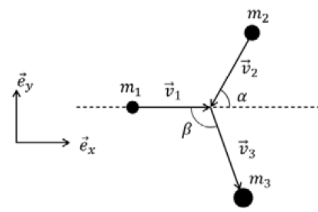
Exercises

Exercise 1 *A shocking duel*

Let there be two Swiss wrestlers with masses m_1 and m_2. The two combatants collide with velocities $\vec{v_1}$ and $\vec{v_2}$ as shown in the diagram.

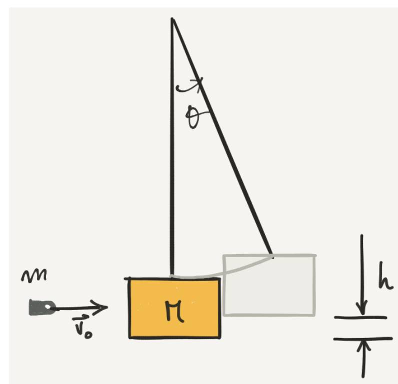


a) Calculate the velocity $\vec{v_3}$ (magnitude and angle β) knowing that after the collision, the two wrestlers remain in contact.

b) Calculate the energy dissipated during the collision. For what value of α is the dissipated energy maximal ?

Exercise 2 *A bulletproof block*

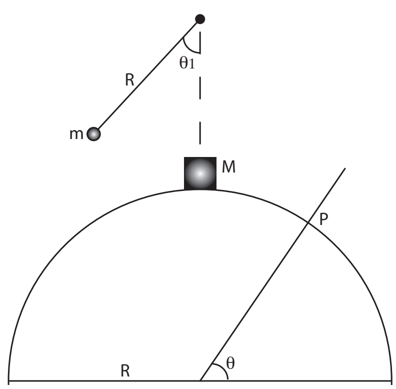
A ballistic pendulum is used to measure the velocity of a bullet of mass m fired from a gun. The bullet is fired into a block of wood of mass M suspended from a string, which is initially at rest. The bullet embeds itself in the block, causing the pendulum to rise to a height h.



1. Show that measuring h allows us to measure v_0 knowing m and M.
2. Assume $m \ll M$. Show that almost all of the ball's kinetic energy is dissipated in the collision.

Exercise 3 *The ice cap is breaking away*

A block of wood with mass M is placed in equilibrium at the top of a hemisphere with radius R. It can slide without friction. A ball of mass m, connected to a string of length R (see diagram), is released at an angle θ_1. The collision with M is perfectly elastic.



1. Determine the angle θ at which mass M leaves the spherical cap.
2. Assume $\theta_1 = 90°$. For what limit value of m does the block lift off immediately, without starting to slide along the sphere ?

Exercise 4 *Un exercice qui nous laisse sur le carreau**
In this exercise, we are looking for "stopping" conditions during an elastic collision. We say that there is "stopping" when a puck thrown at another puck remains stationary after the collision. We conduct the experiments on a perfectly horizontal air cushion table ; the pucks (flat cylinders) slide on it without any friction. The pucks are considered to be solid objects without rotation.

1. A puck of mass M is thrown at velocity $\vec{V}$ against another puck of mass m_a. Show that for a stopping to occur, the pucks must have the same mass ($m_a = M$).

2. We now throw the puck of mass M against two pucks of the same mass m_b. These two pucks are arranged symmetrically, so that after the collision they fly off in opposite directions with the same speed v and at the same angle α to the direction of the throw.

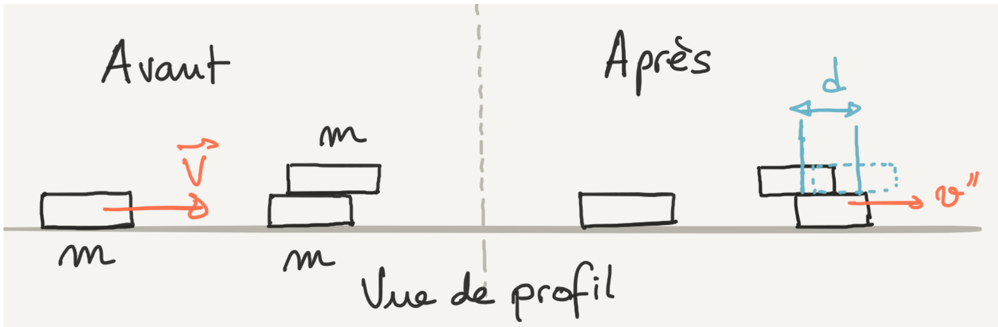
Calculate the value of mass m_b so that there is stopping. Express m_b as a function of the data in the problem.

m_b..

