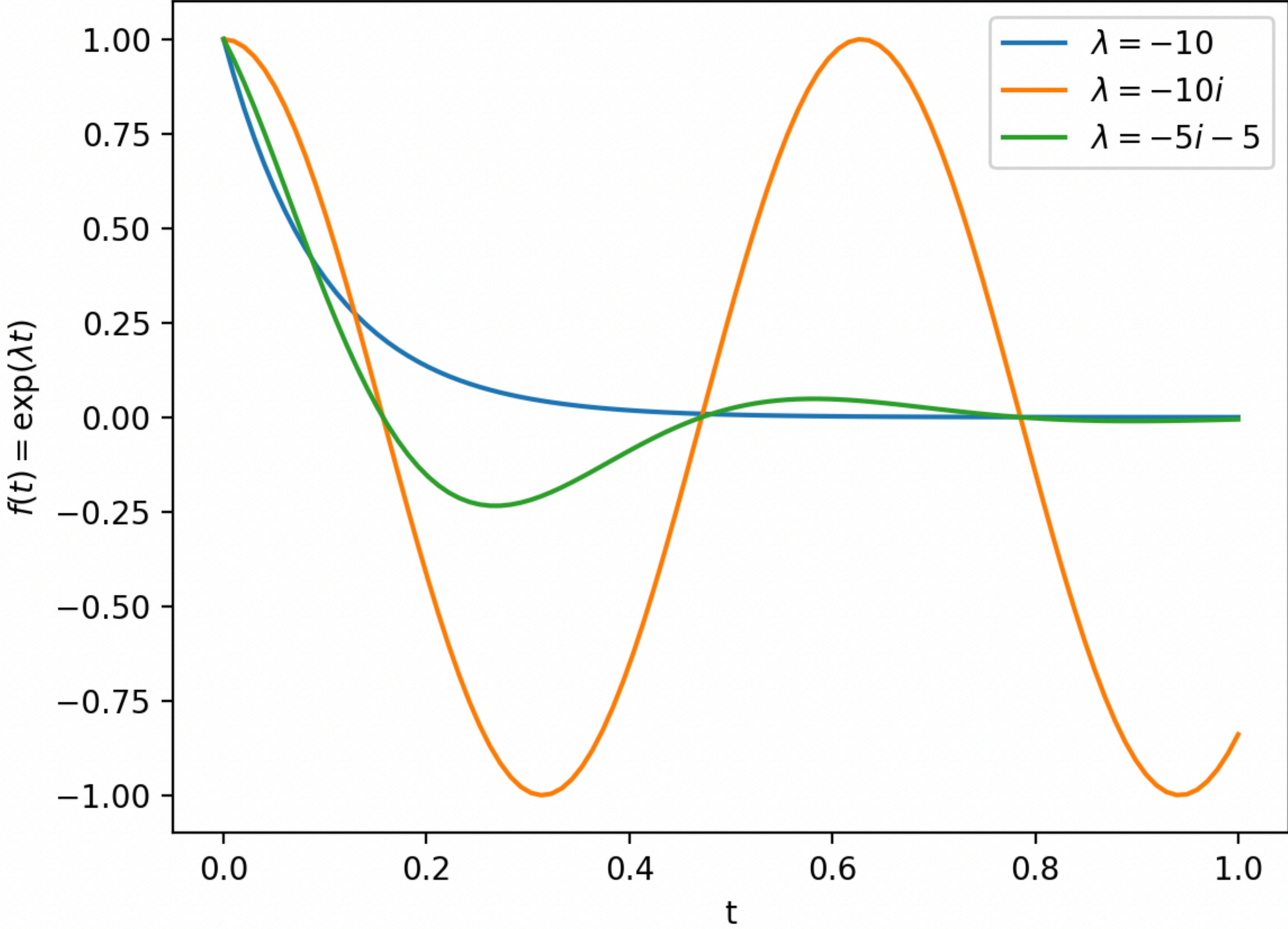
Need to take the real part of f if roots are complex

