Exercises

Exercise 1
A rowdy student shoots paper balls at a speed of $v_0 = 10$ m/s from a height of 1.2 m using his blowgun. He aims at the top of the blackboard 3 m in front of him. The blackboard is hung 1 m above the floor and is 1.50 m high. Does he hit the blackboard ?
Tip : $\frac{1}{\cos\alpha^2} = 1 + \tan\alpha^2$

Exercise 2 , *30 minutes* **
A cannon is placed at the foot of a hill whose slope forms an angle φ with the horizontal. If the cannon is pointed at an angle α from the horizontal and the shell has an initial velocity v_0, find the distance l measured *on the hill* at which it will fall.

Exercise 3 *Winter is coming*
A student in a physics class gets into a snowball fight with a friend. The friend manages to catch the snowballs and throw them back immediately.
The student knows that a snowball can be thrown at two different angles but with the same speed and still hit the same target. However, the flight times are different. So, to win the game, the student decides to throw two balls at different times, one on a higher trajectory than the other. The higher ball will create a diversion, and while the friend prepares to catch it, the second ball will arrive and both balls will strike simultaneously. If the friends are at a distance L from each other and they throw the balls at an initial velocity v_0 :

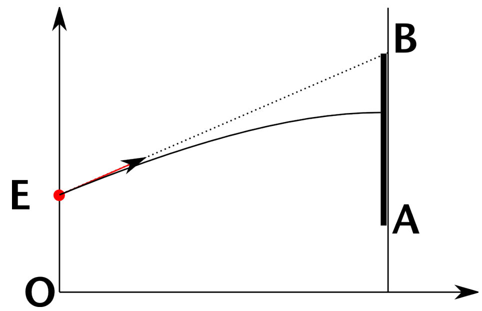
1. What are the firing angles ?
2. How long should you wait before throwing the second ball ?
3. Numerical application : $L = 25$ m and $v_0 = 20$ m/s.

Exercise 4 *David and Goliath*
This is an example of an exercise that requires (a little) modeling of the problem and breaking it down into elements. This approach is part of an engineer's daily routine.

A slingshot consists of a 30 cm long string that holds a stone and is spun quickly in a horizontal plane, then released abruptly. If the slingshot is used from a height of 1.8 meters and the aim is to hit a target on the ground 10 meters away, how fast must the slingshot be spun ? How could you achieve a lower rotation speed (using the same slingshot) ? What is the minimum rotation speed ?

Solutions

Solution 1



$$E = \begin{pmatrix} x_E \\ y_E \end{pmatrix} = \begin{pmatrix} 0 \\ 1.2 \end{pmatrix} \qquad A = \begin{pmatrix} x_A \\ y_A \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \qquad B = \begin{pmatrix} x_B \\ y_B \end{pmatrix} = \begin{pmatrix} 3 \\ 2.5 \end{pmatrix}$$

But,

$$\tan\alpha = \frac{y_B - y_E}{x_B} = \frac{y_B - y_E}{x_A}$$

It thus follows :

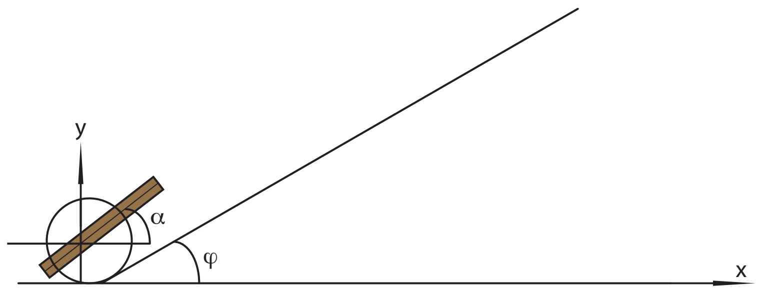
$$\vec{a} = \begin{pmatrix} 0 \\ -g \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} v_0\cos\alpha \\ -gt + v_0\sin\alpha \end{pmatrix} \qquad \vec{r} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} (v_0\cos\alpha)t \\ -\frac{1}{2}gt^2 + (v_0\sin\alpha)t + y_E \end{pmatrix}$$

The ball hits the wall at $t_A = \frac{x_A}{(v_0\cos\alpha)}$. The y-coordinate is then given by

$$\begin{aligned}
y(t_A) &= -\frac{1}{2}gt_A^2 + (v_0\sin\alpha)t_A + y_E \\
&= -\frac{gx_A^2}{2v_0^2\cos^2\alpha} + (\tan\alpha)x_A + y_E \\
&= -\frac{gx_A^2}{2v_0^2}(1+\tan^2\alpha) + (\tan\alpha)x_A + y_E \\
&= -\frac{gx_A^2}{2v_0^2}\left(1+\left(\frac{y_B-y_E}{x_A}\right)^2\right) + \frac{y_B-y_E}{x_A}x_A + y_E \\
&= -\frac{g}{2v_0^2}\left(x_A^2 + (y_B-y_E)^2\right) + y_B
\end{aligned}$$

Note : $y(t_A) = 1.98$ m. The ball therefore hits the wall 1.98 m above the ground.
The board is hit.