3. We now stack two target pucks, both with mass m, as shown in the diagram below : the top puck is slightly offset to the right of the bottom puck. There is solid friction between these two pucks, with μ_c being the coefficient of kinetic friction. A puck of mass m is thrown at the stack at a speed of $\vec{V}$, and we see that there is once again stopping.
After the collision, the two target pucks are still stacked and move at velocity v'' in the direction of the throw. We also observe that the upper puck has shifted to the left by a distance d relative to its initial position on the lower puck.
   (a) Express v'' in terms of the data given in the problem.

v''..

∗. This is a French wordplay : the expression « laisser quelqu'un sur le carreau » means "to leave someone behind", but "condition de carreau" in physics refers to the case of a head-on elastic collision between equal masses, where the projectile stops completely.



Side view of the problem. Left : before collision, right : after collision.

(b) We consider the set of 3 pucks as a system. Is the collision elastic ? Justify your answer.

☐ Yes ☐ No

(c) Calculate the change in kinetic energy during the collision as a function of m and V.

ΔE_k..

(d) Calculate the kinetic friction coefficient μ_c as a function of V and d.

μ_c..

Solutions

Solution 1

a) Before the collision, we have :

$$\vec{p_1} = m_1\vec{v_1} = m_1 v_1 \vec{e}_x$$

$$\vec{p_2} = m_2\vec{v_2} = -m_2 v_2(\cos\alpha\vec{e}_x + \sin\alpha\vec{e}_y)$$

We have a perfectly inelastic (soft) collision, so the two wrestlers remain attached after the collision and momentum is conserved :

$$\vec{p_3} = (m_1 + m_2)\vec{v_3} = \vec{p_1} + \vec{p_2}$$

We deduce the magnitude of $\vec{v_3}$:

$$||\vec{v_3}|| = \frac{1}{m_1 + m_2}||\vec{p_1} + \vec{p_2}|| = \frac{\sqrt{p_1^2 + p_2^2 + 2\vec{p_1}\vec{p_2}}}{m_1 + m_2} = \frac{\sqrt{m_1^2 v_1^2 + m_2^2 v_2^2 - 2m_1 m_2 v_1 v_2 \cos\alpha}}{m_1 + m_2}$$

The following formula is used to calculate the angle :

$$\tan(\beta - \pi) = \tan\beta = \frac{v_{3y}}{v_{3x}} = \frac{p_{3y}}{p_{3x}} = \frac{p_{1y} + p_{2y}}{p_{1x} + p_{2x}}$$

$$\Rightarrow \tan\beta = \frac{-m_2 v_2 \sin\alpha}{m_1 v_1 - m_2 v_2 \cos\alpha}$$

Care must be taken because $\arctan(x)$ returns an angle between $-\pi/2$ and $\pi/2$, whereas β is between 0 and π (see diagram).
If $(m_1 v_1 - m_2 v_2 \cos\alpha) < 0$

$$\beta = \tan^{-1}(\frac{-m_2 v_2 \sin\alpha}{m_1 v_1 - m_2 v_2 \cos\alpha})$$

If $(m_1 v_1 - m_2 v_2 \cos\alpha) > 0$

$$\beta = \pi + \tan^{-1}(\frac{-m_2 v_2 \sin\alpha}{m_1 v_1 - m_2 v_2 \cos\alpha})$$

b) Calculation of energy dissipated during impact. The kinetic energy before the collision is written as :

$$E_{k_{init}} = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$$

After the collision, we have :

$$E_{k_{final}} = \frac{1}{2}(m_1+m_2)v_3^2 = \frac{m_1^2v_1^2 + m_2^2v_2^2 - 2m_1m_2v_1v_2\cos\alpha}{2(m_1+m_2)}$$