$$\lambda_1 = \lambda_2, \quad Y = \Omega_0^2$$

1 root

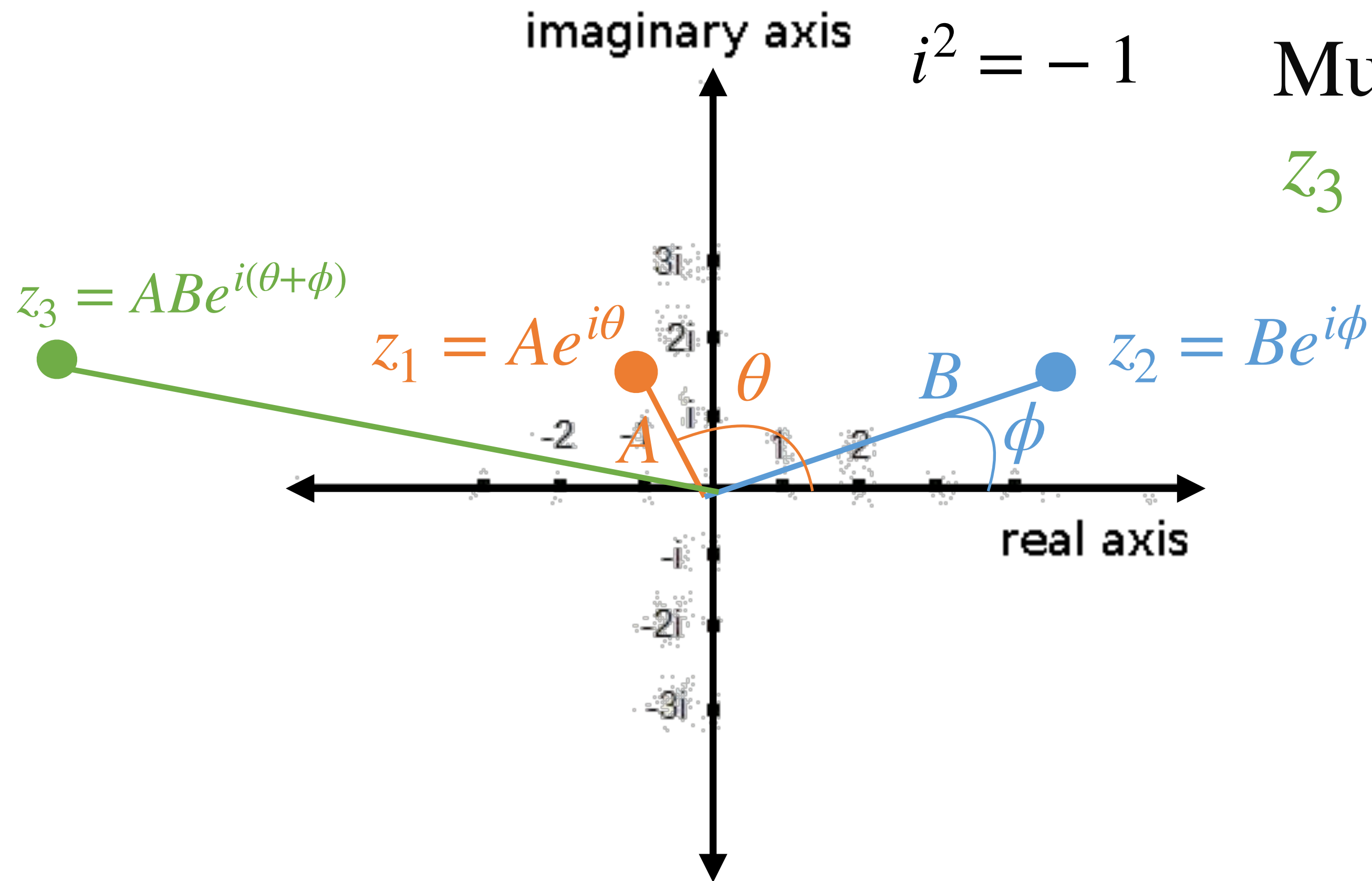
$$f(t) = (A + Bt)e^{\lambda_2 t}$$

DIFFERENTIAL EQUATIONS



$$f(t) = Ae^{\lambda t}$$

COMPLEX PLANE



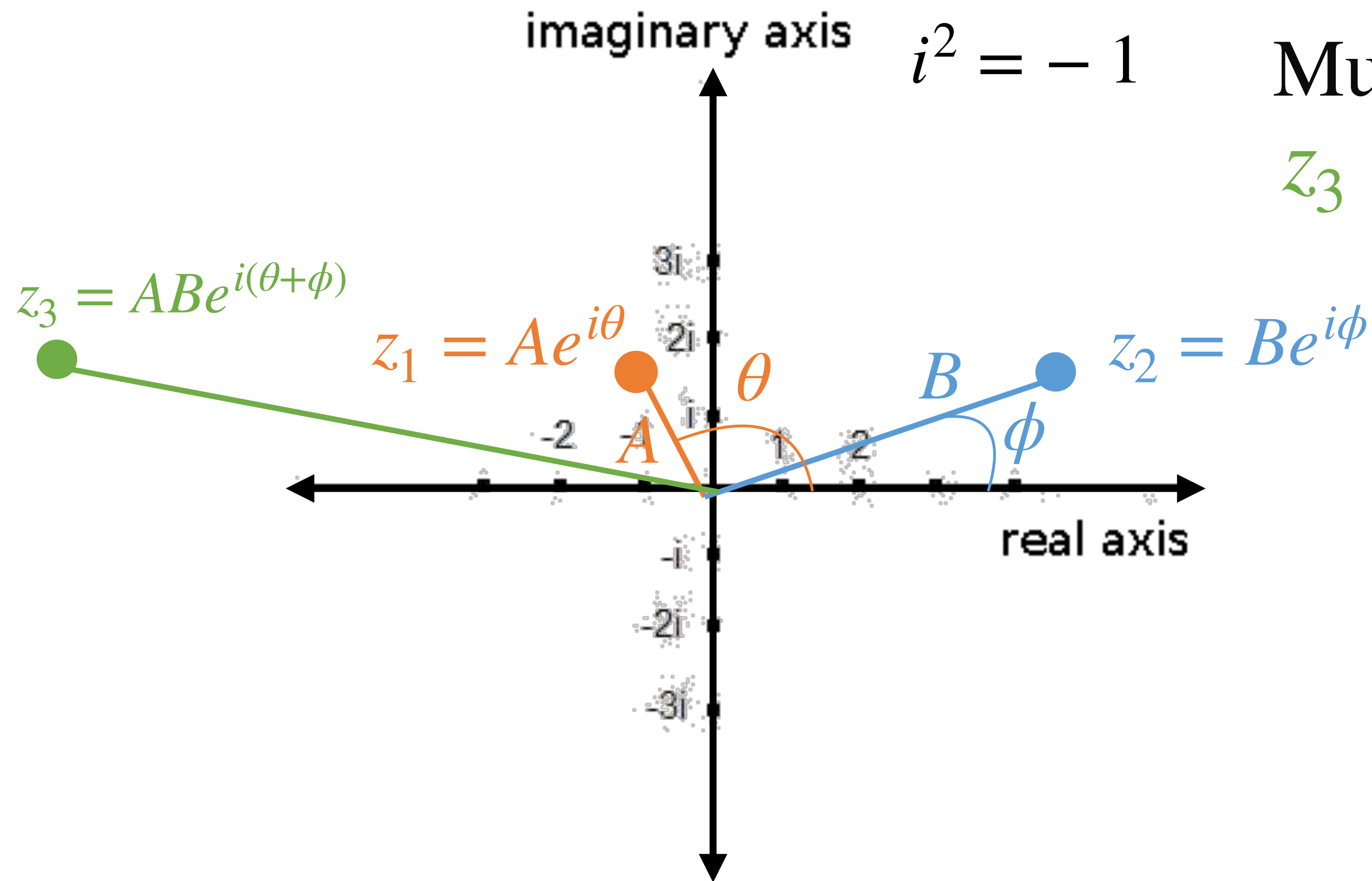
Multiplication

$$z_3 = z_1 z_2 = Ae^{i\theta} Be^{i\phi} = ABe^{i(\theta+\phi)}$$