Solution 2



System : shell ; reference frame : terrestrial ; coordinate system $(O\vec{x}\vec{y})$.
Note : it is better to use this reference point because in this case the equations of motion are easier to write (acceleration according to $(O\vec{y})$ only).
For the shell fired at $t = 0$ from O at $\vec{v}_0$, we have :

$$\vec{a} = \begin{pmatrix} 0 \\ -g \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} v_0 \cos\alpha \\ -gt + v_0 \sin\alpha \end{pmatrix} \qquad \vec{r}_{ob} = \begin{pmatrix} (v_0 \cos\alpha)t \\ -\frac{1}{2}gt^2 + (v_0 \sin\alpha)t \end{pmatrix}$$

The shell falls on the hill when the curve described by the shell intersects the equation of the hill $y = x \tan\varphi$.
The shell falls at $P = \begin{pmatrix} x_p \\ y_p \end{pmatrix}$ and therefore $y_p = x_p \tan\varphi$.

$$\vec{r}_p = \begin{cases} x_p = (v_0 \cos\alpha)t_p & (1) \\ y_p = -\dfrac{1}{2}gt_p^2 + (v_0 \sin\alpha)t_p & (2) \end{cases}$$

$$y_p = x_p \tan\varphi \qquad (3)$$

We have three equations with three unknowns x_p, y_p, t_p. We are looking for x_p or y_p...
Let us eliminate t_p : (1) $\Rightarrow t_p = \frac{x_p}{v_0 \cos\alpha}$.

$$\begin{cases} y_p = -\dfrac{1}{2}g\left(\dfrac{x_p}{v_0 \cos\alpha}\right)^2 + (v_0 \sin\alpha)\dfrac{x_p}{v_0 \cos\alpha} \\ y_p = x_p \tan\varphi \end{cases}$$

Combining these two equations, we get

$$x_p \tan\varphi + \frac{1}{2}g\frac{1}{v_0^2 \cos^2\alpha}x_p^2 - \tan\alpha x_p = 0$$

It follows that :

$$\frac{g}{2v_0^2 \cos^2\alpha}x_p^2 - (\tan\alpha - \tan\varphi)x_p = 0 \Rightarrow x_p = 0 \text{ (not useful)}$$

or

$$\frac{g}{2v_0^2\cos^2\alpha}x_p = \tan\alpha - \tan\varphi$$

Thus :

$$\boxed{x_p = \frac{2v_0^2\cos^2\alpha}{g}[\tan\alpha - \tan\varphi]}$$

so :

$$\boxed{l = \frac{2v_0^2\cos^2\alpha[\tan\alpha - \tan\varphi]}{g\cos\varphi}}$$

With a few trigonometric transformations, we could also write

$$l = \frac{2v_0^2}{g}\frac{\cos\alpha\sin(\alpha-\varphi)}{\cos^2\varphi}$$

Solution 3

1. The equations of motion are :

$$\vec{a} = \begin{pmatrix} 0 \\ -g \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} v_0\cos\alpha \\ -gt + v_0\sin\alpha \end{pmatrix} \qquad \vec{r} = \begin{pmatrix} v_0\cos\alpha\, t \\ -\frac{1}{2}gt^2 + (v_0\sin\alpha)t \end{pmatrix}$$

The equations of motion at point L give :

$$\begin{cases} L = v_0\cos\alpha\, t_f \\ 0 = -\frac{1}{2}gt_f^2 + v_0\sin\alpha\, t_f \end{cases}$$

Variant 1 We combine these two equations to obtain

$$\begin{cases} t_f = \dfrac{L}{v_0\cos\alpha} \\ 2\sin\alpha\cos\alpha = \dfrac{gL}{v_0^2} \end{cases}$$

Knowing that $2\sin\alpha\cos\alpha = \sin 2\alpha$, we obtain $\sin 2\alpha = \frac{gL}{v_0^2}$, which gives us two angles of fire, since $\sin\alpha = \sin(\pi - \alpha)$.

Variant 2 We rearrange these two equations to obtain

$$\begin{cases} v_0\cos\alpha = \dfrac{L}{t_f} \\ v_0\sin\alpha = \frac{1}{2}gt_f \end{cases}$$

By adding the square of each of these equations, we obtain

$$v_0^2 = \frac{L^2}{t_f^2} + \frac{1}{4}g^2t_f^2$$

which allows us to find, using Viet's formula :

$$t_f^2 = \frac{v_0^2 \pm \sqrt{v_0^4 - g^2 L^2}}{\frac{1}{2} g^2}$$

Both solutions give the flight times for each trajectory ; the angles are obtained using the formula $\cos\alpha = \frac{L}{v_0 t_f}$

2. Shooting time for both balls (variant 1) :

$$t_{1,2} = \frac{L}{v_0 \cos\alpha_{1,2}}$$

and so :

$$\Delta t = t_2 - t_1 = \frac{L}{v_0}\left(\frac{1}{\cos\alpha_2} - \frac{1}{\cos\alpha_1}\right)$$

3. Note : The firing angles are : $\alpha_2 = 18.9°$ and $\alpha_1 = 71.1°$; the flight times become $t_1 = 1.32$ s, $t_2 = 3.86$ s, so $\Delta t = 2.54$ s.