The dissipated energy is thus :

$$E_{k_{dissipated}} = E_{k_{init}} - E_{k_{final}} = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 - \frac{m_1^2v_1^2 + m_2^2v_2^2 - 2m_1m_2v_1v_2\cos\alpha}{2(m_1+m_2)}$$

$$E_{k_{dissipated}} = \frac{(m_1v_1^2+m_2v_2^2)(m_1+m_2) - (m_1^2v_1^2 + m_2^2v_2^2 - 2m_1m_2v_1v_2\cos\alpha)}{2(m_1+m_2)}$$

$$E_{k_{dissipated}} = \frac{m_1m_2(v_1^2+v_2^2+2v_1v_2\cos\alpha)}{2(m_1+m_2)}$$

$$E_{k_{dissipated}} = \frac{1}{2}\mu||\vec{v_1}-\vec{v_2}||^2 \text{ with } \mu = \frac{m_1m_2}{m_1+m_2}$$

We note that the energy dissipated is maximal when $\alpha = 0$, that is, when there is a head-on collision between the two wrestlers.

Solution 2

1. Perfectly inelastic (soft) collision $\Rightarrow$ we obtain v velocity after the collision for the system (ball, block).
   Conservation of $\vec{p}$: $\quad mv_0 = (m+M)\,v \quad v = \frac{m}{m+M}v_0$
   Conservation of E_m in the phase "pendulum" :
   $$E_{k1} + E_{p1} = E_{k2} + E_{p2} \tag{1}$$
   $$\frac{1}{2}\cancel{(m+M)}v^2 + 0 = 0 + \cancel{(m+M)}gh \tag{2}$$
   $$v^2 = 2gh \;\Rightarrow\; v = \sqrt{2gh} \tag{3}$$
   and so :
   $$v_0 = \frac{m+M}{m}v = \frac{m+M}{m}\sqrt{2gh} \tag{4}$$
2. $$E_{kf} = \frac{1}{2}\,(m+M)\,v^2 \tag{5}$$
   $$E_{ki} = \frac{1}{2}mv_0^2 \tag{6}$$
   Dissipated energy :
   $$\Delta E = E_{ki} - E_{kf} = \frac{1}{2}mv_0^2 - \frac{1}{2}\,(m+M)\,v^2 \tag{7}$$
   $$\Delta E = \frac{1}{2}mv_0^2 - \frac{1}{2}\cancel{(m+M)}\frac{m^2}{(m+M)^{\cancel{2}}}v_0^2 \tag{8}$$
   $$= \frac{1}{2}mv_0^2\left(1 - \frac{m}{m+M}\right) \tag{9}$$

Since

$$m << M \qquad \frac{m}{m+M} << 1 \qquad \Rightarrow \Delta E \simeq E_{ki} \tag{10}$$

Solution 3

1. Let v_1 be the velocity of m just before the collision. Using the conservation of energy [†], we have $mgh_1 = \frac{1}{2}mv_1^2$, which allows us to find this speed :

$$v_1^2 = 2gh_1 = 2gR(1 - \cos\theta_1)$$

Determination of the velocity of M immediately after the frontal elastic collision : We use the formulas :

$$\vec{v}_1' = \frac{(m_1 - m_2)\vec{v}_1 + 2m_2\vec{v}_2}{m_1 + m_2}$$

$$\vec{v}_2' = \frac{(m_2 - m_1)\vec{v}_2 + 2m_1\vec{v}_1}{m_1 + m_2}$$

With $m_2 = M$ and $m_1 = m$ and since $\vec{v}_2 = \vec{0}$, we gather :

$$\boxed{\vec{v}_2' = \frac{2m}{m+M}\vec{v}_1}$$

The takeoff point P is given by the condition $\vec{N} = \vec{0}$, where $\vec{N}$ is the reaction of the support. Newton's second law, the circular motion described by M on the cap, and a geometric consideration give us $-N\vec{e}_r + Mg\sin\theta\vec{e}_r = M\frac{v^2}{R}\vec{e}_r$, and therefore at P ($\vec{N} = \vec{0}$) :

$$v_P^2 = gR\sin\theta \tag{11}$$

The velocity at P is given by the conservation of energy :

$$\frac{1}{2}Mv_P^2 = \frac{1}{2}Mv_2'^2 + Mgh_2 = \frac{1}{2}Mv_2'^2 + MgR(1 - \sin\theta)$$