Scale by B

Rotate by ϕ

COMPLEX PLANE



Multiplication

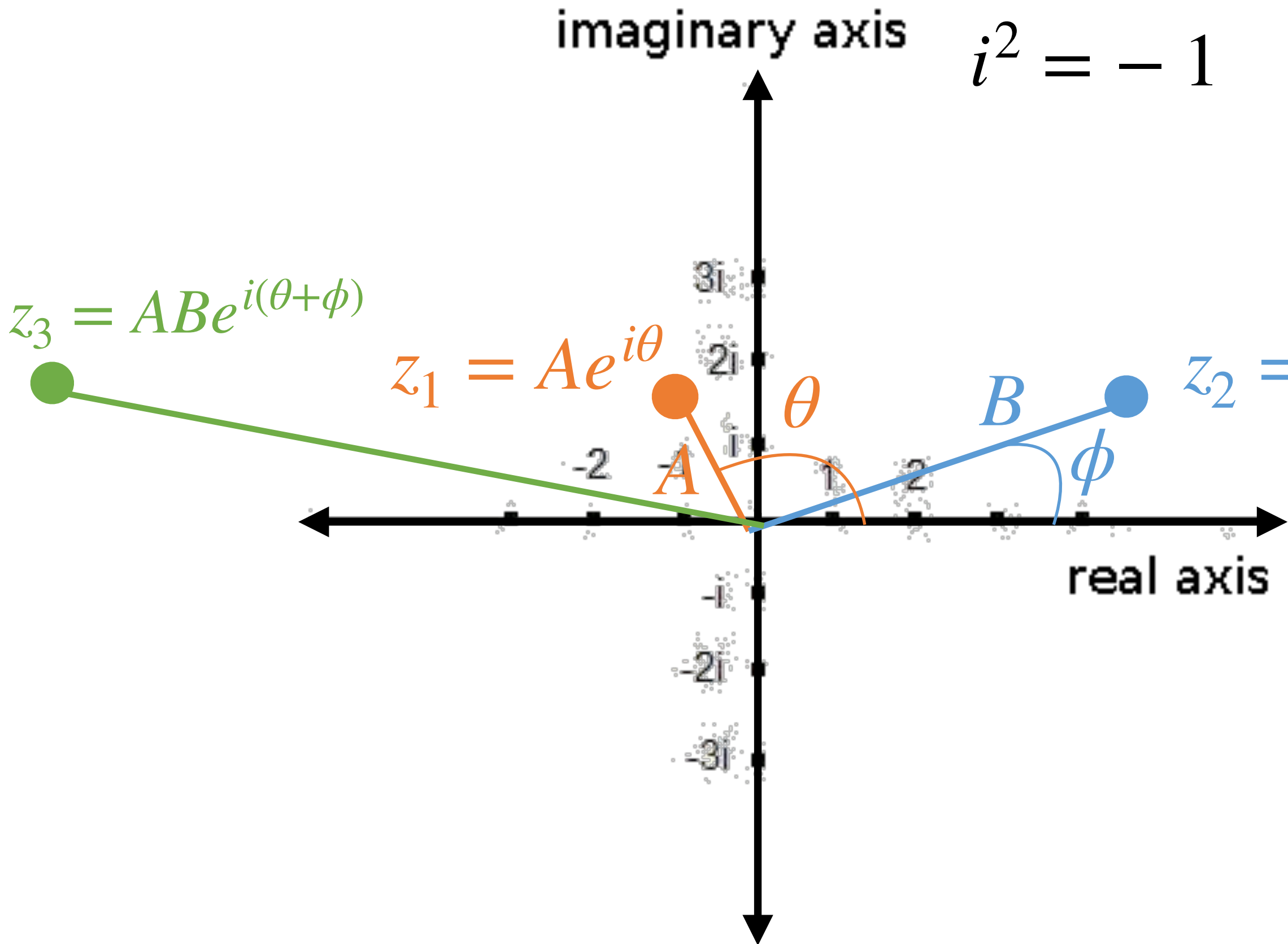
$$z_3 = z_1 z_2 = Ae^{i\theta} Be^{i\phi} = ABe^{i(\theta+\phi)}$$

Scale by B

Rotate by ϕ

$$f(t + dt) = Ae^{\lambda(t+dt)} = Ae^{\lambda t} e^{\lambda dt} = f(t) e^{\lambda dt}$$