Solution 4

First, we assume that the slingshot and the stone are on the same plane parallel to the ground.
The reference frame is terrestrial and the reference point is (O, x, y), where O is the attachment point of the slingshot.
Let P be the point of impact of the stone. Then

$$P = \begin{pmatrix} l \\ -h \end{pmatrix}$$

The first thing to do is to calculate the velocity v_0 that the stone must have in order to reach P. Then, with v_0, we will be able to calculate the angular velocity of the slingshot…

Ballistics gives us :

$$\vec{a} = \begin{pmatrix} 0 \\ -g \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} v_0 \\ -gt \end{pmatrix} \qquad \vec{r} = \begin{pmatrix} v_0 t \\ -\frac{1}{2} g t^2 \end{pmatrix}$$

The arrival at the ground occurs at $P = \begin{pmatrix} l \\ -h \end{pmatrix}$ at t_p, i.e.

$$\vec{r}(t_p) = \begin{pmatrix} v_0 t_p = l \\ -\frac{1}{2} g t_p^2 = -h \end{pmatrix} \qquad \begin{matrix} (4) \\ (5) \end{matrix}$$

From (4), we have $t_p = \frac{l}{v_0}$, which we substitute into (5) : $\frac{1}{2}g\frac{l^2}{v_0^2} = h$, from which we obtain :

$$\boxed{v_0 = l\sqrt{\frac{g}{2h}}}$$

From the relationship $v_0 = \omega_0 R$, we have :

$$\boxed{\omega_0 = \frac{l}{R}\sqrt{\frac{g}{2h}}}$$

Note : $\omega_0 = \frac{10}{0.30}\sqrt{\frac{9.81}{2\cdot 1.8}} \simeq 55$ rad/s.

It is possible to reduce the speed of rotation of the slingshot to hit the target by changing the angle of fire. To determine this, the speed must be minimized in relation to this angle. By adding a defined angle of fire relative to the horizontal to the problem (as a reminder, $-\arctan\frac{h}{l} < \alpha < \frac{\pi}{2}$), we find the equation for velocity as follows :

$$\vec{a} = \begin{pmatrix} 0 \\ -g \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} v_0\cos\alpha \\ -gt + v_0\sin\alpha \end{pmatrix} \qquad \vec{r}(t) = \begin{pmatrix} v_0 t\cos\alpha \\ -\frac{1}{2}gt^2 + v_0 t\sin\alpha \end{pmatrix}$$

$$\vec{r} = \overrightarrow{OP} \Rightarrow \begin{cases} v_0 t_p \cos\alpha = l & (6) \\ -\dfrac{1}{2}gt_p^2 + v_0 t_p \sin\alpha = -h & (7) \end{cases}$$

From (6), we get : $t_p = \frac{l}{v_0\cos\alpha}$, which in substitute into (7) : $-\frac{1}{2}g\frac{l^2}{v_0^2\cos^2\alpha} + l\tan\alpha = -h$, so $\frac{1}{2}g\frac{l^2}{v_0^2\cos^2\alpha} = h + l\tan\alpha$, and finally :

$$\boxed{v_0 = \frac{l}{\cos\alpha}\sqrt{\frac{g}{2(h + l\tan\alpha)}} \Rightarrow \omega_1 = \frac{l}{R\cos\alpha}\sqrt{\frac{g}{2(h + l\tan\alpha)}}}$$

Minimizing v_0 corresponds to determining $\frac{dv_0}{d\alpha} = 0$, which can be solved numerically with a good calculator...or literally after a clever expansion. Let us rewrite v_0 :

$$v_0(\alpha) = \sqrt{\frac{gl^2/2}{\cos^2\alpha(h + l\tan\alpha)}} = \sqrt{\frac{cte}{f(\alpha)}}$$

with $f(\alpha) = \cos^2\alpha(h + l\tan\alpha)$. Since the function $1/x$ is monotonically decreasing for $x > 0$ and the function $\sqrt{x}$ is monotonically increasing, minimizing v_0 is equivalent to maximizing $f(x)$. We therefore seek the value of α such that $df/d\alpha = 0$. This is already less daunting than working with v_0 ...

$$f(\alpha) = h\cos^2\alpha + l\sin\alpha\cos\alpha$$

$$\frac{df}{d\alpha} = -2h\sin\alpha\cos\alpha + l(\cos^2\alpha - \sin^2\alpha) = -h\sin(2\alpha) + l\cos(2\alpha)$$

This expression is only canceled when $l\cos 2\alpha - h\sin 2\alpha = 0$, i.e., when $\tan 2\alpha = \frac{l}{h}$. The expression for the optimal firing angle then becomes relatively simple

$$\boxed{\alpha_{opt} = \frac{1}{2}\arctan\frac{l}{h}}$$

Note : $\alpha_{opt} = 39.8°$, $\omega_1 = 30,2$ rad/s.