It thus follows :

$$\begin{aligned} v_P^2 &= 2gR(1 - \sin\theta) + v_2'^2 \\ &= 2gR(1 - \sin\theta) + \left(\frac{2m}{m+M}\right)^2 \underbrace{2gR(1 - \cos\theta_1)}_{v_1^2} \end{aligned}$$

†. To simplify the calculations, the "0 potential" is placed at the top of the spherical cap for m and at point P for M.

and (11) gives $v_P^2 = gR\sin\theta$. So,

$$3\sin\theta = 2 + 2\left(\frac{2m}{m+M}\right)^2 (1-\cos\theta_1)$$

or finally :

$$\boxed{\sin\theta = \frac{2 + 2\left(\frac{2m}{m+M}\right)^2 (1-\cos\theta_1)}{3}}$$

2. Since $\theta_1 = 90°$, $\cos\theta_1 = 0$, and so

$$\sin\theta = \frac{2 + 2\left(\frac{2m}{m+M}\right)^2}{3}$$

For the block to fly off directly, we must have $\sin\theta = 1$ ($\theta = 90°$) ! Thus, it comes

$$2\left(\frac{2m}{m+M}\right)^2 = 1$$

or

$$\frac{2m}{m+M} = \frac{1}{\sqrt{2}}$$

By rearranging the terms of this last expression, we obtain

$$m = M\frac{1}{2\sqrt{2}-1}$$

The lower limit of m is therefore $m_{lim} = 0.54M$ for M to take off directly.

Solution 4

1. Conservation of $\vec{p}$ and E_k (elastic collision) :

$$\vec{p} : M\vec{v} + \vec{0} = \vec{0} + m_a\vec{v}_2' \Rightarrow MV = m_a\vec{v}_2'$$
$$E_k : \frac{1}{2}Mv^2 = \frac{1}{2}m_a v_2'^2 \Rightarrow \cancel{MV}V = \cancel{m_a v_2'}v_2' \Rightarrow V = v_2'$$
$$MV = m_a v_2' \text{ et } V = v_2' \Rightarrow M = m_a$$

2. Now we must take into account the vector nature of $\vec{p}$:

$$M\vec{V} + \vec{0} = \vec{0} + m_b\vec{v}_1 + m_b\vec{v}_2$$

Projection on (Ox) :
$$MV = 2m_b v \cos\alpha \tag{12}$$
Conservation of E_k :
$$\frac{1}{2}MV^2 = \frac{1}{2}m_b v^2 + \frac{1}{2}m_b v^2 = m_b v^2 \tag{13}$$

12 squared : $M^2V^2 = 4m_b^2 v^2 \cos^2\alpha$

13 : $MV^2 = 2m_b v^2$

13 in 12 : $M^2V^2 = M \cdot MV^2 = M \cdot 2\cancel{m_b v^2} = 4m_b^{\cancel{2}}\cancel{v^2}\cos^2\alpha \Rightarrow m_b = \dfrac{M}{2\cos^2\alpha}$

3. a) Conservation of $\vec{p}$ (always valid) :

$$mV = (m+m)v'' = 2mv'' \Rightarrow v'' = \frac{V}{2}$$

b) Friction between the pucks dissipates energy $\Rightarrow$ No, the collision is not elastic.

c) Before : $E_k = \dfrac{1}{2}mV^2$

After : $E_k = \dfrac{1}{2}(2m)v''^2 = \dfrac{1}{2}(2m)\left(\dfrac{V}{2}\right)^2 = \dfrac{1}{4}mV^2$

$$\Delta E_k = E_f - E_i = \frac{1}{4}mV^2 - \frac{1}{2}mV^2$$

d) Friction force : $F_F = \mu_c R = \mu_c mg$
$W_F = \int_{\text{displacement}} \vec{F}_f \cdot \vec{r} = -\mu_c mgd = \Delta E_k = -\frac{1}{4}mV^2$

$$\Rightarrow \mu_c = \frac{V^2}{4gd}$$