COMPLEX PLANE



$i^2 = -1$

Multiplication

$$z_3 = z_1 z_2 = A e^{i\theta} B e^{i\phi} = A B e^{i(\theta+\phi)}$$

Scale by B

Rotate by ϕ

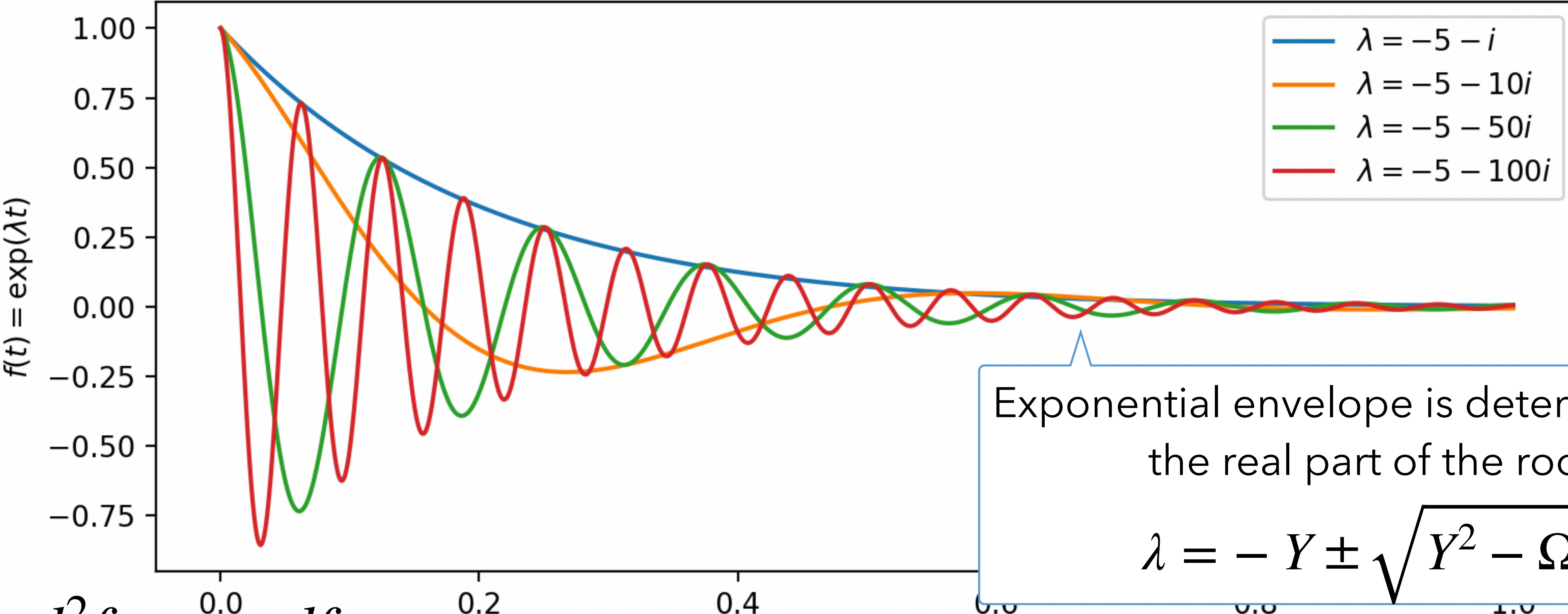
$$f(t + dt) = A e^{\lambda(t+dt)} = A e^{\lambda t} e^{\lambda dt} = f(t) e^{\lambda dt}$$

Imaginary part rotates

$$\lambda = a + ib \quad \longrightarrow \quad f(t + dt) = f(t) e^{b dt} e^{ia dt}$$

Real part scales

DIFFERENTIAL EQUATIONS



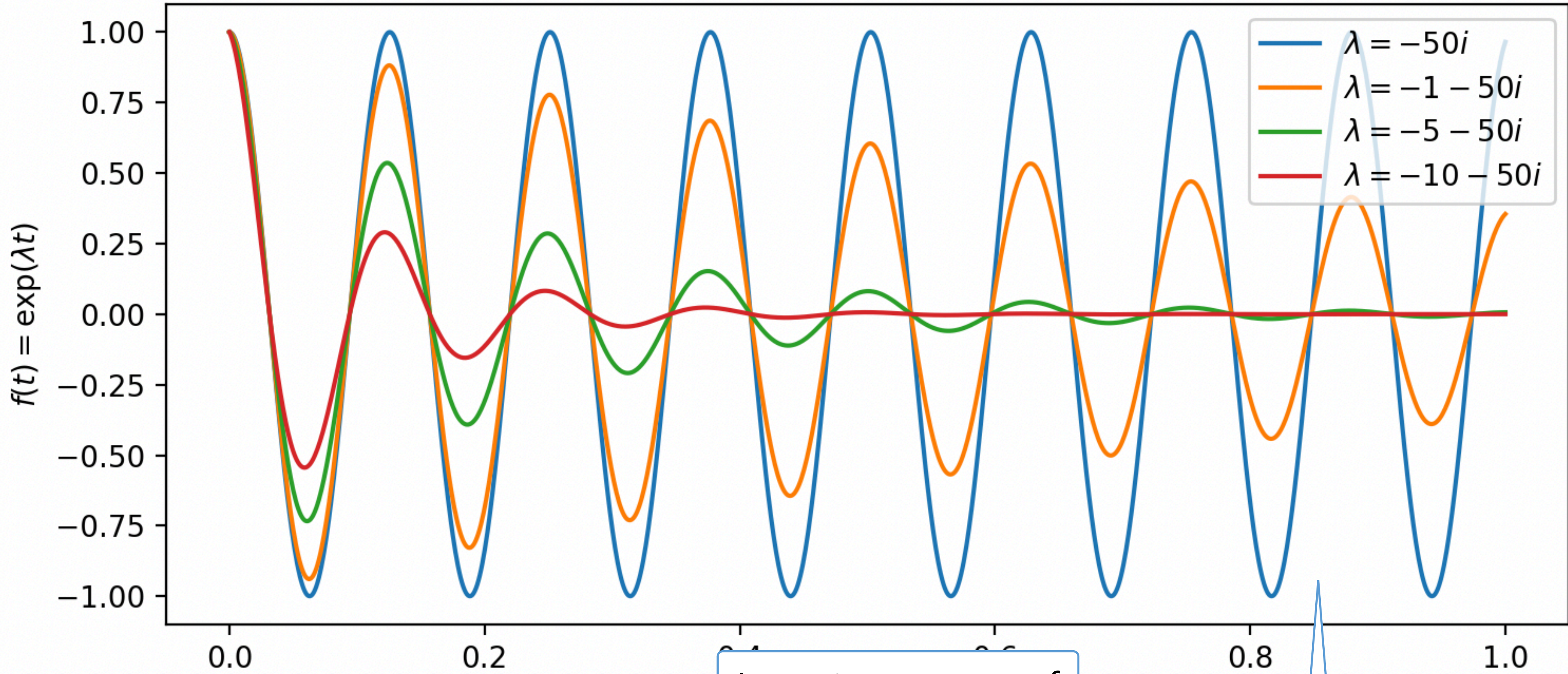
Exponential envelope is determined by the real part of the root

$$\lambda = -Y \pm \sqrt{Y^2 - \Omega_0^2}$$

$$\frac{d^2f}{dt^2} + 2Y \frac{df}{dt} + \Omega_0^2 f = 0$$

Real part of the root comes from damping term

DIFFERENTIAL EQUATIONS



$$\frac{d^2f}{dt^2} + 2Y\frac{df}{dt} + \Omega_0^2 f = 0$$

Imaginary part of the root determined by natural frequency & damping

Oscillation period is determined by the imaginary part of the root

$$\lambda = -Y \pm \sqrt{Y^2 - \Omega_0^2}$$

GENERAL STRATEGY

Step 1: analyze the forces, write Newton's 2nd law, project along relevant axes

Step 2: from the resulting differential equation, use the appropriate solution

Simple harmonic oscillator, equation of motion : $\ddot{x}(t) + \Omega_0^2 x(t) = 0$. Solution : $x(t) = A * \cos(\Omega_0 t) + B * \sin(\Omega_0 t)$

Damped harmonic oscillator, equation of motion : $\ddot{x} + 2\gamma\dot{x} + \Omega_0^2 x = 0$

Case 1 : Critical regime ($\Delta' = 0, \gamma = \Omega_0$)

$$x(t) = (A + Bt)e^{-\gamma t}$$

Case 2 : Overdamped regime ($\Delta' > 0, \gamma^2 > \Omega_0^2$)

$$x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

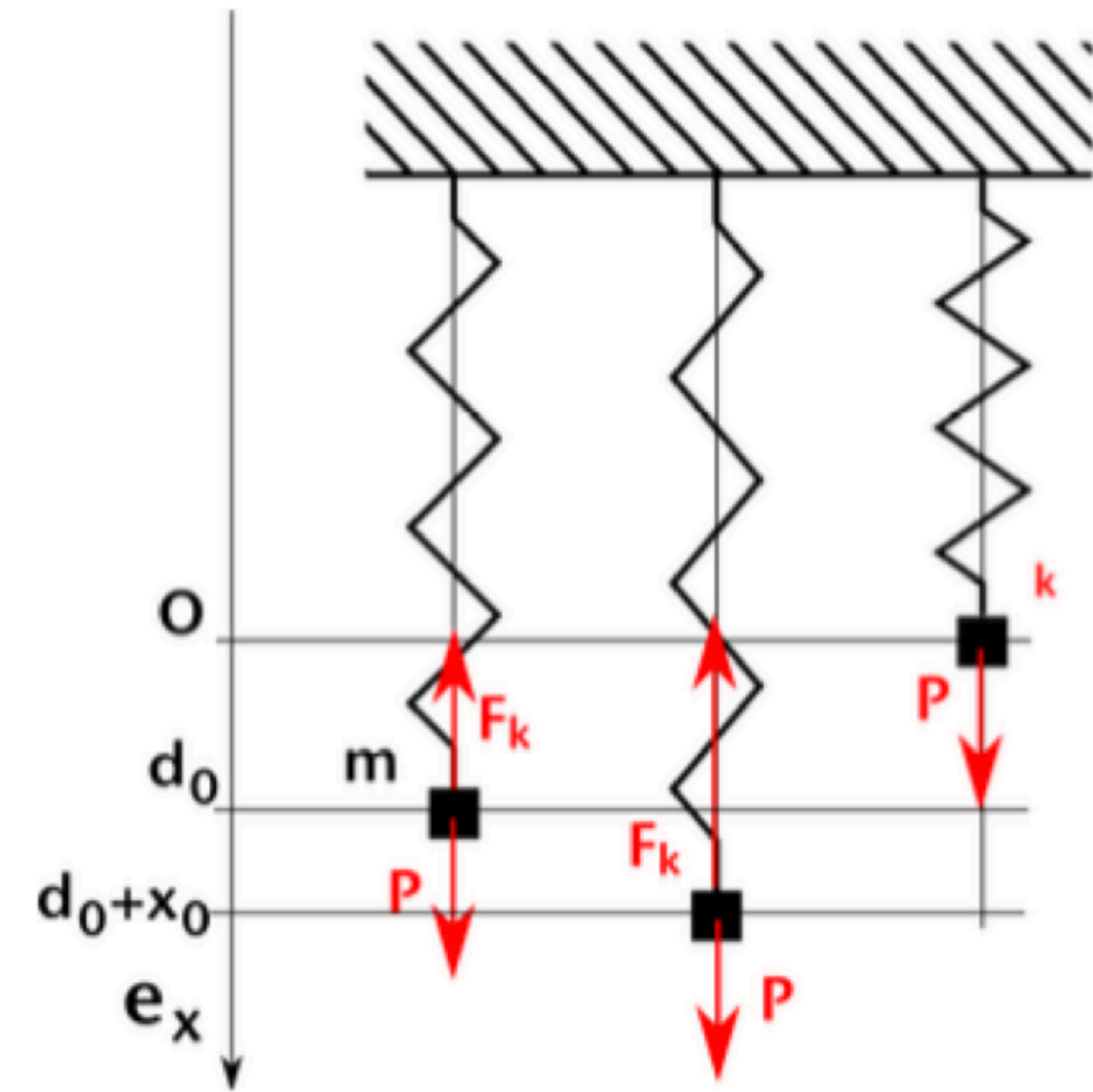
Case 3 : Underdamped regime ($\Delta' < 0, \gamma^2 < \Omega_0^2$)

$$\omega = \sqrt{\Omega_0^2 - \gamma^2} \quad x(t) = e^{-\gamma t} * (A' \cos \omega t + B' \sin \omega t)$$

FOUCAULT PENDULUM



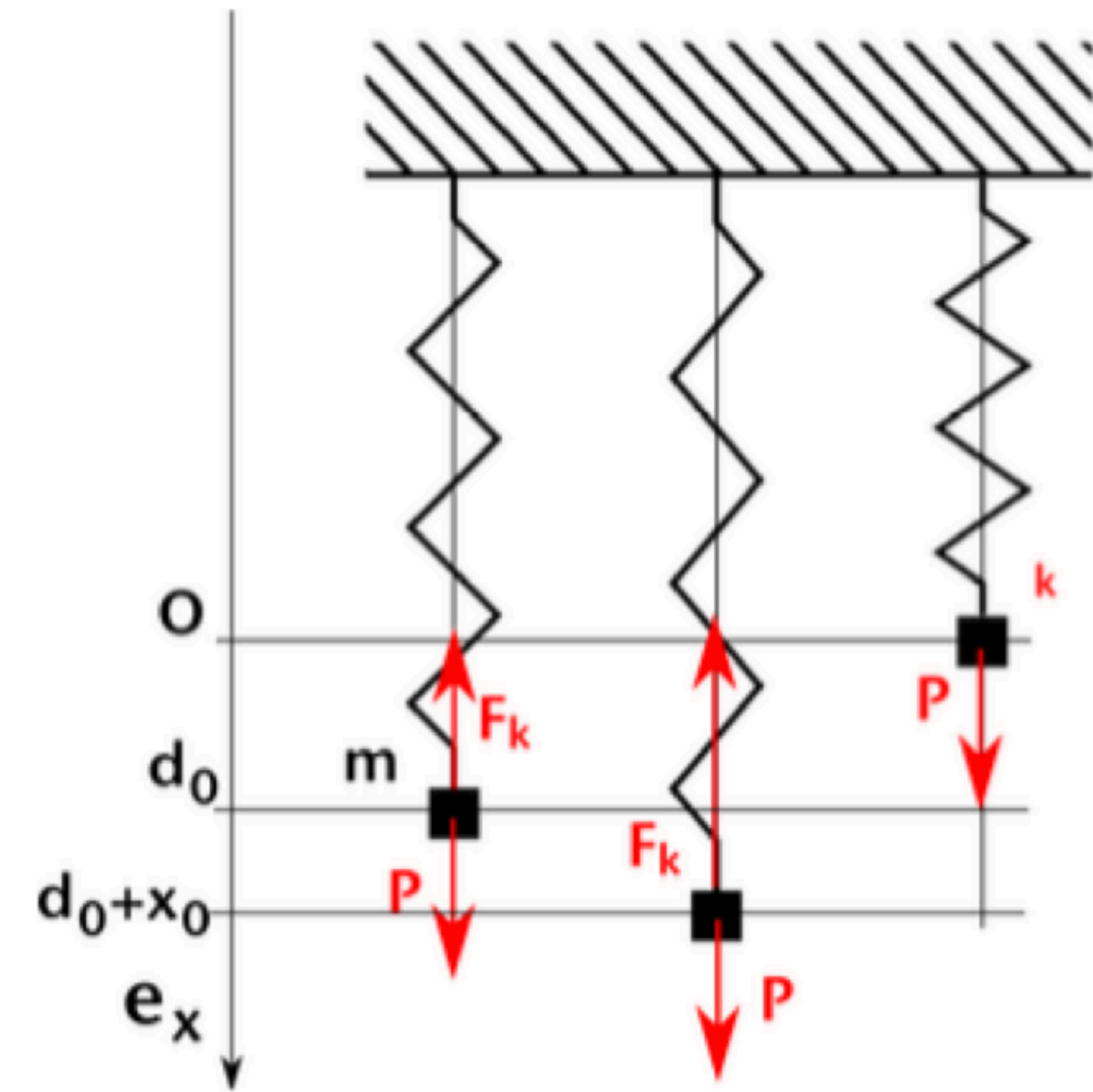
EXERCISE: VERTICAL SPRING



Consider a mass m suspended from a spring of rest length l_0 and spring constant k , attached vertically. The origin of the coordinate system O is placed at the position of the end of the spring when the spring is at rest. When the mass is attached to the spring, the equilibrium position is denoted d_0 . The mass is pulled downwards by a distance x_0 , in addition to d_0 , and released from rest.

Calculate the position of the mass as a function of time

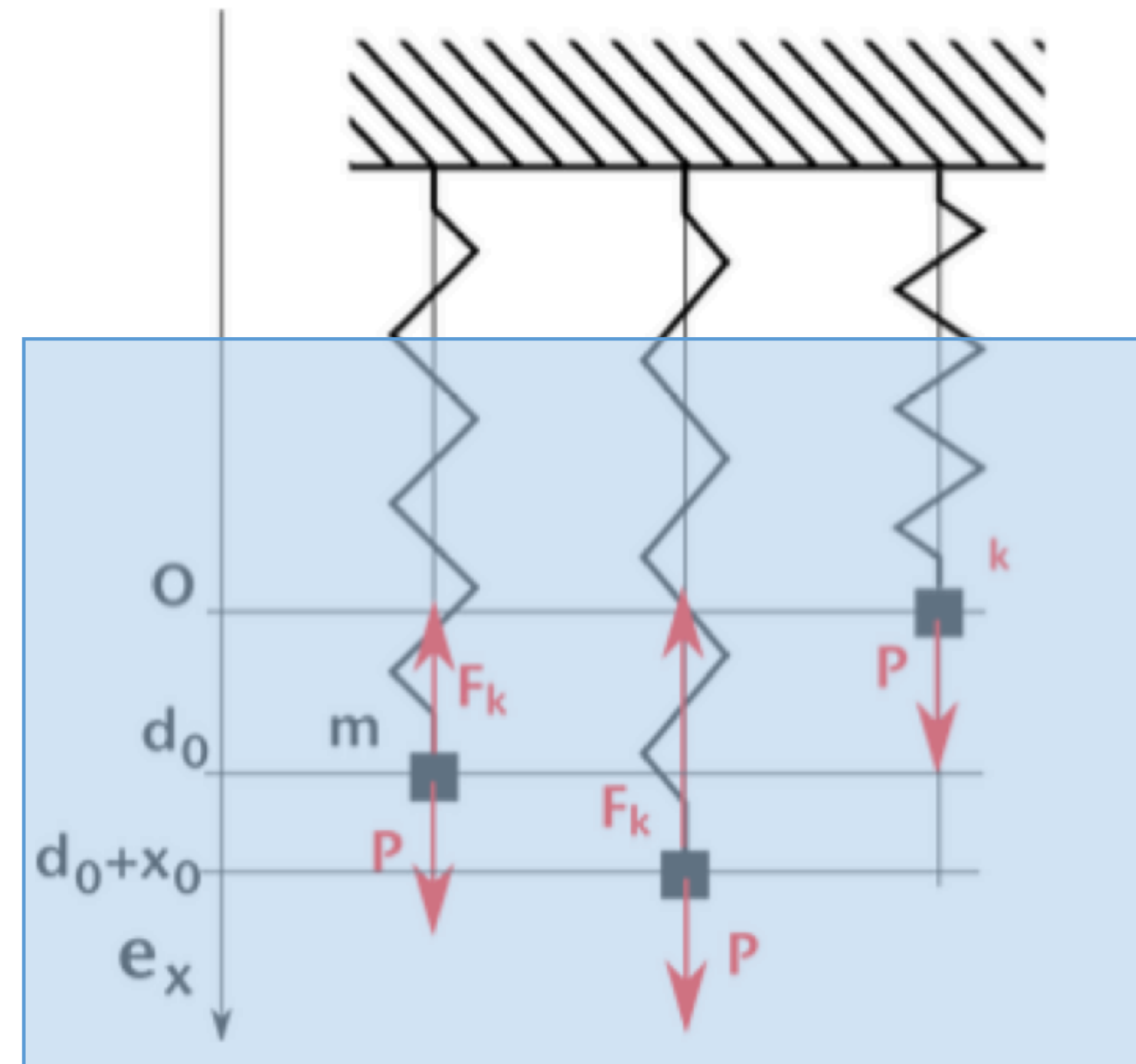
EXERCISE: VERTICAL SPRING



Mass m , spring of rest length l_0 , spring constant k , mass position d_0 , initial displacement to $x_0 + d_0$, and released from rest.

1. Calculate d_0 as a function of the given data.
2. Find the differential equation of motion using the in the coordinate system shown in the figure.
3. Make a change of variable on x to obtain the usual differential equation and solve it.
4. Calculate and draw the potential energy in the reference frame of the figure.
5. Use this potential energy to find the previous solution

QUIZ: UNDERWATER VERTICAL SPRING

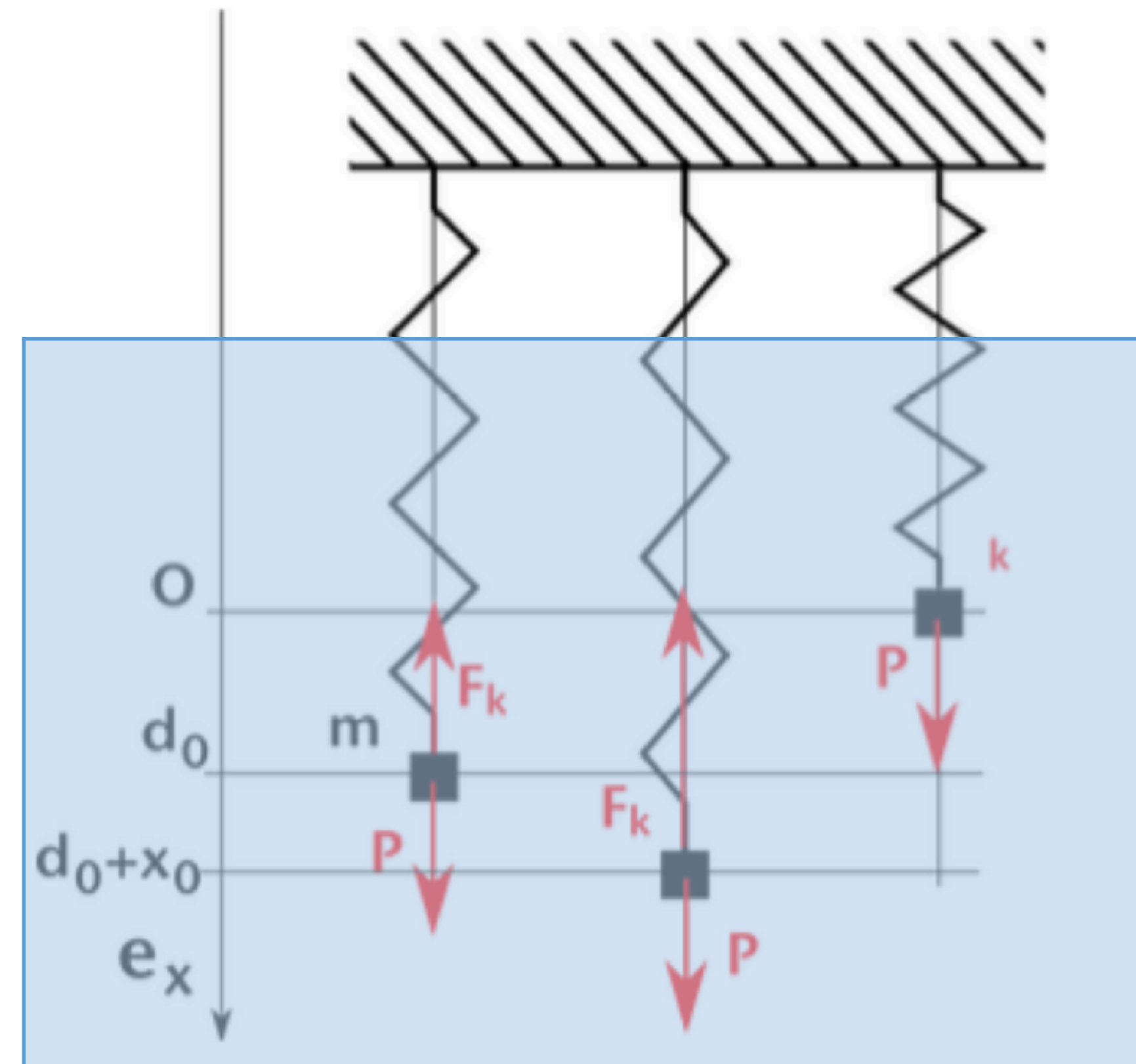


Consider a damped oscillator consisting of a mass m immersed in a fluid and attached to a spring. If we take into account the buoyant force (Archimedes' principle) compared to the case where we neglect it, this changes:

- The natural frequency 0%
- The pseudo-frequency 0%
- The equilibrium position 0%
- Nothing changes 0%



QUIZ: UNDERWATER VERTICAL SPRING



Consider a damped oscillator consisting of a mass m immersed in a fluid and attached to a spring. If we take into account the buoyant force (Archimedes' principle) compared to the case where we neglect it, this changes:

- The natural frequency 0%
- The pseudo-frequency 0%
- The equilibrium position 0%
- Nothing changes 0%



APPENDIX: CRITICAL DAMPING

Differential equation

$$\frac{d^2f}{dt^2} + 2Y\frac{df}{dt} + \Omega_0^2 f = 0$$

General solution

$$f(t) = u(t)e^{\lambda_1 t}?$$

Assume that λ_1 is a root of

$$\lambda^2 + 2Y\lambda + \Omega_0^2$$

$$\lambda_1 = -Y + \sqrt{Y^2 - \Omega_0^2} \quad \lambda_2 = -Y - \sqrt{Y^2 - \Omega_0^2}$$

Bonus exercise: show that if $\lambda_1 = \lambda_2 = -Y = -\Omega_0$ (Critical damping)

$$f(t) = (A + Bt)e^{\lambda_1 t}$$

APPENDIX: CRITICAL DAMPING

Differential equation

$$\frac{d^2f}{dt^2} + 2Y\frac{df}{dt} + \Omega_0^2 f = 0$$

General solution

$$f(t) = u(t)e^{\lambda_1 t}?$$

$$\dot{f} = \dot{u}e^{\lambda_1 t} + \lambda_1 u e^{\lambda_1 t}$$

$$\ddot{f} = \ddot{u}e^{\lambda_1 t} + 2\lambda_1 \dot{u}e^{\lambda_1 t} + \lambda_1^2 u e^{\lambda_1 t}$$

$$(\ddot{u}e^{\lambda_1 t} + 2\lambda_1 \dot{u}e^{\lambda_1 t} + \lambda_1^2 u e^{\lambda_1 t}) + 2Y(\dot{u}e^{\lambda_1 t} + \lambda_1 u e^{\lambda_1 t}) + \Omega_0^2 (u e^{\lambda_1 t}) = 0$$

$$\ddot{u} + 2\lambda_1 \dot{u} + \lambda_1^2 u + 2Y\dot{u} + 2Y\lambda_1 u + \Omega_0^2 u = 0$$

$$\ddot{u} + \dot{u}(2\lambda_1 + 2Y) + u(\lambda_1^2 + 2Y\lambda_1 + 2Y\Omega_0^2) = 0$$

Zero because λ is a root of this term

$$\ddot{u} + \dot{u}(2\lambda_1 + 2Y) = 0$$

APPENDIX: CRITICAL DAMPING

Differential equation

$$\frac{d^2f}{dt^2} + 2Y\frac{df}{dt} + \Omega_0^2 f = 0$$

General solution

$$f(t) = u(t)e^{\lambda_1 t}?$$

$$\ddot{u} + \dot{u}(2\lambda_1 + 2Y) = 0$$

$$\lambda_1 = -Y + \sqrt{Y^2 - \Omega_0^2}$$

$$\lambda_2 = -Y - \sqrt{Y^2 - \Omega_0^2}$$

$$\Rightarrow \lambda_1 + \lambda_2 = -2Y$$

$$2\lambda_1 + 2Y = 2\lambda_1 - \lambda_1 - \lambda_2 = \lambda_1 - \lambda_2$$

$$\ddot{u} + \dot{u}(\lambda_1 - \lambda_2) = 0$$

APPENDIX: CRITICAL DAMPING

Differential equation

$$\frac{d^2f}{dt^2} + 2Y\frac{df}{dt} + \Omega_0^2 f = 0$$

Over damped or underdamped

$$\text{if } \lambda_1 \neq \lambda_2$$

General solution

$$f(t) = u(t)e^{\lambda_1 t}?$$

$$\ddot{u} + \dot{u}(\lambda_1 - \lambda_2) = 0$$

$$\dot{u} = Ae^{-(\lambda_1 - \lambda_2)t}$$

$$u = \frac{A}{\lambda_1 - \lambda_2} e^{-(\lambda_1 - \lambda_2)t} + B$$

$$f(t) = \frac{A}{\lambda_1 - \lambda_2} e^{-(\lambda_1 - \lambda_2)t} e^{\lambda_1 t} + Be^{\lambda_1 t}$$

$$f(t) = \frac{A}{\lambda_1 - \lambda_2} e^{\lambda_2 t} + Be^{\lambda_1 t}$$

APPENDIX: CRITICAL DAMPING

Differential equation

$$\frac{d^2f}{dt^2} + 2Y\frac{df}{dt} + \Omega_0^2 f = 0$$

Critically damped

$$\text{if } \lambda_1 = \lambda_2$$

General solution

$$f(t) = u(t)e^{\lambda_1 t}?$$

$$\ddot{u} + \dot{u}(\lambda_1 - \lambda_2) = 0$$

$$\ddot{u} + \dot{u} \times (0) = 0$$

$$\ddot{u} = 0$$

$$u = A + Bt$$

$$f(t) = (A + Bt)e^{\lambda_1 t}$$